Spectral theory of ordinary and partial linear differential operators on finite intervals

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Abstract

A new, unified transform method for boundary value problems on linear and integrable nonlinear partial differential equations was recently introduced by Fokas. We consider initial-boundary value problems for linear, constant-coefficient evolution equations of arbitrary order on a finite domain. We use Fokas' method to fully characterise well-posed problems. For odd order problems with non-Robin boundary conditions we identify sufficient conditions that may be checked using a simple combinatorial argument without the need for any analysis. We derive similar conditions for the existence of a series representation for the solution to a well-posed problem.

We also discuss the spectral theory of the associated linear two-point ordinary differential operator. We give new conditions for the eigenfunctions to form a complete system, characterised in terms of initial-boundary value problems.

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Declaration

I confirm that this is my own work and all material from other sources has been fully and properly acknowledged.

CHAPTER 1

Introduction

1.1. Background and motivation

This thesis is concerned with the theory of linear two-point initial-boundary value problems, the spectral theory of linear differential operators and the connections between the two fields. The boundary value problems we study are posed for linear, constant-coefficient, evolution partial differential equations in one space and one time variable. One of the best known examples of such a problem is the heat equation for a finite rod,

$$q_t = q_{xx}, \quad x \in [0,1], \quad t \in [0,T].$$

The primary interest in this work is not second order partial differential equations, such as the heat equation, but third and higher odd order equations. Indeed we study equations of the form

$$\partial_t q - (\partial_x)^n q = 0, \quad x \in [0,1], \quad t \in [0,T], \tag{1.1.1}$$

for any $n > 3$, n an odd integer.

To define an initial-boundary value problem for the partial differential equation (1.1.1) one must specify the initial state of the system, by prescribing $q(x,0)$ to be equal to some known function, and impose some conditions on the value of q and its x-derivatives at the left and right ends of the space interval. The problem is then to find a sufficiently smooth function $q : [0,1] \times [0,T] \to \mathbb{C}$ which satisfies the partial differential equation (1.1.1), the initial condition and the boundary conditions. It is reasonable to ask two questions relating to such problems:

(1) Does a solution exist and is that solution unique?

(2) If the answer is yes, how can the solution be expressedtlye0.9091 Tf 8.2s(g)-334(to)-Guou87(in

The wave equation was introduced and solved by d'Alembert [11], albeit under strict restrictions on the boundary conditions. The method was refined by Euler [21]. Bernoulli [2] introduced the idea that a solution of the wave equation might be expressed as an infinite series and Fourier [30] studied the heat equation similarly.

A form of Laplace transform method for partial differential equations was introduced by Euler in a paper [22], first presented in 1779 but not published until 1813. The integral Euler used had indefinite limits. Lagrange [38], originally published in 1759, used a Fourier transform method with definite integrals to solve the wave equation. Laplace himself solved a linear evolution partial differential equation using his eponymous transform with definite limits in Section V of [43], originally published in 1810, where he also derived an inverse transform. A survey of the history of the Laplace transform is given in [17, 18].

Fokas' transform method was originally developed for solving boundary value problems for non-linear partial differential equations [23] but has been successfully applied to elliptic [60] as well as evolution [24] linear partial differential equations. A good introduction to the significance of Fokas' method is given in [23] but it should be noted that the method was not fully refined at this stage. Sections 1.1–1.3 of [24] give a good overview of the method for linear, constant-coefficient boundary value problems.

Separation of variables

We aim to find a solution to a partial differential equation subject to an initial condition and some boundary conditions. To solve such an initial-boundary value problem using the method of separation of variables [30] one must make two assumptions: that a solution exists and that a solution is separable, in the sense that there exist sequences of functions ${}_k(x)$, ${}_k(t)$, whose products ${}_k(x)\,{}_k(t)$ satisfy the partial differential equation and boundary conditions, such that the solution may be expressed as a series with uniform convergence,

$$(x,t) = \sum_{k\in\mathbb{N}} {}_k(x)\,{}_k(t)$$

It is trivial to find the general solutions of these equations in terms of the common spectral parameter, $\in \mathbb{C}$. The boundary conditions then restrict to a sequence of discrete points $_k$, defining the $_k$, $_k$. Under the assumption that the series (1.1.2) converges uniformly, Fourier transform methods are used to determine the constants $_k$ in terms of the initial datum. It is well-known that the family of solutions $_k$ obtained from particular spectral problems forms an eigenfunction basis for the x-differential operator, with eigenvalues ^n_k, but for partial differential equations of third or higher order with any but the simplest boundary conditions this is not always true. This connection is critical in our work.

Laplace transform

In the Laplace transform method, separability of the solution is not assumed directly but it is necessary to assume that the Laplace transform can be inverted. The first step is to apply the time Laplace transform to the partial differential equation (1.1.1). Using the properties of this transform and the initial datum, this yields an inhomogeneous ordinary differential equation of order n in the Laplace transform of q. Solving this equation subject to the boundary conditions yields an expression for the Laplace transform of the solution.

The final step is to reconstruct the solution from its Laplace transform. If the domain is semi-infinite in time, if $T = \infty$, and the boundary data have sufficient decay then the transform may be invertible. An example is given in Appendix C of [28]. However, we study initial-boundary value problems on a finite domain so the solution at final time appears in the representation. To remove the effects of this function, it is necessary to make arguments similar to those we make for Fokas' method. However these arguments are more complex than their equivalents below because of the presence of fractional powers in the integrands.

Fokas' unified transform method

The first step of Fokas' method is to construct a Lax pair for the partial differential equation. The term `Lax pair' is usually reserved for nonlinear partial differential equations, following is trivial to find an integral solution with lower limit at an arbitrary point in the domain of the original partial differential equation. In Proposition 3.1 of [24] it is argued that, by taking the lower limit at each corner of the domain, a sectionally analytic function in the auxiliary parameter is defined in the whole complex the Laplace transform method uses a transform in only the time variable. Different partial differential equations and different boundary conditions require different transforms and finding a transform that will work for a particular initial-boundary value problem is not a simple task. It is particularly problematic when the partial differential equation is of third or higher order, particularly odd order, or the boundary conditions are complex.

In Fokas' method, as a simultaneous spectral analysis in both the space and the time variable is performed, a different type of transform is used. This simplifies the process of choosing the relevant transform as it may be immediately deduced from the Lax pair and is independent of the boundary conditions. It is therefore unsurprising that such a method should yield novel results, not only for nonlinear but also for linear partial differential equations.

One great advantage of the universal applicability of Fokas' method in the linear, constant-coefficient context is that for it to produce a solution one only has to guarantee that the problem is well posed, whereas separation of variables requires an extra assumption on the solution, that it be separable or that the x-differential operator admits a suitable basis of eigenfunctions. This means that, armed with Fokas' method, question (2) on page 2 may be considered fully resolved for any initial-boundary value problem posed for the partial differential equation (1.1.1). Question (1) may be expressed as the question *Is the problem well-posed?* This is one of the major topics of the present work.

Another great difference between the methods presented above is the representation of the result. Separation of variables yields a discrete series representation of the solution whereas Fokas' method gives the solution as a contour integral. The use of the definite article to describe `*the* solution' in the previous sentence is intentional as both of the methods are applied to problems known to be well-posed. This means that for separable, well-posed problems we now have two methods which yield two different representations of the same solution.

A method for converting the integral representation to a series representation for third order problems with particular boundary conditions is discussed in [9, 54]. In any attempt to generalise this argument to higher order problems and those with more exotic boundary conditions it is certainly necessary to consider another question, supplementary to the two questions on page 2: *Which well-posed initial-boundary value problems have the property that their solutions may also be expressed as discrete series?* The answer to this question is the second major topic of this thesis.

It is shown in [54] that there is no series representation of the solution for a particular example. Algebraic methods are used in [36] to show that some linear partial differential equations are inseparable for *any* boundary conditions but this requires either non-constant coefficients or systems of constant-coefficient equations. There is an important distinction between the work of Johnson et al. and our work — the partial differential equations we study are all separable because separation of variables always yields a solution for periodic boundary conditions, it is particular sets of boundary conditions that may make the initial-boundary value problem inseparable by preventing the eigenfunctions of the differential operator from forming a basis.

1.1.2. Spectral theory of two-point ordinary di er-ential operators

Birkho [3, 4

- Investigate the existence of a series representation of the solution to well-posed problems in general, giving both necessary and sufficient conditions.
- Investigate inseparable boundary conditions by linking the initial-boundary value problem to the study of the ordinary differential operator.
- Contribute to the spectral theory of degenerate irregular non-self-adjoint two-point linear ordinary differential operators.

Chapter 2

As noted above, it is known that one may use Fokas' transform method to find a solution to any well-posed initial-boundary value problem on a linear, constant-coefficient evolution partial differential equation on a rectangular domain. In view of this it is perhaps surprising that any improvement may be made to the means of derivation of a solution but we have some small contributions in this area beyond the overview of the established method in Section 2.1.

Chapter 2 provides a modest development upon the method in the following way. While it is established that a system of linear equations for the boundary functions must exist in the method as presented in [27], we derive that system explicitly and in general. The *reduced global relation* is given in Lemma 2.17. Further, we explicitly solve the system to yield, in Theorem 2.20, the general expression for the solution in terms of the initial and boundary data and the solution at final time. Mathematically this is elementary linear algebra but the explicit determination of these functions is necessary to support the remainder of the thesis.

Chapter 3

Chapter 3 contains a discussion of well-posedness of initial-boundary value problems and the existence of a series representation of their solutions using only analytic techniques.

We make a pair of assumptions on the decay of certain meromorphic functions, which are the general analogues of the functions appearing in the integrands of equation (1.1.4). In Section 3.1 we work under those assumptions, removing the effects of the solution at final time and obtaining a series representation for the the solution. The second and third sections are devoted to discussing those assumptions.

In Section 3.2 one of the aforementioned assumptions is shown to be equivalent to well-posedness of the initial-boundary value problem. This new condition of well-posedness is at once much simpler to check than the characterisation by admissible functions of [27] and more general than the result for simple, uncoupled boundary conditions of [53] and [55]. We also give the final result of Fokas' method in Theorem 3.29, an integral representation for the solution involving only the initial and boundary data. In the case of odd-order problems with non-Robin boundary conditions, we give a pair of conditions sufficient for well-posedness and demonstrate their use for a variety of examples.

For well-posed problems, it is shown that the other decay assumption is equivalent to the existence of a series representation of the solution in Section 3.3. We also give a pair of sufficient conditions for a well-posed odd-order problem with non-Robin boundary conditions to have a solution that admits representation by a series. These conditions mirror those in the previous section.

In order to discuss complete, biorthogonal and basic systems of eigenfunctions it is necessary to understand the established theory of these concepts in Banach spaces. We give an overview of the essential definitions and a few theorems in Section 4.4, following the construction in [15]. More complete treatments of the subject are given in the excellent two-part survey article [56, 57] and the lecture notes [58]; these sources have large bibliographies containing the original research upon which they draw.

Chapter 5

In Chapter 5 we present two examples, one of which has degenerate irregular boundary conditions. We prove that the eigenfunctions of this operator do not form a basis, following a method of Davies [14, 15]. Indeed, we show that certain projection operators, defined in terms of the eigenfunctions, are not uniformly bounded in norm. The exponential blow-up of these norms is of the same rate as the divergence of the meromorphic function from the initial-boundary value problem.

Chapter 6

In the final chapter we draw together some conclusions and present some directions for further work.

CHAPTER 2

Initial-boundary value problems

In this chapter we give an account of Fokas' uni ed transform method for solving initial-

for the function $(x; t; \ )$*, where* $c_j(\ )$ *are the functions defined in equations* (2.1.6)*.*

Proof. We take the x partial derivative of equation (2.1.8),

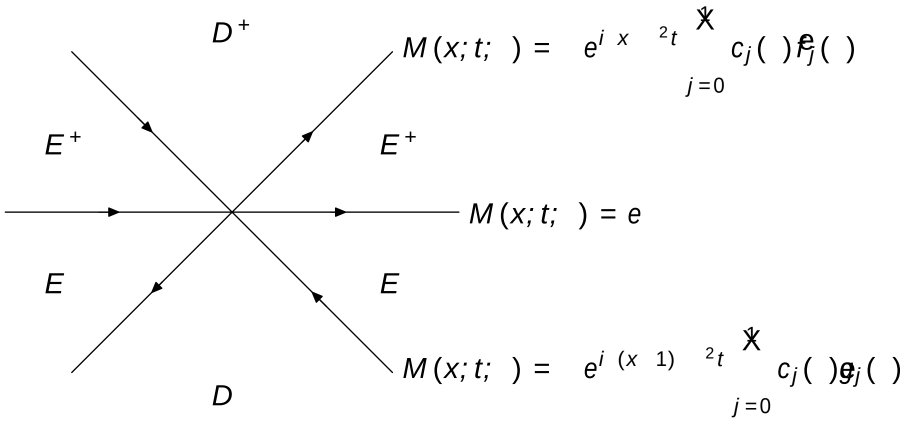
$$
\begin{aligned}
\partial_x\partial_t \ &= a \ ^n \left[\partial_x \ + \sum_{j=1}^{n} (i \ )^j \partial_x^j q\right] \\
&= a \ ^n \left[q + i \ + \sum_{j=1}^{n} (i \ )^j \partial_x^j q\right];
\end{aligned}
\tag{2.1.10}
$$

the latter equality being justified by equation (2.1.9). Similarly, we take the t partial derivative of equation (2.1.9),

$$\partial_t\partial_x \ = \partial_t q + i \ \partial_t \tag{2.1.11}$$

Following Proposition 3.1 of [24] we choose the the points $(x^\star, t^\star)$ to be the four corners of Ω, defining the functions $\nu_Y(x,t;\lambda)$ for $Y \in \{D^\pm, E^\pm\}$:

$$\nu_{E^+}(x,t;\lambda) = \int_0^x e^{i\lambda(x-y)} q(y,t)\,\mathrm{d}y + e^{i\lambda x}\int_0^t e^{-a\lambda^n(t-s)} \sum_{j=0}^{n-1} c_j(\lambda)$$


D^+
$M(x;t;\ ) = e^{i\ x\ \ ^2 t} \sum_{j=0}^{\infty} c_j(\ )\tilde{f}_j(\ )$
E^+
E^+
$M(x;t;\ ) = e$
E
E
$M(x;t;\ ) = e^{i\ (x\ \ 1)\ \ ^2 t} \sum_{j=0}^{\infty} c_j(\ )\tilde{g}_j(\ )$
D

initial datum, $q_0(x)$, and *final function*, $q_T(x) = q(x;T)$. We define these Fourier transforms as

$$\hat{q}_0(\lambda) = \int_0^1 e^{-i\lambda x} q_0(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-i\lambda x} q_0(x)\chi_{[0,1]}\,\mathrm{d}x \qquad \lambda \in \mathbb{C};$$
$$\hat{q}_T(\lambda) = \int_0^1 e^{-i\lambda x} q(x;T)\,\mathrm{d}x; \qquad \lambda \in \mathbb{C}.$$

Now we derive the global relation.

Lemma 2.3 (Global relation). *Let $q : \Omega \to \mathbb{R}$ be a formal solution to an initial-boundary value problem specified by the partial differential equation* (2.1.1) *and initial condition* (2.1.2)*. Then the functions $\hat{q}_0$, $\hat{q}_T$ defined above and the functions $\tilde{f}_j$ and $\tilde{g}_j$, given by* (2.1.6) *satisfy*

$$\sum_{j=0}^{n-1} c_j(\lambda)\left[\tilde{f}_j(\lambda) - e^{-i\lambda}\tilde{g}_j(\lambda)\right] = \hat{q}_0(\lambda) - e^{a\lambda^n T}\hat{q}_T(\lambda); \qquad \lambda \in \mathbb{C}. \tag{2.1.16}$$

Proof. For $(x;t) \in \Omega$ and $\lambda \in \mathbb{C}$ let

$$X(x;t;\lambda) = e^{-i\lambda x + a\lambda^n t} q(x;t); \qquad Y(x;t;\lambda) = e^{-i\lambda x + a\lambda^n t}\sum_{j=0}^{n-1} c_j(\lambda)\partial_x^j q(x;t).$$

Then

$$\partial_t X(x;t;\lambda) = e^{-i\lambda x + a\lambda^n t}(a\lambda^n + \partial_t)q(x;t);$$
$$\partial_x Y(x;t;\lambda) = e^{-i\lambda x + a\lambda^n t}(-i\lambda + \partial_x)\sum_{j=0}^{n-1} c_j(\lambda)\partial_x^j q(x;t)$$

hence

$$(\partial_t X - \partial_x Y)(x;t;\lambda) = e^{-i\lambda x + a\lambda^n t}\left[(a\lambda^n + \partial_t) + (i\lambda - \partial_x)\sum_{j=0}^{n-1} c_j(\lambda)\partial_x^j\right] q(x;t)$$
$$= e^{-i\lambda x + a\lambda^n t}\left[(a\lambda^n - a(-i\partial_x)^n) - a\lambda^n(i\lambda - \partial_x)\sum_{j=0}^{n-1}(i\lambda)^{-(j+1)}\partial_x^j\right] q(x;t);$$

using the differential equation (2.1.1) and the definition of the polynomials c_j,

$$= e^{-i\lambda x + a\lambda^n t} a\left(\lambda^n - (-i\partial_x)^n - \lambda^n\left(1 - (i\lambda)^{-n}\partial_x^n\right)\right) q(x;t)$$
$$= 0.$$

If we apply Green's Theorem B.1 to Ω then we see that

$$\iint_\Omega (\partial_t X - \partial_x Y)(x;t;\lambda)\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial\Omega} (Y\,\mathrm{d}t + X\,\mathrm{d}x)$$

where $\tilde{f}_j$, $\tilde{g}_j$ are de ned in (2.1.6), from which the result follows.

The global relation is useful because of the particular form of the spectral transforms of the boundary functions. The transformed boundary functions may be considered as functions not of but of ^{n}. This means that the transforms are invariant under the map $\mapsto \omega^j$, for $\omega = e^{2 \; i}$

By examining the de nitions (2.1.21) we see that the transformed boundary functions are functions of $\ ^2$. This means that they are invariant under the map $\ \mapsto\ $, that is

$$\begin{aligned} \tilde f_0(\ \ ) &= \tilde f_0(\ ); & \tilde g_0(\ \ ) &= \tilde g_0(\ ); \\ \tilde f_1(\ \ ) &= \tilde f_1(\ ); & \tilde g_1(\ \ ) &= \tilde g_1(\ ). \end{aligned} \tag{2.1.22}$$

Since the global relation (2.1.20) is valid for any $\ \in \mathbb{C}$, evaluating it at $\ $ we obtain

$$i\ \ \tilde f_0(\ \ )\ \ e^{i}\ \tilde g_0(\ \ ) + \tilde f_1(\ \ )\ \ e^{i}\ \tilde g_1(\ \ ) = \hat q_0(\ \ )\ \ e^{(\ \ )^2 T}\hat q_T(\ \ );$$

which, by equations (2.1.22), is

$$i\ \ \tilde f_0(\ )\ \ e^{i}\ \tilde g_0(\ ) + \tilde f_1(\ )\ \ e^{i}\ \tilde g_1(\ ) = \hat q_0(\ \ )\ \ e^{\ ^2 T}\hat q_T(\ \ ). \tag{2.1.23}$$

The global relation equations (2.1.20) and (2.1.23) may now be written in matrix form

$$B(\ )\begin{pmatrix} \tilde f_1(\ ) \\ \tilde g_1(\ ) \\ i\ \ \tilde f_0(\ ) \\ i\ \ \tilde g_0(\ ) \end{pmatrix} = \begin{pmatrix} \hat q_0(\ ) \\ \hat q_0(\ \ ) \end{pmatrix}\ \ e^{\ ^2 T}\begin{pmatrix} \hat q_T(\ ) \\ \hat q_T(\ \ ) \end{pmatrix} \tag{2.1.24}$$

where

$$B(\ ) = \begin{pmatrix} 1 & e^{\ i} & 1 & e^{\ i} \end{pmatrix}$$

for each $k = 0, 1, 2$. The global relation equations (2.1.28) may now be written in matrix form

$$B(\lambda)\begin{pmatrix} \tilde{f}_2(\lambda) \\ \tilde{g}_2(\lambda) \\ i\lambda \tilde{f}_1(\lambda) \\ i\lambda \tilde{g}_1(\lambda) \\ -\lambda^2 \tilde{f}_0(\lambda) \\ -\lambda^2 \tilde{g}_0(\lambda) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\lambda) \\ \hat{q}_0(\omega\lambda) \\ \hat{q}_0(\omega^2\lambda) \end{pmatrix} - e^{\lambda^3 T}\begin{pmatrix} \hat{q}_T(\lambda) \\ \hat{q}_T(\omega\lambda) \\ \hat{q}_T(\omega^2\lambda) \end{pmatrix} \tag{2.1.29}$$

where

$$B(\lambda) = \begin{pmatrix} 1 & -e^{-i\lambda} & 1 & -e^{-i\lambda} & 1 & -e^{-i\lambda} \\ 1 & -e^{-i\omega\lambda} & \omega & -\omega e^{-i\omega\lambda} & \omega^2 & -\omega^2 e^{-i\omega\lambda} \\ 1 & -e^{-i\omega^2\lambda} & \omega^2 & -\omega^2 e^{-i\omega^2\lambda} & \omega & -\omega e^{-i\omega^2\lambda} \end{pmatrix}.$$

Equation (2.1.29) corresponds to Corollary 2.4.

2.1.5. A classification of boundary conditions

In Definition 2.7 we provide a rough classification of boundary values. We classify the boundary conditions in terms of the representation used in Locker's work [47] on differential operators.

Definition 2.7 (Classification of boundary conditions).

Example 2.8. The boundary conditions

$$q_x(0,t) = q_x(1,t) \qquad q(0,t) = q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence these boundary conditions are homogeneous and non-Robin but coupled.

Example 2.9. The boundary conditions

$$q_x(0,t) = t(T-t) \qquad q(0,t) = q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = t(T-t)$, $h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

into the two vectors V and W. The entries in W are the transform, $\widetilde{f}_j$ or $\widetilde{g}_j$, of a boundary function that is, in equation (2.1.32), multiplied by a pivot of A, where the entries in V are the other entries in the vector (2.2.1) and overall we preserve the order of the entries in the original vector (2.2.1).

2.2.1.1. Developing some notation

Notation 2.12. Given boundary conditions defined by equations (2.1.32) and (2.1.33) such that A is in reduced row-echelon form, we define the following index sets and functions.

$J^{0+} = \{j \in \{0, 1$

Example 2.13. If $n = 3$ and the boundary conditions are specified by equation (2.1.32) where

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2.2}$$

then $c_2(\lambda) = -ai$, $c_1(\lambda) = a\lambda$, $c_0(\lambda) = ai\lambda^2$,

$$W(\lambda) = \begin{pmatrix} \widetilde{f}_1(\lambda) \\ \widetilde{f}_0(\lambda) \\ \widetilde{g}_0(\lambda) \end{pmatrix} \text{ and } V(\lambda) = \begin{pmatrix} \widetilde{f}_2(\lambda) \\ \widetilde{g}_2(\lambda) \\ \widetilde{g}_1(\lambda) \end{pmatrix}.$$

Indeed, comparing equations (2.1.33) and (2.2.2) we see that

$$\begin{pmatrix} \alpha_{12} & \beta_{12} & \alpha_{11} & \beta_{11} & \alpha_{10} & \beta_{10} \\ \alpha_{22} & \beta_{22} & \alpha_{21} & \beta_{21} & \alpha_{20} & \beta_{20} \\ \alpha_{32} & \beta_{32} & \alpha_{31} & \beta_{31} & \alpha_{30} & \beta_{30} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pivots in this boundary coefficient matrix are α_{11}, α_{20} and β_{30} so

$$\widehat{J}^+ = \{0, 1\}; \qquad \widehat{J}^- = \{0\};$$
$$\widetilde{J}^+ = \{2\} \text{ and } \qquad \widetilde{J}^- = \{1, 2\}.$$

Following through Notation 2.12 in order we see that

$$J = \{2, 4, 5\}; \qquad J' = \{0, 1, 3\};$$
$$(J_j)_{j=1}^3 = (5, 4, 2); \qquad (J'_j)_{j=1}^3 = (3, 1, 0);$$
$$V(\lambda) = \begin{pmatrix} \widetilde{f}_2(\lambda) \\ \widetilde{g}_2(\lambda) \\ \widetilde{g}_1(\lambda) \end{pmatrix} \text{ and } \qquad W(\lambda) = \begin{pmatrix} \widetilde{f}_1(\lambda) \\ \widetilde{f}_0(\lambda) \\ \widetilde{g}_0(\lambda) \end{pmatrix}.$$

We also note that, defining the sequences

$$\widehat{J}^+_j = \begin{cases} 2 & \text{if } j = 0, \\ 1 & \text{if } j = 1; \end{cases} \qquad \widehat{J}^-_j = 3 \text{ for } j = 0;$$

the pivots in A are

$$\alpha_{\widehat{J}^+_{(J'_1-1)/2}\,(J'_1-1)/2} = \alpha_{11}; \quad \alpha_{\widehat{J}^+_{(J'_2-1)/2}\,(J'_2-1)/2} = \alpha_{20} \text{ and } \beta_{\widehat{J}^-_{J'_3/2}\,J'_3/2} = \beta_{30}.$$

Indeed the aim of the definition of sequences $(\widehat{J}^+_j)_{j\in\widehat{J}^+}$ and $(\widehat{J}^-_j)_{j\in\widehat{J}^+}$

2.2.1.2. The main lemma

We may now state the result.

Lemma 2.14. *Let $q : [0,\infty) \times [0,T] \to \mathbb{R}$ be a solution of the initial-boundary value problem specified by the partial differential equation* (2.1.1)*, the initial condition* (2.1.2) *and the homogeneous, non-Robin boundary conditions* (2.1.32)*. Assume the matrix A, whose entries are defined by equation* (2.1.33)*, is in reduced row-echelon form. Then the vectors V and W from Notation 2.12 satisfy*

$$A(\lambda)\begin{pmatrix} V_1(\lambda) \\ V_2(\lambda) \\ \vdots \\ V_n(\lambda) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\lambda) \\ \vdots \\ \hat{q}_0(\omega^{n-1}\lambda) \end{pmatrix} - e^{a\lambda^n T}\begin{pmatrix} \hat{q}_T(\lambda) \\ \vdots \\ \hat{q}_T(\omega^{n-1}\lambda) \end{pmatrix} \text{ and} \tag{2.2.3}$$

$$\begin{pmatrix} W_1(\lambda) \\ W_2(\lambda) \\ \vdots \\ W_n(\lambda) \end{pmatrix} = \hat{A}\begin{pmatrix} V_1(\lambda) \\ V_2(\lambda) \\ \vdots \\ V_n(\lambda) \end{pmatrix}; \tag{2.2.4}$$

where

$$A_{kj}(\lambda) = \begin{cases} \omega^{(n-1-[J_j-1]/2)(k-1)} c_{(J_j-1)/2}(\lambda) & J_j \\ \end{cases}$$

with the simple, homogeneous boundary conditions

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \\ f_1 \\ g_1 \\ f_0 \\ g_0 \end{pmatrix} = 0. \tag{2.2.8}$$

Then, as in Example 2.13,

$$W(\lambda) = \begin{pmatrix} \tilde{f}_1(\lambda) \\ \tilde{f}_0(\lambda) \\ \tilde{g}_0(\lambda) \end{pmatrix} \text{ and } V(\lambda) = \begin{pmatrix} \tilde{f}_2(\lambda) \\ \tilde{g}_2(\lambda) \\ \tilde{g}_1(\lambda) \end{pmatrix}.$$

The boundary conditions (2.2.8) may be rewritten

$$I_3 \begin{pmatrix} f_1 \\ f_0 \\ g_0 \end{pmatrix} + 0_3 \begin{pmatrix} f_2 \\ g_2 \\ g_1 \end{pmatrix} = 0,$$

where I_3 is the 3×3 identity matrix and 0_3 is the 3×3 zero matrix, which yields

$$\begin{pmatrix} f_1(t) \\ f_0(t) \\ g_0(t) \end{pmatrix} = 0 \qquad t \in [0, T].$$

Applying the t-transform (2.1.30) entrywise we see that

$$W(\lambda) = \begin{pmatrix} \tilde{f}_1(\lambda) \\ \tilde{f}_0(\lambda) \\ \tilde{g}_0(\lambda) \end{pmatrix} = 0, \qquad \lambda \in \mathbb{C} \tag{2.2.9}$$

This corresponds to the reduced boundary conditions (2.2.4) in the lemma.

The fact we have exploited here is that, because it is in reduced row-echelon form, the boundary coefficient matrix has I_3 as a maximal square submatrix. This allows us to break the boundary coefficient matrix into two parts: the identity and the rest of it, which we call the reduced boundary coefficient matrix. In this example the reduced boundary coefficient matrix is the zero matrix. This need not be the case but, provided the boundary conditions are non-Robin, this matrix must be diagonal. Of course, this process will work for any regularised boundary coefficient matrix, the only requirement being that the boundary coefficient matrix has the identity as a maximal square submatrix, which is guaranteed by the reduced row-echelon form it is assumed to take.

We still have to find the other three boundary functions, those that appear in the vector V. To do this we will make use of the global relation in the form of Corollary 2.4. The partial differential equation (2.2.7) studied in this example defines $n = 3$ and $a = i$ so the corollary may be written

$$\begin{pmatrix} 1 & e^{-i\lambda} & -i\lambda & -i\lambda e^{-i\lambda} & -\lambda^2 & -\lambda^2 e^{-i\lambda} \\ 1 & e^{-i\omega\lambda} & -\omega i\lambda & -\omega i\lambda e^{-i\omega\lambda} & -\omega^2\lambda^2 & -\omega^2\lambda^2 e^{-i\omega\lambda} \\ 1 & e^{-i\omega^2\lambda} & -\omega^2 i\lambda & -\omega^2 i\lambda e^{-i\omega^2\lambda} & -\omega\lambda^2 & -\omega\lambda^2 e^{-i\omega^2\lambda} \end{pmatrix} \begin{pmatrix} \tilde{f}_2(\lambda) \\ \tilde{g}_2(\lambda) \\ \tilde{f}_1(\lambda) \\ \tilde{g}_1(\lambda) \\ \tilde{f}_0(\lambda) \\ \tilde{g}_0(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} \hat{q}_0(\lambda) \\ \hat{q}_0(\omega\lambda) \\ \hat{q}_0(\omega^2\lambda) \end{pmatrix} - e^{i\lambda^3 T} \begin{pmatrix} \hat{q}_T(\lambda) \\ \hat{q}_T(\omega\lambda) \\ \hat{q}_T(\omega^2\lambda) \end{pmatrix};$$

the right hand side of which is the right hand side of the reduced global relation (2.2.3) from the lemma. The left hand side must be simplified. Substituting the reduced boundary conditions (2.2.9) into the global relation gives

$$\begin{pmatrix} 1 & e^{-i\lambda} & -i\lambda & -i\lambda e^{-i\lambda} & -\lambda^2 & -\lambda^2 e^{-i\lambda} \\ 1 & e^{-i\omega\lambda} & -\omega i\lambda & -\omega i\lambda e^{-i\omega\lambda} & -\omega^2\lambda^2 & -\omega^2\lambda^2 e^{-i\omega\lambda} \\ 1 & e^{-i\omega^2\lambda} & -\omega^2 i\lambda & -\omega^2 i\lambda e^{-i\omega^2\lambda} & -\omega\lambda^2 \tilde{e}(\end{pmatrix}$$

entries in W in terms of the entries in V hence in terms of the Fourier transforms of the initial datum and final function.

2.2.1.4. Proof of the main lemma

Using Example 2.16 as a model, we give the full proof of Lemma 2.14.

Proof. Because A is in reduced row-echelon form it has the $n \times n$ identity matrix, I_n, as a submatrix. That submatrix is the one obtained by taking all n rows of A but only the columns which contain pivots. These are the columns of A indexed by $2n - j$, where $j \in J'$. Any such column multiplies the boundary function $f_{(j-1)/2}$ or $g_{j/2}$, for j odd or j even respectively, in the boundary conditions (2.1.32). The columns of A not appearing in the identity submatrix are those indexed by $2n - k$ for $k \in J$. Any such column multiplies the boundary function $f_{(k-1)/2}$ or $g_{k/2}$, for k odd or k even respectively, in the boundary conditions (2.1.32). The sequences $(J_j)_{j=1}^n$ and $(J'_j)_{j=1}^n$ simply ensure the entries in the vectors V and W appear in the correct order. We may now break the $n \times 2n$ matrix A into two square matrices, rewriting the boundary conditions in the form

$$I_n \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} + \hat{A} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = 0, \tag{2.2.11}$$

where

$$X_j = \begin{cases} f_{(J_j-1)/2} & J_j \text{ odd,} \\ g_{J_j/2} & J_j \text{ even,} \end{cases} \qquad Y_j = \begin{cases} f_{(J'_j-1)/2} & J'_j \text{ odd,} \\ g_{J'_j/2} & J'_j \text{ even,} \end{cases} \tag{2.2.12}$$

and $\hat{A}$ is initially defined as the square matrix given by

$$\hat{A}_{kj} = \begin{cases} \alpha_{k(J_j-1)/2} & J_j \text{ odd,} \\ \beta_{kJ_j/2} & J_j \text{ even.} \end{cases}$$

If J_j is odd then there does not exist $k \in \{1, 2, \dots, n\}$ such that $\alpha_{k(J_j-1)/2}$ is a pivot of A. Because the boundary conditions are non-Robin, this implies

$$\alpha_{k(J_j-1)/2} = 0 \quad \forall k \in \{1, 2, \dots, n\}, \quad \forall j \text{ odd.}$$

If J_j is even then there does not exist $k \in \{1, 2, \dots, n\}$ such that $\beta_{kJ_j/2}$ is a pivot. If it happens that there does exist some $k \in \{1, 2, \dots, n\}$ such that $\beta_{kJ_j/2}$ is a pivot, that is $J_j + 1 \in J'$ hence $J_j/2 \in$

column in A whose entries are given by powers of ω multiplied by the sum of exponential powers of and a constant (type (3)) then that is the only column with those powers of ω.

Consider boundary conditions that are all specified at the end $x = 1$, that is the boundary coefficient matrix has the form

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{J}^{+} = \{0, 1, \dots, n-1\}$ and $\mathcal{J}^{+} = \emptyset$ so A' is a Vandermonde matrix which has rank n, as is shown in Section 1.4 of [50]. This matrix contains all columns of type (1) that may appear in any A, so given any A the columns of the corresponding A of type (1) are linearly independent.

If instead the boundary conditions are all specified at $x = 0$, that is

$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

then the determinant of A'' is equal to the determinant of the same Vandermonde matrix. This matrix contains all columns of type (2) that may appear in any A, so given any A the columns of the corresponding A of type (2) are linearly independent.

Other columns of any A, that is a column of type (3), can be written as the sum of two columns: one of type (1) and one of type (2). But we have already established that neither of these may appear in A and neither may be written as a linear combination of columns that do appear in A. This establishes that the column rank of any reduced global relation matrix is n.

2.2.2. General boundary conditions

reduced row-echelon form. Then the vectors V and W from Notation 2.12 satisfy

$$A(\ )\begin{pmatrix} V_1(\ ) \\ V_2(\ ) \\ \vdots \\ V_n(\ ) \end{pmatrix} = U(\ ) \quad e^{a\ \ ^{n}T}\begin{pmatrix} \hat{q}_T(\ ) \\ \vdots \\ \hat{q}_T(!^{n\ \ 1}\ ) \end{pmatrix} \text{ and} \tag{2.2.15}$$

$$\begin{pmatrix} W_1(\ ) \\ W_2(\ ) \\ \vdots \\ W_n(\ ) \end{pmatrix} = \begin{pmatrix} \hbar_1(\ ) \\ \hbar_2(\ ) \\ \end{pmatrix}$$

where X and Y are given by equations (2.2.12) in the previous proof and the reduced boundary coefficient matrix, $\hat{A}$, is defined by equation (2.2.20). Now the t-transform (2.1.30) may be applied to each line of equation (2.2.21) and by the linearity of the transform we obtain equation (2.2.16).

We may rewrite equation (2.2.16) in the form

$$\tilde{f}_j(\lambda) = \tilde{h}_{\hat{J}^+_j}(\lambda) - \sum_{r \in \tilde{J}^+} \hat{a}_{\hat{J}^+_j\, r} \tilde{f}_r(\lambda) - \sum_{r \in \tilde{J}^-} \hat{a}_{\hat{J}^+_j\, r} \tilde{g}_r(\lambda); \qquad \text{for } j \in \hat{J}^+ \text{ and} \tag{2.2.22}$$

$$\tilde{g}_j(\lambda) = \tilde{h}_{\hat{J}^-_j}(\lambda) - \sum_{r \in \tilde{J}^+} \hat{a}_{\hat{J}^-_j\, r} \tilde{f}_r(\lambda) - \sum_{r \in \tilde{J}^-} \hat{a}_{\hat{J}^-_j\, r} \tilde{g}_r(\lambda); \qquad \text{for } j \in \hat{J}^- : \tag{2.2.23}$$

Corollary 2.4 may be rewritten as the system of linear equations

$$\sum_{j=0}^{n-1} c_j(\lambda)\, \omega^{(n-1-j)r} \tilde{f}_j(\lambda) - \sum_{j=0}^{n-1} e^{-i\omega^r \lambda} c_j(\lambda)\, \omega^{(n-1} \sum_{j=0}$$

hence

$$\sum_{j\in\widetilde{J}^+}\tilde f_j(\lambda)\left[c_j(\lambda)\omega^{(n-1-j)r}-\sum_{k\in\widehat{J}^+}\widehat{J}^{+}_{kj}c_k(\lambda)\omega^{(n-1-k)r}+e^{-i\omega^r\lambda}\sum_{k\in\widehat{J}^-}\widehat{J}^{-}_{kj}c_k(\lambda)\omega^{(n-1-k)r}\right]$$
$$-\sum_{j\in\widetilde{J}^-}\tilde g_j(\lambda)\left[e^{-i\omega^r\lambda}c_j(\lambda)\omega^{(n-1-j)r}+\sum_{k\in\widehat{J}^+}\widehat{J}^{+}_{kj}c_k(\lambda)\omega^{(n-1-k)r}\right.$$
$$\left.-e^{-i\omega^r\lambda}\sum_{k\in\widehat{J}^-}\widehat{J}^{-}_{kj}c_k(\lambda)\omega^{(n-1-k)r}\right]$$
$$=\hat q_0(\omega^r\lambda)-\sum_{j\in\widehat{J}^+}c_j(\omega^r\lambda)\tilde h_{\widehat{J}^+_j}(\lambda)+\sum_{j\in\widehat{J}^-}e^{-i\omega^r\lambda}c_j(\omega^r\lambda)\tilde h_{\widehat{J}^-_j}(\lambda)-e^{a\lambda^nT}\hat q_T(\omega^r\lambda);$$

for $r\in\{0,1,\dots,n-1\}$. Taking a factor of $c_j(\lambda)$ out of each square bracket and using the identity

$$\frac{c_k(\lambda)}{c_j(\lambda)}=(i\lambda)^{j-k},$$

we establish

$$\sum_{j\in\widetilde{J}^+}\tilde f_j(\lambda)c_j(\lambda)\Big[\omega^{(n-1-j)}\,{}^{1-j)}$$

into two matrices

$$X_{kj} = \begin{cases} c_{(J_j-1)/2}(\;)\,\omega^{(n-1-[J_j-1]/2)(k-1)} - \displaystyle\sum_{r\in J^{+}} J_r^{+}{}_{(J_j-1)/2}\, c_r(\;)\,\omega^{(n-1-r)(k-1)} & J_j \text{ odd}, \\ -\displaystyle\sum_{r\in J^{+}} J_r^{+} & \end{cases}$$

and

$$_{\mathrm{PDE}}(\ ) = \det\begin{pmatrix} 1 & e^{\ i\ } & i\ e^{\ i\ } \\ 1 & e^{\ i\omega\ } & \omega i\ e^{\ i\omega\ } \\ 1 & e^{\ i\omega^2\ } & \omega^2 i\ e^{\ i\omega^2\ } \end{pmatrix}$$

in accordance with the following Definition 2.19. Indeed $_j$ may be found from $_j$ by replacing $\hat{q}_0$ with $\hat{q}_T$. Applying Theorem B.2 to the reduced global relation (2.2.10) and observing that the reduced boundary coefficient matrix is 0_3 we obtain

$$\tilde{f}_2(\ ) = \frac{\hat{\ }_1(\ ) \quad e^{i\ ^3T}\hat{\ }_1(\ )}{_{\mathrm{PDE}}(\ )};$$

$$\tilde{g}_2(\ ) = \frac{\hat{\ }_2(\ ) \quad e^{i\ ^3T}\hat{\ }_2(\ )}{_{\mathrm{PDE}}(\ )};$$

$$\tilde{g}_1(\ ) = i\ \frac{\hat{\ }_3(\ ) \quad e^{i\ ^3T}\hat{\ }_3(\ )}{_{\mathrm{PDE}}(\ )};$$

$$\tilde{f}_1(\ ) = \tilde{f}_0(\ ) = \tilde{g}_0(\ ) = 0.$$

Substituting the abovhe(

We define further

$$\zeta_j(\lambda)=\begin{cases} c_{(J_j-1)/2}(\lambda)\hat{\zeta}_j(\lambda) & J_j \text{ odd},\\ c_{J_j/2}(\lambda)\hat{\zeta}_j(\lambda) & J_j \text{ even},\\ c_{(J'_{j-n}-1)/2}(\lambda)\hat{\zeta}_j(\lambda) & J'_{j-n} \text{ odd},\\ c_{J'_{j-n}/2}(\lambda)\hat{\zeta}_j(\lambda) & J'_{j-n} \text{ even},\end{cases} \qquad \eta_j(\lambda)=\begin{cases} c_{(J_j-1)/2}(\lambda)\hat{\eta}_j(\lambda) & J_j \text{ odd},\\ c_{J_j/2}(\lambda)\hat{\eta}_j(\lambda) & J_j \text{ even},\\ c_{(J'_{j-n}-1)/2}(\lambda)\hat{\eta}_j(\lambda) & J'_{j-n} \text{ odd},\\ c_{J'_{j-n}/2}(\lambda)\hat{\eta}_j(\lambda) & J'_{j-n} \text{ even},\end{cases} \tag{2.3.2}$$

for $\lambda \in \mathbb{C}$, *where the monomials* c_k *are defined in equations* (2.1.5). *Define the index sets*

$$J^+ = \{j : J_j \text{ odd}\} \cup \{n+j : J'_j \text{ odd}\};$$
$$J^- = \{j : J_j \text{ even}\} \cup \{n+j : J'_j \text{ even}\}.$$

Also let

$$\Delta_{\mathrm{PDE}}(\lambda) = \det A(\lambda); \qquad \lambda \in \mathbb{C}. \tag{2.3.3}$$

Note that, for homogeneous boundary conditions, the η_j *are simply the* ζ_j *with* $\hat{q}_T$ *replacing* $\hat{q}_0$.

Now by Lemma 2.17 and Cramer's rule, Theorem B.2, we may obtain expressions for the boundary functions:

$$\frac{\hat{\zeta}_j(\lambda)-e^{a\lambda^n T}\hat{\eta}_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}=\begin{cases} \tilde{f}_{(J_j-1)/2}(\lambda) & J_j \text{ odd},\\ \tilde{g}_{J_j/2}(\lambda) & J_j \text{ even},\\ \tilde{f}_{(J'_{j-n}-1)/2}(\lambda) & J'_{j-n} \text{ odd},\\ \tilde{g}_{J'_{j-n}/2}(\lambda) & J'_{j-n} \text{ even},\end{cases}$$

hence

$$\frac{\zeta_j(\lambda)-e^{a\lambda^n T}\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}=\begin{cases} c_{(J_j-1)/2}(\lambda)\tilde{f}_{(J_j-1)/2}(\lambda) & J_j \text{ odd},\\ c_{J_j/2}(\lambda)\tilde{g}_{J_j/2}(\lambda) & J_j \text{ even},\\ c_{(J'_{j-n}-1)/2}(\lambda)\tilde{f}_{(J'_{j-n}-1)/2}(\lambda) & J'_{j-n} \text{ odd},\\ c_{J'_{j-n}/2}(\lambda)\tilde{g}_{J'_{j-n}/2}(\lambda) & J'_{j-n} \text{ even},\end{cases} \tag{2.3.4}$$

and

$$\sum_{j=0}^{n-1} c_j(\lambda)\tilde{f}_j(\lambda) = \sum_{j\in J^+}\frac{\zeta_j(\lambda)-e^{a\lambda^n T}\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)},$$
$$\sum_{j=0}^{n-1} c_j(\lambda)\tilde{g}_j(\lambda) = \sum_{j\in J^-}\frac{\zeta_j(\lambda)-e^{a\lambda^n T}\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}.$$

This establishes the following theorem, the main result of this chapter.

Theorem 2.20. *Assume that there exists a unique* $q : [0,1]\times[0,T] \to \mathbb{R}$ *solving the initial-boundary value problem specified by the partial differential equation* (2.1.1), *the initial condition* (2.1.2) *and the boundary conditions* (2.1.32). *Then* $q(x,t)$ *may be expressed in terms of* *contour integrals of transforms of the boundary data, initial datum and nal function as follows:*

$$2\ q(x;t) = \int_{\mathbb{R}} e^{i\ x\ a\ ^n t}\hat{q}_0(\ )\,\mathrm{d}\ \ \int_{\partial D^+} e^{i\ x\ a\ ^n t} \sum_{j\in J^+} \frac{{}_j(\ )\ \ e^{a\ ^n T}\ {}_j(\ )}{{}_{\mathrm{PDE}}(\ )}\,\mathrm{d}$$
$$\int_{\partial D} e^{i\ (x\ 1)\ a\ ^n t} \sum_{j\in J} \frac{{}_j(\ )\ \ e^{a\ ^n T}\ {}_j(\ )}{{}_{\mathrm{PDE}}(\ )}\,\mathrm{d}\ ; \qquad (2.3.5)$$

where $D\ = \mathbb{C}\ \setminus \{\ \in \mathbb{C} : \mathrm{Re}(a\ ^n) < 0\}$.

Example 2.21. We give another example to illustrate De nition 2.19 and Theorem 2.20. The boundary value problem we consider is the same is in Example 2.13; $n = 3$, $a = i$ and the boundary conditions are given by equation (2.1.32) with $h_j = 0$ and a boundary coe cient matrix)n-9967 .16 .8370-2913m) eJn-91 401.515 0CTd [5 (n-6 1 224.741Amatrix)]TJ/F43 10.9090 lows

CHAPTER 3

Series representations and well-posedness

While Chapter 2 is concerned with deriving an integral representation for the solution to a well-posed initial-boundary value problem, the present chapter is devoted to investigating well-posedness of such a problem and the related question of finding a discrete series representation of its solution. We continue in the general setting of Chapter 2 with a partial differential equation (2.1.1) specified by its order $n > 2$ and the parameter a. The form of our results depends upon the value of a; we present them in the three cases $a = i$, $a = -i$ and $\mathrm{Re}(a) > 0$.

Theorem 3.1. *Let the homogeneous initial-boundary value problem* (2.1.1)–(2.1.3) *obey Assumptions 3.2 and 3.3. Then the solution to the problem may be written in series form as follows:*

$$q(x;t) = \frac{i}{2} \sum_{\substack{k\in K^+\\ \cup K^{D+}\cup K^{E+}\\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda) + \frac{i}{2} \sum_{\substack{k\in K^-\\ \cup K^{D-}\cup K^{E-}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\hat{P}(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda). \tag{3.0.6}$$

If n is odd and $a = -i$,

$$q(x;t) = \frac{i}{2} \sum_{\substack{k\in K^+\\ \cup K^{D+}\cup K^{E+}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda) + \frac{i}{2} \sum_{\substack{k\in K^-\\ \cup K^{D-}\cup K^{E-}\\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\hat{P}(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda). \tag{3.0.7}$$

If n is even and $a = -i$,

$$\begin{aligned} q(x;t) = \frac{i}{2} \sum_{\substack{k\in K^+\\ \cup K^{D+}\cup K^{E+}\\ \cup K^{D}_{E}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda) &+ \frac{i}{2} \sum_{\substack{k\in K^-\\ \cup K^{D-}\cup K^{E-}\\ \cup K^{E}_{D}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\hat{P}(\lambda;x;t)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda) \\ &+ \frac{i}{4} \operatorname*{Res}_{\lambda=0} \frac{1}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \sum_{j\in J^+} \end{aligned}$$

assumptions, once stated, are considered to hold throughout Section 3.1. Particular examples are not discussed as, for any but the most trivial examples, a lengthy calculation of bounds on zeros of certain exponential polynomials is required in order to perform any meaningful simplification of the general definitions or argument. Instead, the definitions and subsequent derivation are broken up depending upon the value of the parameter a.

3.1. Derivation of a series representation

In this section we apply Jordan's Lemma B.3 to deform the contours of integration in the integral representation given by Theorem 2.20. We do not investigate whether the conditions of Jordan's Lemma are met, instead we assume that they are met and show that this implies we may perform a residue calculation, obtaining a series representation of the solution. Sections 3.2 and 3.3 are concerned with investigating the validity of these assumptions.

Consider the same initial-boundary value problem studied in Chapter 2. That is, we wish to find q which satisfies the partial differential equation (2.1.1) subject to initial condition (2.1.2) and boundary conditions (2.1.32) where the boundary coefficient matrix A, given by equation (2.1.33), is in reduced row-echelon form. We assume throughout this section that such a function q exists and is unique hence the initial-boundary value problem is well-posed. The criteria for Theorem 2.20 are now met.

Definition 3.4. *Let the functions $P, \hat{P} : \mathbb{C} \to \mathbb{C}$ be defined by*

$$P(\lambda; x, t) = e^{i\lambda x - a\lambda^n t} \quad \text{and} \quad \hat{P}(\lambda; x, t) = e^{-i\lambda(1-x) - a\lambda^n t}$$

We shall usually omit the x and t dependence of these functions, writing simply $P(\lambda)$ and $\hat{P}(\lambda)$.

The aim of Definition 3.4 is that we may write the result of Theorem 2.20 in a way that emphasises the λ-dependence of the integrands, instead of their dependence on x and t. Indeed as x and t are both bounded real numbers they are treated as parameters in what follows.

We also define the five integrals

$$I_1 = \int_{\mathbb{R}} P(\lambda)\hat{q}_0(\lambda)\,\mathrm{d}\lambda, \tag{3.1.1}$$

$$I_2 = \int_{\partial D^+} P(\lambda) \sum_{j\in J^+} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda, \qquad I_3 = \int_{\partial D^+} P(\lambda)e^{a\lambda^n T} \sum_{j\in J^+} \frac{\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda,$$

$$I_4 = \int_{\partial D^-} \hat{P}(\lambda) \sum_{j\in J^-} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda, \qquad I_5 = \int_{\partial D^-} \hat{P}(\lambda)e^{a\lambda^n T} \sum_{j\in J^-} \frac{\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda,$$

where ζ_j, η_j, Δ_{PDE}, J^+ and J^- are given in Definition 2.19. We may now rewrite the result of Theorem 2.20 in the form

$$2\pi q = \sum_{k=1}^{5} I_k. \tag{3.1.2}$$

3.1.1. The behaviour of the integrands

We put aside I_1 for this subsection and investigate the behaviour of the integrands in the other four integrals in the regions to the left of the contours of integration. The results of this subsection are summarised in the following lemma.

Lemma 3.5. *Let q be the solution of the well-posed initial-boundary value problem studied in this section. Under Assumptions 3.2 and 3.3 the following hold:*

- *The integrand of I_2 is analytic within $\widetilde{E}^+$ and decays as $\lambda \to \infty$ within $\widetilde{E}^+$.*
- *The integrand of I_3 is analytic within $\widetilde{D}^+$ and decays as $\lambda \to \infty$ within $\widetilde{D}^+$.*

The integrand of I_4 is analytic within $\hat{E}$ and decays as ! 1 within $\hat{E}$.

The integrand of I_5 is analytic within $\hat{D}$ and decays as ! 1 within $\hat{D}$.

The open sets $\hat{D}$; $\hat{E}$

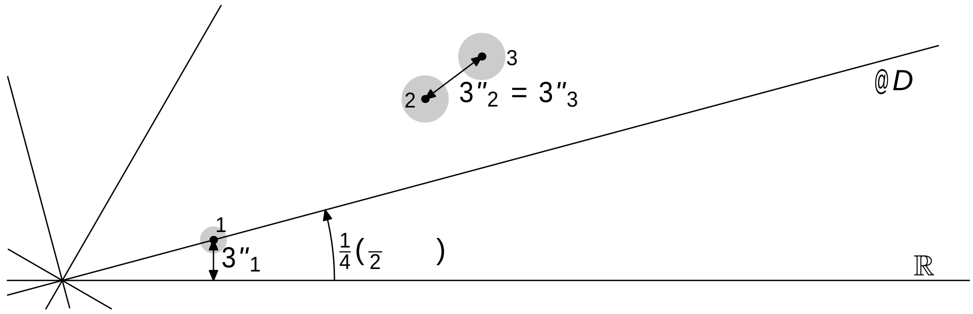


Figure 3.1. The bounds on $"_k$

$\overline{B}(\ _k; "_k) \setminus \overline{B}(\ _j; "_j)$ is empty for $j \not= k$. Also, for $k > 0$, the closed disc $\overline{B}(\ _k; "_k)$ does not touch any part of $@D$ except, when $k\ 2\ K^+ [\ K\quad [\ K^{\mathbb{R}}$, the half line on which $\ _k$ lies. Choosing such small $"_k$ is not necessary for this subsection but it is useful for simplifying the residue calculations of Subsection 3.1.3. Figure 3.1 shows the suprema of $"_k$ given some particular $\ _k$; the shaded regions are the discs $B(\ _k; "_k)$.

The de nition must be split into two cases, depending upon the value of a. In either case it is justi ed as we know that $\ _{\mathrm{PDE}}$ is holomorphic on $\mathbb{C}$ so its zeros are isolated. For each $k\ 2\ K^X$ we de ne a small disc around $\ _k$ that is wholly contained within X. This disc is labeled $B(\ _k; "_k)$, using the \ball" notation to avoid confusion with the notation D, representing the subset of the complex plane for which Re$(a\ ^n) < 0$.

Definition 3.8 ($"_k$). *Let* $a = \ i$. *For each* $k\ 2\ \mathbb{N}$ *we de ne* $"_k > 0$ *as follows:*

For each $k\ 2\ K^+ [\ K\quad [\ K^{\mathbb{R}}$, *we select* $"_k > 0$ *such that*

$$3"_k < j\ _k j \sin(\overline{n}) \text{ and } \overline{B}(\ _k; 3"_k) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _k g:$$

For each $k\ 2\ K^{D^+} [\ K^D\quad [\ K^{E^+} [\ K^E$ *we select* $"_k > 0$ *such that*

$$3"_k < \mathrm{dist}(\ _k; @D) \text{ and } \overline{B}(\ _k; 3"_k) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _k g:$$

We de ne $"_0 > 0$ *such that*

$$\overline{B}(0; 3"_0) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _0 g:$$

Let $a = e^i$ *for some* $\ 2\ (\ \overline{2}; \overline{2})$. *For each* $k\ 2\ \mathbb{N}$ *we de ne* $"_k > 0$ *as follows:*

For each $k\ 2\ K^+ [\ K\quad [\ K^{\mathbb{R}}$, *we select* $"_k > 0$ *such that*

$$3"_k < j\ _k j \sin(\tfrac{1}{n}(\overline{2}\quad j\ j)) \text{ and } \overline{B}(\ _k; 3"_k) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _k g:$$

For each $k\ 2\ K^{D^+} [\ K^D\quad [\ K^{E^+} [\ K^E$ *we select* $"_k > 0$ *such that*

$$3"_k < \mathrm{dist}(\ _k; @D\ [\ \mathbb{R}) \text{ and } \overline{B}(\ _k; 3"_k) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _k g:$$

We de ne $"_0 > 0$ *such that*

$$\overline{B}(0; 3"_0) \setminus f\ _j : j\ 2\ \mathbb{N}^0 g = f\ _0 g:$$

The next de nition uses De nitions 3.6, 3.7 and 3.8 to de ne subsets of D and E on which the functions

Definition 3.9. *We define the sets of complex numbers*

$$\widehat{D} = D \setminus \bigcup_{k\in\mathbb{N}_0} \overline{B}(\sigma_k, \varepsilon_k) \text{ and } \widehat{E} = E \setminus \bigcup_{k\in\mathbb{N}_0} \overline{B}(\sigma_k, \varepsilon_k);$$

and observe that $\frac{\zeta_j}{\Delta_{\mathrm{PDE}}}$ *is analytic on* $\widehat{E}$ *and* $\frac{\eta_j}{\Delta_{\mathrm{PDE}}}$ *is analytic on* $\widehat{D}$*.*

Because the positions of the zeros of Δ_{PDE} are affected by the boundary conditions, the sets $\widehat{D}$, $\widehat{E}$ depend upon the boundary conditions. This is in contrast to the sets D and E which depend only upon the partial differential equation (that is upon n and a) and are independent of the boundary conditions.

To complete this subsection, we give an example for which Assumptions 3.2 and 3.3 hold.

Example 3.10. Consider the initial-boundary value problem of Example 2.21; $n = 3$, a

Definition 3.11. *We define the index sets*

$$K_E^D = \{k \in \mathbb{N} \text{ such that } \lambda_k \in \mathbb{R} \cap \partial D^+ \cap \partial E^-\};$$
$$K_D^E = \{k \in \mathbb{N} \text{ such that } \lambda_k \in \mathbb{R} \cap \partial E^+ \cap \partial D^-\};$$
$$K_E^E = \{k \in \mathbb{N} \text{ such that } \lambda_k \in \mathbb{R} \cap E\}.$$

Note that in Definition 3.11 we do not define a set K_D^D as such a set is guaranteed to be empty. This is because $a \neq e^{i\theta}$ for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$.[1] It is also clear from the definition and the fact that $D^\pm$ and $E^\pm$ are open sets that the index sets K_E^D, K_D^E and K_E^E are disjoint with union $K^{\mathbb{R}}$.

Definition 3.12. *Let $(\lambda_k)_{k \in \mathbb{N}}$ be the PDE discrete spectrum of an initial-boundary value problem, and ε_k be the associated radii from Definition 3.8. We define the following contours, whose traces are circles or the boundaries of semicircles or circular sectors. Each is oriented such that the corresponding λ_k lies to the left of the circular arc which forms part of the contour; so that they enclose a finite region.*

For $k \in K^{D^+} \cup K^{E^+} \cup K^{D^-} \cup K^{E^-}$ we define the contour

$\Gamma_k = \partial D(\lambda_k, \varepsilon_k)$.

For $k \in K^+ \cup K^- \cup K_E^D \cup K_D^E$ we define the contours

$\Gamma_k = \partial D(\lambda_k, \varepsilon_k)$,

$\Gamma_k^D = \partial(D(\lambda_k, \varepsilon_k) \cap D)$ *and*

$\Gamma_k^E = \partial(D(\lambda_k, \varepsilon_k) \cap E)$.

For $k \in K_E^E$ we define the contours

$\Gamma_k = \partial D(\lambda_k, \varepsilon_k)$,

$\Gamma_k^+ = \partial(D(\lambda_k, \varepsilon_k) \cap \mathbb{C}^+)$ *and*

$\Gamma_k^- = \partial(D(\lambda_k, \varepsilon_k) \cap \mathbb{C}^-)$.

We define the contours

$\Gamma_0 = \partial D(0, \varepsilon_0)$,

$\Gamma_0^{D^+} = \partial(D(0, \varepsilon_0) \cap D^+)$,

$\Gamma_0^{E^+} = \partial(D(0, \varepsilon_0) \cap E^+)$,

$\Gamma_0^{D^-} = \partial(D(0, \varepsilon_0) \cap D^-)$,

$\Gamma_0^{E^-} = \partial(D(0, \varepsilon_0) \cap E^-)$,

$\Gamma_0^+ = \partial(D(0, \varepsilon_0) \cap \mathbb{C}^+)$ *and*

$\Gamma_0^- = \partial(D(0, \varepsilon_0) \cap \mathbb{C}^-)$.

Some of the contours in Definition 3.12 are shown in Figure 3.3. In this example $1 \in K^{D^+}$ and $2 \in K_E^E$ and the partial differential equation is the heat equation, $q_t = q_{xx}$. We do not claim that there exists any particular set of boundary conditions for the heat equation such that these particular λ_1 and λ_2 are in the PDE discrete spectrum; the figure is purely to illustrate Definition 3.12. The contours associated with 0 and λ_2 are shown slightly away from these points for clarity on the figure but they do pass through the points. Indeed $\Gamma_0^{E^+}$ and $\Gamma_0^{E^-}$ each self-intersect at 0.

[1] See Figures 3.4, 3.5 and 3.6.

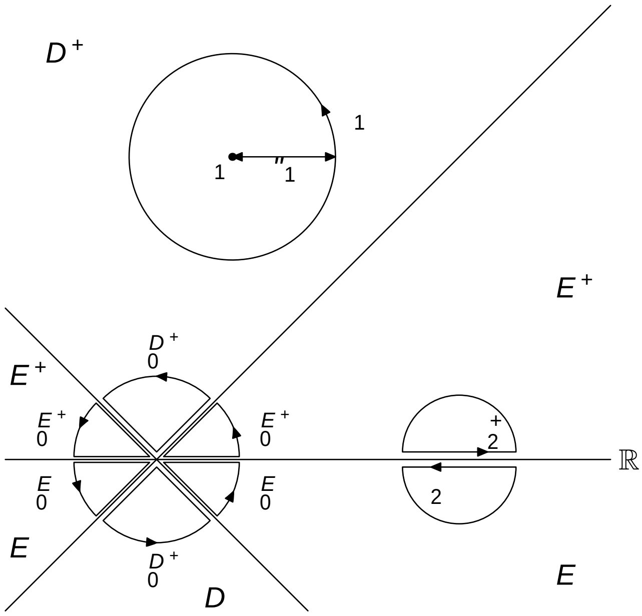
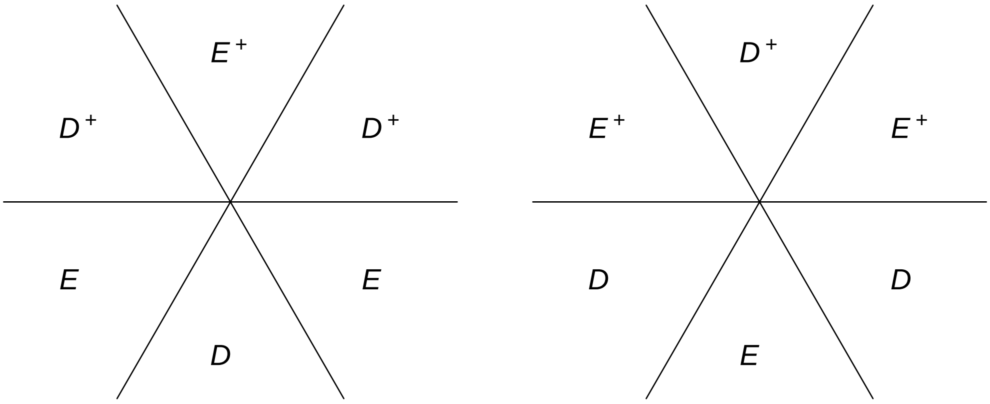


Figure 3.3. Some contours from Definition 3.12

The first step is to rewrite the integrals I_k for $k \in \{2; 4\}$ found in equations (3.1) as

$$I_2 = \int_{\mathbb{R}} + \int_{\partial E^+} P(\ ) \sum_{j \in J^+} \frac{{}_j(\ )}{}$$

Figure 3.3.

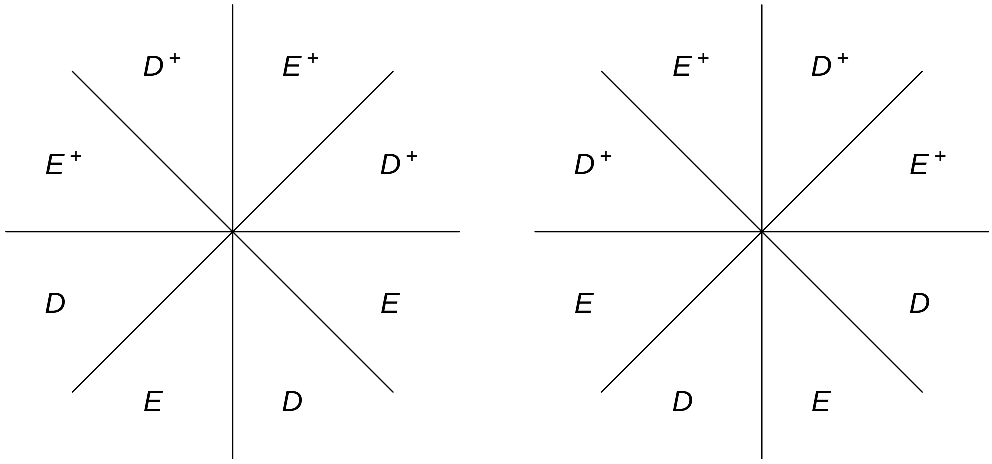


$n = 3$, $a = i$ $\qquad$ $n = 3$, $a = \;\; i$

Figure 3.4. The regions D and E for n odd and $a = \;\; i$

Using De nition 3.12 we may rewrite equations (3.1.5){(3.1.8) as

$$I_2 = \begin{cases} \displaystyle\int_{\mathbb{R}} + \int_{E} \end{cases}$$



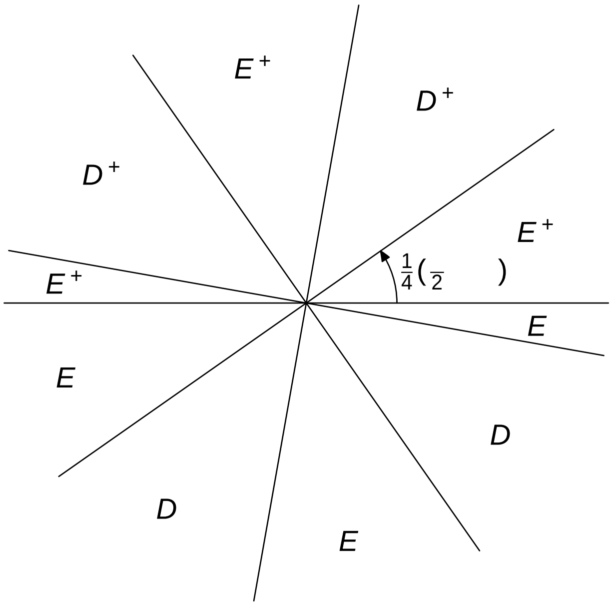
$n = 4, a = i$ $\qquad$ $n = 4, a = \quad i$

Figure 3.5. The regions D and E for $n = 4$ and $a = \quad i$

Hence $K_E^E = K_E^D = ;$ and $K_E^D = K^{\mathbb{R}}$. The right of Figure 3.4 shows the positions of D and E for $a = \quad i$ when $n = 3$.

3. **n even, a = i.** Then statement (3.1.13) holds hence $K_E^E = ;$, $K_E^D = fk\ 2\ K^{\mathbb{R}}$ such that $_k > 0g$ and $K_D^E = fk\ 2\ K^{\mathbb{R}}$ such that $_k < 0g$. The left of Figure 3.5 shows the positions of D and E for $a = i$ when $n = 4$.

4. **n even, a =**



$n = 4$, $a = e^{i}$

Figure 3.6. The regions D and E for n even

If n is odd and $a = i$,

$$q(x;t) = \frac{i}{2} \sum_{\substack{k\in K^+\\ \cup K^{D^+}\cup K^{E^+}\\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{=\,k} \frac{P(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ ) + \frac{i}{2} \sum_{\substack{k\in K\\ \cup K^{D}\ \cup K^{E}}} \operatorname*{Res}_{=\,k} \frac{\flat(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )$$

$$\frac{1}{2}\left\{ \sum_{k\in K^{\mathbb{R}}} \int \right.$$

If n is even and $a = -i$,

$$q(x;t) = \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K_{D^+}\cup K_{E^+}\\ \cup K_E^D}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\sum_{j\in J^+}\zeta_j(\lambda) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K_{D^-}\cup K_{E^-}\\ \cup K_D^E}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\sum_{j\in J^-}\zeta_j(\lambda)$$

$$+\frac{i}{4}\frac{1}{\Delta_{\mathrm{PDE}}'(0)}\frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{j\in J^+\cup J^-}\zeta_j(\lambda)$$

$$+\frac{1}{2}\left\{\sum_{k\in K_E^D}\int_{\Gamma_k^E}+\frac{1}{2}\int_{\Gamma_0}-\int_{0}^{\infty}-\sum_{k\in K_D^E}\int_{\Gamma_k^E}-\frac{1}{2}\int_{\Gamma_0^+}+\int_{0}^{\infty}\right\}$$

$$P(\lambda)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)}-1\right)H(\lambda)\,\mathrm{d}\lambda \qquad (3.1.18)$$

If n is even and $a = e^{i\theta}$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$q(x;t) = \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K_{D^+}\cup K_{E^+}\\ \cup K_{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\sum_{j\in J^+}\zeta_j(\lambda) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K_{D^-}\cup K_{E^-}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\sum_{j\in J^-}\zeta_j(\lambda)$$

$$-\frac{1}{2}\left\{\sum_{k\in K_{\mathbb{R}}}\int_{\Gamma_k}+\int_{\Gamma_0}+\int_{\mathbb{R}}\right\}P(\lambda)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)}-1\right)H(\lambda)\,\mathrm{d}\lambda; \qquad (3.1.19)$$

The proofs of these theorems are mathematically simple but, partly due to the range of values of a, take a large amount of space. For this reason, they are relegated to the Appendix Section B.2.

3.2. Well-posed IBVP

In this section we investigate Assumption 3.2. Specifically, we give a sufficient condition for and reduced boundary coefficient matrix

$$\hat{A} = \begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Following Definition 2.19 we calculate

$$\Delta_{\mathrm{PDE}}(\xi) = (\omega^2 - \omega)c_2^2(\xi)c_1(\xi)\Big[(e^{i\xi} - e^{-i\xi}) + \omega(e^{i\omega\xi} - e^{-i\omega\xi}) + \omega^2(e^{i\omega^2\xi} - e^{-i\omega^2\xi})$$

is decaying as $\lambda \to \infty$ from within D_1 because, as noted above, the exponentials $e^{i\lambda(1-x)}$ and $e^{i\omega\lambda(1-x)}$ are decaying for $x \in (0,1)$. Hence the ratio

$$\frac{\zeta_1(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$$

also decays as $\lambda \to \infty$ from within D_1. The same calculation can be performed to check the other ζ_j. Indeed the dominant terms in the ratio $\frac{\zeta_2(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ have ratio

$$\frac{1}{\omega^2-\omega}\int_0^1 \left\{ \omega e^{i\lambda(1-x)} - e^{i\omega\lambda(1-x)} \right\} q_T(\lambda)(x)\,\mathrm{d}x$$

and the dominant terms in the ratio $\frac{\zeta_3(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ have ratio

$$\frac{1}{\omega^2-\omega}\int_0^1 \left\{ e^{i\lambda(1-x)} - e^{i\omega\lambda(1-x)} \right\} q_T(\lambda)(x)\,\mathrm{d}x,$$

both of which decay as $\lambda \to \infty$ from within D_1.

We do not present the calculation for D_2 or D_3 or for $a = -i$ but it may be checked in the same way, case-by-case.

Remark 3.15. Although in Example 3.14 the full calculation is not presented for each case it is not true that

$$\frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \to 0 \text{ as } \lambda \to \infty \text{ from within } D_p \Rightarrow \frac{\zeta_k(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \to 0 \text{ as } \lambda \to \infty \text{ from within } D_r$$

for any j, k, p, r and it is not true that if Assumption 3.2 holds for a particular initial-boundary value problem then it holds for the initial-boundary value problem with the same boundary conditions but with a different value of a. Specific counterexamples are given in Example 3.16 (see Remark 3.17) and the uncoupled example of Chapter 5.

Example 3.16. We consider the 3$^{\mathrm{rd}}$ order initial-boundary value problem with $a = i$ and boundary conditions specified by the boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This gives reduced global relation matrix

$$\mathcal{A}(\lambda) = \begin{pmatrix} c_2(\lambda) & c_2(\lambda)e^{-i\lambda} & c_1(\lambda) \\ c_2(\lambda) & c_2(\lambda)e^{-i\omega\lambda} & \omega c_1(\lambda) \\ c_2(\lambda) & c_2(\lambda)e^{-i\omega^2\lambda} & \omega^2 c_1(\lambda) \end{pmatrix}$$

and reduced boundary coefficient matrix

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Following Definition 2.19 we calculate

$$_{\mathrm{PDE}}(\ ) = (\omega^2 \quad \omega) c_2^2(\ ) c_1(\ ) \sum_{k=0}^{2} \omega^k e^{\ i\omega^k}\ ;$$

$$_1(\ ) = c_2^2(\ ) c_1(\ ) \sum_{k=0}^{2} \omega^{k+2} \left(e^{\ i\omega^k}\ \hat{q}_T(\omega^{k+1}\ ) \quad e^{\ i\omega^{k+1}}\ \hat{q}_T(\omega^k\ ) \right)\ ;$$

$$_2(\ ) = \quad (\omega^2 \quad \omega) c_2^2(\ ) c_1(\ ) \sum_{k=0}^{2} \omega^k \hat{q}_T(\omega^k\ );$$

$$_3(\ ) = \quad c_2^2(\ ) c_1(\ ) \sum_{k=0}^{2} \left(e^{\ i\omega^k}\ \hat{q}_T(\omega^{k+1}\ ) \quad e^{\ i\omega^{k+1}}\ \hat{q}_T(\omega^k\ ) \right) \quad \text{and}$$

so, provided q_T is not identically zero, the numerator approaches infinity hence

$$\frac{\ _3(\ _j)}{\ _{\mathrm{PDE}}(\ _j)} \to \infty \text{ as } j \to \infty.$$

Hence the ratio (3.2.1) is unbounded for $\ \in \mathcal{D}_1$.

This establishes that Assumption 3.2 does not hold.

Remark 3.17. Although Assumption 3.2 does not hold in Example 3.16, it may be seen that the ratio

$$\frac{\ _2(\ )}{\ _{\mathrm{PDE}}(\ )}$$

is bounded within $\mathcal{D}_3$ and decaying as $\ \to \infty$ from within $\mathcal{D}_3$. Clearly the ratios

$$\frac{\ _4(\ )}{\ _{\mathrm{PDE}}(\ )}, \quad \frac{\ _6(\ )}{\ _{\mathrm{PDE}}(\ )}$$

both evaluate to 0 and $J\ = \{2, 4, 6\}$ so

$$\frac{\sum_{j \in J}\ _j(\ )}{\ _{\mathrm{PDE}}(\ )} = \frac{\ _2(\ )}{\ _{\mathrm{PDE}}(\ )}$$

so it is possible to make the necessary contour deformations in the lower half plane, that is in $\mathcal{D}_3$, just not in the upper half plane, that is $\mathcal{D}_1$ and $\mathcal{D}_2$. This is not particularly interesting in this example, except to give one of the counterexamples for Remark 3.15, as the problem is still ill-posed but a similar fact may be exploited in the uncoupled example of Chapter 5 to give a partial series representation of a solution to a well-posed problem; see Remark 5.9.

3.2.1. n odd, homogeneous, non-Robin

A sufficient condition for homogeneous, non-Robin boundary conditions to specify a problem that satisfies Assumption 3.2 may be written as two conditions of the form:

(1) There are enough boundary conditions that couple of the ends of the interval and, of the remaining boundary conditions, roughly the same number are specified at the left hand side of the interval as are specified at the right hand side.
(2) Certain coefficients are non-zero.

More precise formulations of these conditions are given below.

3.2.1.1. The first condition

To formally give the first condition we require the following:

Notation 3.18. Define

$$L = |\{j : \;_{rj} = 0 \; \forall r\}| \quad \begin{array}{c}\text{The number of left-hand boundary functions} \\ \text{that do not appear in the boundary conditions}\end{array} \tag{3.2.2}$$

$$R = |\{j : \;_{rj} = 0 \; \forall r\}| \quad \begin{array}{c}\text{The number of right-hand boundary functions} \\ \text{that do not appear in the boundary conditions}\end{array} \tag{3.2.3}$$

$$C = |\{j : \exists r : \;_{rj}, \;_{rj} \neq 0\}| \quad \begin{array}{c}\text{The number of boundary conditions that couple the} \\ \text{ends of the } x \text{ interval}\end{array} \tag{3.2.4}$$

Indeed, there are C boundary conditions that couple the ends of the interval, L boundary conditions prescribed at the right end of the interval and R boundary conditions prescribed at the left end of the interval. Clearly $n = L + R + C$. We now state the first condition.

Condition 3.19. If $a = i$ then the 2 1 boundary conditons are such that

$$R \leq \quad \leq R + C$$

and if $a = \quad i$ then the 2 1 boundary conditons are such that

$$R \leq \quad 1 \leq R + C$$

where R and C are defined by (3.2.3) and (3.2.4).

The remainder of this subsubsection is devoted to showing the relevance of the above condition. Consider the ratio

$$\frac{\;_j(\;)}{\;_{\mathrm{PDE}}(\;)}. \tag{3.2.5}$$

The denominator is an exponential polynomial, hence it is a sum of terms of the form

$$Z(\;)e^{\;i\;\sum_{r=1}^{k}\;!}$$

If $a = -i$ then $D = \bigcup_{j=1}^{n} D_j$ where

$$D_j = \{\lambda \in \mathbb{C} : \tfrac{\pi}{n}(2j-1) < \arg(\lambda) < \tfrac{\pi}{n}2j\}.$$

For concreteness, let $a = i$. If $\lambda \in D_j$ then, for all $\sigma \in S_n$ and for all $k \in \{1, 2, \ldots, n\}$,

$$\mathrm{Re}\left(-i\lambda\sum_{r=1-j}^{\kappa-j}\omega^{r}\right) \geq \mathrm{Re}\left(-i\lambda\sum_{r=1}^{k}\omega^{\sigma(r)}\right)$$

with equality if and only if $k = \kappa$ and the first κ entries in σ are some permutation of $(1-j, 2-j, \ldots, \kappa-j)$ (modulo n). Hence the exponential

$$e^{-i\lambda\sum_{r=1-j}^{\kappa-j}\omega^{r}} \tag{3.2.6}$$

dominates all other exponentials of the form

$$e^{-i\lambda\sum_{r=1}^{k}\omega^{\sigma(r)}}$$

and all functions of the form

$$Z(\lambda)\int_0^1 e^{-i\lambda\sum_{r=1}^{k}\omega^{\sigma(r)}} e^{-i\lambda x\omega^{\sigma(k+1)}} q_T(x)\,\mathrm{d}x.$$

Hence, if the exponential (3.2.6) multiplied by some polynomial appears in $\Delta_{\mathrm{PDE}}(\lambda)$ then Assumption 3.2 must hold. The conditions are necessary and sufficient for this exponential to appear in $\Delta_{\mathrm{PDE}}(\lambda)$.

By Lemma 2.14, we know that we may express the matrix A in the form

$$A_{kj}(\lambda) = \begin{cases} \omega^{(n-1-[J_j-1]/2)(k-1)} c_{(J_j-1)/2}(\lambda) & J_j \text{ odd}, \\ \omega^{(n-1-J_j/2)(k-1)} c_{J_j/2}(\lambda)\left(e^{-i\lambda\omega^{k-1}} + b^{+}_{J_j/2}\beta_{J_j/2}\right) & J_j \text{ even}, \end{cases}$$

but we may express this in terms of the three possible kinds of columns that A may contain. Indeed, using Notation 3.18, A has L columns of the form

$$c_{(n-1-j)}(\lambda)(1, \omega^{j}, \ldots, \omega^{j(n-1)})^{\mathsf{T}},$$

R columns of the form

$$c_{(n-1-j)}(\lambda)(e^{-i\lambda}, \omega^{j}e^{-i\lambda\omega}, \ldots, \omega^{j(n-1)}e^{-i\lambda\omega^{n-1}})^{\mathsf{T}}$$

and C columns of the form

$$c_{(n-1-j)}(\lambda)((e^{-i\lambda} + \beta_{rj}), \omega^{j}(e^{-i\lambda\omega} + \beta_{rj}), \ldots, \omega^{j(n-1)}(e^{-i\lambda\omega^{n-1}} + \beta_{rj}))^{\mathsf{T}},$$

where j ranges over L, R and C values within $\{0, 1, \ldots, n-1\}$ respectively.

Hence $\Delta_{\mathrm{PDE}}(\lambda) = \det A(\lambda)$ has terms

$$\sum P_l(\omega)e^{-i\lambda\sum_{r=1}^{R+l}\omega^{\sigma(r)}}$$

for each $l \in \{0, 1, \dots, C\}$ and $\sigma \in S_n$ where P_l are polynomials and X is some (fixed) integer. The terms appearing in $\phi_k(\sigma)$ are

$$\text{if } L > 1 \qquad \omega^{Z_l} L_l(\omega) \int_0^\infty e^{-i\sum_{r=1}^{R+l}\omega^{(r)}} e^{-i\omega^{(R+l+1)}x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C\}$$

$$\text{if } R > 1 \qquad \omega^{\widehat{Z}_l} R_l(\omega) \int_0^\infty e^{-i\sum_{r=1}^{R-1+l}\omega^{(r)}} e^{-i\omega^{(R+l)}x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C\}$$

$$\text{if } C > 1 \qquad \omega^{\overline{Z}_l} C_l(\omega) \int_0^\infty e^{-i\sum_{r=1}^{R+l}\omega^{(r)}} e^{-i\omega^{(R+l+1)}x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C-1\}$$

for each $\sigma \in S_n$ where R_l, L_l and C_l are polynomials and Z_l, $\widehat{Z}_l$ and $\overline{Z}_l$ are integers.

Remark 3.21. It should be noted that the polynomials P_l and R_l and the integer $\widehat{Z}_l$ depend not upon σ but upon the 9O831 is nonzero if $k > 1$ or the expression

$$\sum_{\substack{\sigma\in S_n:\\ \forall m\in R\ \exists p\in R:\\ \sigma_j r(m)=\sigma r(p)}} \operatorname{sgn}(\sigma)\,\omega^{\sum_{m\in R}\sigma r(m)m - \sum_{m\in C}\sigma c(m)m - \sum_{m\in L}\sigma l(m)m} \tag{3.2.9}$$

is nonzero if $k = 0$.

Note that in the case $\widetilde{\alpha}_j = \alpha$ for all $j \in C$ expression (3.2.8) simplifies to

$$\sum_{\substack{\sigma\in S_n:\ \exists\sigma'\in S_C:\\ (\sigma;\sigma')\in S_{k\,j}'}} \operatorname{sgn}(\sigma)\,\omega^{\sum_{m\in R}\sigma r(m)m - \sum_{m\in C}\sigma c(m)m - \sum_{m\in L}\sigma l(m)m}. \tag{3.2.10}$$

The set S_k', the functions l, r and c and their domains L, R and C are given in Definition B.7 and Lemma B.8.

This condition is checked for particular boundary conditions in the examples of Subsubsection 3.2.1.4.

3.2.1.3. Sufficient conditions for Assumption 3.2

Theorem 3.23. *Assume n is odd. If the boundary conditions of initial-boundary value problem* (2.1.1)–(2.1.3) *are homogeneous and non-Robin, and obey Conditions 3.19 and 3.22, then Assumption 3.2 holds.*

Proof. If the boundary conditions obey Condition 3.19 then $0 \leqslant k \leqslant C$ in Condition 3.22 so the set $S_{k\,j}'$ and the relevant expression (3.2.8) or (3.2.9) are all well defined.

Fix $j \in \{1, 2, \dots, n\}$ and let $\lambda \in \widetilde{D}_j$. Then the modulus of

$$e^{-i\lambda\sum_{y\in Y}\omega^y} \tag{3.2.11}$$

is uniquely maximised for the index set

$$Y = \{j-1, j, \dots, j-1+R+k-1\}.$$

By Condition 3.19 and Lemma B.8, $\Delta_{\mathrm{PDE}}(\lambda)$ has a term given by that exponential multiplied by a polynomial coefficient given by the right hand side of equation (B.3.6) if $k > 1$ or equation (B.3.7) if $k = 0$, with α replaced by α_j. These expressions are monomial multiples of expressions (3.2.8) and (3.2.9) respectively. As $\lambda \in D_j$, $\lambda \neq 0$ so the coefficient is guaranteed to be nonzero by Condition 3.22.

As Y *uniquely* maximises the exponential (3.2.11) this exponential dominates all other terms in $\Delta_{\mathrm{PDE}}(\lambda)$. But it also dominates all terms in $\zeta_j(\lambda)$, that is those of the form

$$Z(\lambda)e^{-i\lambda\sum_{p\in P}\omega^p}\int_0^1 e^{-i\lambda\omega^{p'}x}q_T(x)\,\mathrm{d}x$$

where $P \subsetneq \{0, 1, \dots, n-1\}$ and $p' \notin P$. Hence the ratio (3.2.5) is bounded in $\widetilde{D}_j$ for each $j \in \{1, 2, \dots, n\}$ and decaying as $\lambda \to \infty$ from within $\widetilde{D}_j$.

3.2.1.4. Checking Assumption 3.2 for particular examples

We now give three examples of how Theorem 3.23 can be used to check that a particular set of boundary conditions specifies a problem in which Assumption 3.2 holds. The first, Example 3.24, shows the necessity of checking Condition 3.22 by describing a class of pseudoperiodic boundary conditions for which Condition 3.19 holds but Condition 3.22 does not. This is the only known 3rd order example.

Example 3.24. Let $n = 3$ and the boundary coefficient matrix be given by

$$A = \begin{pmatrix} 1 & \widetilde{\beta}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \widetilde{\beta}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \widetilde{\beta}_1 \end{pmatrix}; \tag{3.2.12}$$

for $\widetilde{\beta}_j \in \mathbb{R}\setminus\{0\}$ so that the problem is *pseudoperiodic*. Indeed the boundary conditions are

$$\begin{aligned} q_{xx}(0;t) + \widetilde{\beta}_3 q_{xx}(1;t) &= 0;\\ q_x(0;t) + \widetilde{\beta}_2 q_x(1;t) &= 0 \text{ and}\\ q(0;t) + \widetilde{\beta}_1 q(1;t) &= 0. \end{aligned}$$

We check for which values of $\widetilde{\beta}_j$ Assumption 3.2 holds, first if $a = i$ and then if $a = -i$.

All three boundary conditions couple the ends of the space interval so $L = R = 0$ and $C = 3$. This ensures that, for $a = -i$, Condition 3.19 holds.

We adopt the notation of Condition 3.22, with c' the identity permutation on $\{1, 2, 3\}$ and $c(m) = 4 - m$ on the same domain, hence $\tau_j c(m) = m - j$. We simplify expression (3.2.8) to

$$\sum_{(\sigma,\sigma')\in S_{k\,\tau_j\,\sigma'}} \operatorname{sgn}(\sigma\sigma')\,\omega^{\sum_{m=1}^{3} \sigma(m)(4-\sigma'(m))} \prod_{m=k+1}^{3} \widetilde{\beta}_{\sigma'(m)} \tag{3.2.13}$$

for each j.

Assume first $a = i$, hence $k = 2$ and expression (3.2.13) simplifies further to

$$\sum_{(\sigma,\sigma')\in S_{2\,\tau_j\,\sigma'}} \operatorname{sgn}(\sigma\sigma')\,\omega^{\sum_{m=1}^{3} \sigma(m)(4-\sigma'(m))}\,\widetilde{\beta}_{\sigma'(3)} \tag{3.2.14}$$

The definition (B.3.4) of $S_{2\,\tau_j\,\sigma'}$ simplifies here to $(\sigma,\sigma') \in S_{2\,\tau_j\,\sigma'}$ if and only if

$$\begin{aligned} \{\sigma c c' \sigma'(p) : p \in \{1,2\}\} &= \{\tau_j c c' \sigma'(p) : p \in \{1,2\}\}\\ \Leftrightarrow \quad \{\sigma(4-\sigma'(p)) : p \in \{1,2\}\} &= \{1-j, 2-j\}\\ \Leftrightarrow \quad \sigma(4-\sigma'(3)) &= 3-j\\ \Leftrightarrow \quad \sigma'(3) &= 4-\sigma^{-1}(3-j) \end{aligned}$$

so the $\sim_2$ equivalence class of (τ_j, σ') is shown in Table 1.

Using this characterisation of $S_{2\,\tau_j\,\sigma'}$ we see that expression (3.2.14) does not evaluate to 0 provided

$$\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3 \neq 0. \tag{3.2.15}$$

$=$	$\begin{pmatrix} 1 \;\vert \\ 2 \;\vert \\ 3 \;\vert \end{pmatrix}$	$\begin{pmatrix} 2 \;\vert \\ 3 \;\vert \\ 1 \;\vert \end{pmatrix}$	$\begin{pmatrix} 3 \;\vert \\ 1 \;\vert \\ 2 \;\vert \end{pmatrix}$	$\begin{pmatrix} 2 \;\vert \\ 1 \;\vert \\ 3 \;\vert \end{pmatrix}$	$\begin{pmatrix} 1 \;\vert \\ 3 \;\vert \\ 2 \;\vert \end{pmatrix}$	$\begin{pmatrix} 3 \;\vert \\ 2 \;\vert \\ 1 \;\vert \end{pmatrix}$
$^{0}\ 2$	$\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$					

so that

$$A(\lambda) = \begin{pmatrix} c_2(\lambda)(e^{-i\lambda} - 1) & c_1(\lambda)(e^{-i\lambda} - 1) & c_0(\lambda)(e^{-i\lambda} + 2) \\ c_2(\lambda)(e^{-i\omega\lambda} - 1) & \omega c_1(\lambda)(e^{-i\omega\lambda} - 1) & \omega^2 c_0(\lambda)(e^{-i\omega\lambda} + 2) \\ c_2(\lambda)(e^{-i\omega^2\lambda} - 1) & \omega^2 c_1(\lambda)(e^{-i\omega^2\lambda} - 1) & \omega c_0(\lambda)(e^{-i\omega^2\lambda} + 2) \end{pmatrix}.$$

We calculate

$$\Delta_{\mathrm{PDE}}(\lambda) = (\omega - \omega^2)c_2(\lambda)c_1(\lambda)c_0(\lambda)\Big[9 + (2 - 2)(e^{i\lambda} + e^{i\omega\lambda} + e^{i\omega^2\lambda}) \\ + (1 - 4)(e^{-i\lambda} + e^{-i\omega\lambda} + e^{-i\omega^2\lambda})\Big];$$

as expected, the failure of Condition 3.22 causes the coefficients of $e^{i\omega^j}$ to cancel one another for each j, for each j, where

$$k = \begin{cases} \quad a = i; \\ 1 \quad a = \quad i: \end{cases}$$

As R is empty $(\sigma;\sigma') \in S_{k_j\sigma'}$ if and only if

$$\begin{aligned} &\{cc'\sigma'(p) : p \in \{1,2,\dots,k\}\} = \{{}_jcc'\sigma'(p) : p \in \{1,2,\dots,k\}\} \\ \Leftrightarrow\; &\{(n+1-\sigma'(p)) : p \in \{1,2,\dots,k\}\} = \{1-j, 2-j, \dots, k-j\} \end{aligned} \tag{3.2.20}$$

but if $(\sigma;\sigma') \in S_{k_j\sigma'}$ then $(\sigma;\sigma'') \in S_{k_j\sigma'}$ if and only if

$$\begin{aligned} &\forall q \in \{1,2,\dots,k\}\ \exists p \in \{1,2,\dots,k\} : \sigma''(q) = \sigma'(p) \\ \Leftrightarrow\; &\forall q \in \{k+1,k+2,\dots,n\}\ \exists p \in \{k+1,k+2,\dots,n\} : \sigma''(q) = \sigma'(p). \end{aligned}$$

Hence, for any given $\sigma \in S_n$ there exists a σ' for which $(\sigma;\sigma') \in S_{k_j\sigma'}$ but the choice of such a σ' does not affect the product

$$\prod_{m=k+1} e_{\sigma'(m)}$$

in expression (3.2.19) and there are $k!(n-k)! = \alpha!(\alpha-1)!$ choices of σ'. So any particular choice of σ' will suffice, provided we multiply by $\alpha!(\alpha-1)!$. Given $\sigma \in S_n$, define $\sigma' \in S_n$ such that

$$(n+1-\sigma'(p)) = p-j.$$

It is clear that $(\sigma;\sigma')$ satisfies condition (3.2.20) but as ρ is a bijection we may obtain an explicit expression

$$\sigma'(p) = n+1-\rho^{-1}(p-j).$$

Expression (3.2.19) may now be simplified to

$$\alpha!(\alpha-1)! \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \omega^{\sum_{m=1}^{n} m\,\sigma(n+1-m)} \prod_{m=k+1} e_{n+1-\rho^{-1}(m-j)}. \tag{3.2.21}$$

Making the substitution $\tau(m) = \rho^{-1}(m-j)$, for which $\sigma(n+1-m) = \tau^{-1}(n+1-m)-j$ and $\operatorname{sgn}(\tau) = (-1)^{(n-1)j}\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma)$, expression (3.2.21) may be written

$$\alpha!(\alpha-1)! \sum_{\tau \in S_n} \operatorname{sgn}(\tau)\, \omega^{\sum_{m=1}^{n} m(\tau^{-1}(n+1-m)-j)} \prod_{m=k+1} e_{n+1-\tau(m)}. \tag{3.2.22}$$

Expression (3.2.22) evaluates to zero if and only if

$$\sum_{\tau \in S_n} \operatorname{sgn}(\tau)\, \omega^{\sum_{m=1}^{n} m\,\tau^{-1}(n+1-m)} \prod_{m=k+1} e_{n+1-\tau(m)} \tag{3.2.23}$$

evaluates to zero. By Theorem 3.23, a sufficient condition for Assumption 3.2 to hold is that expression (3.2.23) is nonzero for

$$k = \begin{cases} \quad a = i; \\ 1 \quad a = \quad i: \end{cases}$$

Example 3.27. Let the boundary conditions be simple (hence uncoupled and non-Robin) and such that

$$R = \begin{cases} \emptyset; & \\ 1; & \end{cases} \qquad L = \begin{cases} 1; & a = i; \\ \emptyset; & a = -i. \end{cases} \tag{3.2.24}$$

Note these conditions on R and L are precisely those proven to be necessary and sufficient for well-posedness of the boundary value problem in [53].

Clearly Condition 3.19 holds. To show these boundary conditions satisfy Condition 3.22 we must show that expression (3.2.9), that is

$$\sum_{\substack{\sigma\in S_n: \\ \forall\, m\in R\ \exists\, p\in R: \\ \zeta_j r(m) = \sigma r(p)}} \operatorname{sgn}(\sigma)\, \omega^{\sum_{m\in R} \sigma r(m) m - \sum_{m\in L} \sigma l(m) m} \tag{3.2.25}$$

does not evaluate to zero for any j.

By definition (3.2.7) of ζ_j, the requirements on the $\sigma \in S_n$ indexing the first sum in expression (3.2.25) are equivalent to

$$\sigma \in S_n : \forall\, m \in \{1, 2, \dots, R\}\ \exists\, p \in R : m - j = \sigma r(p).$$

3.2.2.1. Assumption 3.2 implies well-posedness

Theorems 3.1 and 3.13 give an explicit representation of a unique solution to the initial-boundary value problem in terms of only known data provided Assumptions 3.2 and 3.3 both hold. It remains to be shown that Assumption 3.3 is not necesary.

Without Assumption 3.3 the expressions for I_2 and I_4 in equations (3.1.5) and (3.1.7) are not valid hence we must replace their representations in equations (3.1.9) and (3.1.11) with

$$I_2 = \left\{ \int_{\mathbb{R}} + \left[\int_{\partial E^+} - \int_{E_0^+} - \sum_{k\in K^+\cup K_D^E} \int_{E_k} \right] + \left[\int_{E_0^+} + \sum_{k\in K^+\cup K_D^E} \int_{E_k} \right] \right\} P(\lambda) \sum_{j\in J^+} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \, d\lambda ;$$

$$I_4 = \left\{ \int_{\mathbb{R}} + \left[\int_{\partial E^-} - \int_{E_0^-} - \sum_{k\in K^-\cup K_E^D} \int_{E_k} \right] + \left[\int_{E_0^-} + \sum_{k\in K^-\cup K_E^D} \int_{E_k} \right] \right\} \hat{P}(\lambda) \sum_{j\in J^-} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \, d\lambda :$$

With this adjustment to the calculation in Section 3.1, we may derive an integral representation of the solution in terms of the known data only.

Theorem 3.29. *Let the initial-boudary value problem* (2.1.1)–(2.1.3) *be well-posed and obey Assumption 3.2. Then its solution may be expressed as follows:*

If n **is odd and** $a = i$,

$$\begin{aligned} q(x;t) = {} & \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D-}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\hat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda_k) \\ & + \frac{1}{2} \left\{ \int_{\partial E^+} - \int_{E_0^+} - \sum_{k\in K^+\cup K_D^E} \int_{E_k} \right\} P(\lambda) \sum_{j\in J^+} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \, d\lambda \\ & + \frac{1}{2} \left\{ \int_{\partial E^-} - \int_{E_0^-} - \sum_{k\in K^-\cup K_E^D} \int_{E_k} \right\} \hat{P}(\lambda) \sum_{j\in J^-} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \, d\lambda \\ & - \frac{1}{2} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{E_k} + \int_{\Gamma_0} + \int_{\mathbb{R}} \right\} P(\lambda) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)} - 1 \right) H(\lambda) \, d\lambda ; \end{aligned} \quad (3.2.26)$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K_D^+}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda_k) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K_D^-\\ \cup K_{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda_k)$$
$$+ \frac{1}{2}\left\{\int_{\partial E^+} - \int_{\partial E_0^+} - \sum_{\substack{k\in\\ K^+\cup K_D^E}} \int_{\partial E_k}\right\} P(\lambda) \sum_{j\in J^+} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda$$
$$+ \frac{1}{2}\left\{\int_{\partial E^-} - \int_{\partial E_0^-} - \sum_{\substack{k\in\\ K^-\cup K_E^D}} \int_{\partial E_k}\right\} \widehat{P}(\lambda) \sum_{j\in J^-} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda$$
$$+ \frac{1}{2}\left\{\sum_{k\in K_{\mathbb{R}}} \int_{\partial E_k} + \int_{\partial E_0^+} - \int_{\mathbb{R}}\right\} P(\lambda)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)} - 1\right) H(\lambda)\,d\lambda; \quad (3.2.27)$$

If n is even and $a = -i$,

$$q(x,t) = \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K_D^+\\ \cup K_E^D}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda_k) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K_D^-\\ \cup K_D^E}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} \zeta_j(\lambda_k)$$
$$+ \frac{i}{4}\,\frac{1}{\Delta_{\mathrm{PDE}}'(0)}\,d \sum_{j\in J^+\cup J^-} \zeta_j(0)$$
$$+ \frac{1}{2}\left\{\int_{\partial E^+} - \int_{\partial E_0^+} - \sum_{\substack{k\in\\ K^+\cup K_D^E}} \int_{\partial E_k}\right\} P(\lambda) \sum_{j\in J^+} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda$$
$$+ \frac{1}{2}\left\{\int_{\partial E^-} - \int_{\partial E_0^-} - \sum_{\substack{k\in\\ K^-\cup K_E^D}} \int_{\partial E_k}\right\} \widehat{P}(\lambda) \sum_{j\in J^-} \frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda$$
$$+ \frac{1}{2}\left\{\sum_{k\in K_E^D} \int_{\partial E_k} + \frac{1}{2}\int_{\partial E_0^-} - \int_{0}^{\infty} - \sum_{k\in K_D^E} \int_{\partial E_k} - \frac{1}{2}\int_{\partial E_0^+} + \int_{-\infty}^{0}\right\}$$
$$P(\lambda)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)} - 1\right) H(\lambda)\,d\lambda; \quad (3.2.28)$$

If n is even and $a = e^{i\theta}$,

$$
\begin{aligned}
q(x;t) = {} & \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K_D^+\\ \cup K_{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+}\zeta_j(\lambda_k) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K_D^-}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-}\zeta_j(\lambda_k) \\
& + \frac{1}{2}\left\{\int_{\partial E^+} - \int_{E_0^+} - \sum_{k\in K^+\cup K_D^E}\int_{E_k^+}\right\} P(\lambda)\sum_{j\in J^+}\frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda \\
& + \frac{1}{2}\left\{\int_{\partial E^-} - \int_{E_0^-} - \sum_{k\in K^-\cup K_E^D}\int_{E_k^-}\right\} \widehat{P}(\lambda)\sum_{j\in J^-}\frac{\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,\mathrm{d}\lambda \\
& - \frac{1}{2}\left\{\sum_{k\in K_{\mathbb{R}}}\int_{\Gamma_k} + \int_{\Gamma_0} + \int_{\mathbb{R}}\right\} P(\lambda)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\lambda)} - 1\right) H(\lambda)\,\mathrm{d}\lambda. \qquad (3.2.29)
\end{aligned}
$$

Note that if the boundary conditions are homogeneous then $H(\lambda) = 0$ *so the last integral evaluates to* 0 *in each case.*

As indicated above, the proof of Theorem 3.29 for well-posed problems is a simple derivation. It remains to be shown that Assumption 3.2 implies well-posedness of the initial-boundary value problem a priori. Using the following Lemma, we appeal to the arguments presented in [27] and [53].

Lemma 3.30. *Let* $n \in \mathbb{N}$ *and let* $a \in \mathbb{C}$ *be such that* $a = \pm i$ *if* n *is odd and* $\mathrm{Re}(a) > 0$ *if* n *is even. Let* $D = \{\lambda \in \mathbb{C}$ *such that* $\mathrm{Re}(a\lambda^n) < 0\}$ *and let the polynomials* c_j *be defined by* $c_j(\lambda) = -a\lambda^n(i\lambda)^{-(j+1)}$. *Let* $\alpha_{jk}, \beta_{jk} \in \mathbb{R}$ *be such that the matrix*

$$
A = \begin{pmatrix}
\alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \cdots & \alpha_{10} & \beta_{10} \\
\alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \cdots & \alpha_{20} & \beta_{20} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & & &
\end{pmatrix}
$$

Corollary 3.31. *Let the initial-boundary value problem specified by equations* (2.1.1)–(2.1.3) *obey Assumption 3.2. Then the problem is well-posed and its solution may be found using Theorem 3.29.*

Corollary 3.31 is a restatement of Theorems 1.1 and 1.2 of [27]. For this reason we refer the reader to the proof presented in Section 4 of that paper. The only difference is that we make use of the above Lemma 3.30 in place of Proposition 4.1. We have not yet shown the reverse, that well-posedness of the initial-boundary value problem implies Assumption 3.2 holds.

3.2.2.2. Assumption 3.2 holds for a well-posed problem

We now present the converse of Corollary 3.31.

Theorem 3.32. *If the initial-boundary value problem* (2.1.1)–(2.1.3) *is well-posed, in the sense that it admits a unique solution* $q \in C^\infty([0,1] \times [0,T])$*, then Assumption 3.2 holds.*

Proof.

is bounded within $\hat{E}_1$ and decaying as $\to \infty$ from within $\hat{E}_1$. Indeed similar calculations may be performed for each ${}_k$ and for each $\hat{E}_j$ for $a = \ i$ to show that Assumption 3.3 holds.

Example 3.34. We present an example with the same partial differential equation and

The ratio

$$\frac{c_2(\;)c_1(\;)\left\{ !^2 \; e^{i} \hat{q}_0(!\;) \quad e^{i!} \hat{q}_0(\;) + ! \quad e^{i!^2} \hat{q}_0(\;) \quad e^{i} \hat{q}_0(!^2\;) \right\}}{(!^2 \quad !)c(\;)\,{}_1(\;)\,! \quad e^{i!}}$$

the boundary conditions are such that for each $j \in \{1, 2, \dots, n\}$ the expression

$$\sum_{(\sigma,\sigma')\in S_{k_j}'} \operatorname{sgn}(\sigma)\,\omega^{\sum_{m\in R} r(m)m \;\; \sum_{m\in C} c(m)m \;\; \sum_{m\in L} l(m)m} \prod_{m=k+1} \tilde{e}_{c'\sigma'(m)} \tag{3.3.3}$$

is nonzero if $k > 1$ or the expression

$$\sum_{\substack{\sigma\in S_n:\\ \forall m\in R\ \exists p\in R:\\ \sigma r(m)= r(p)}} \operatorname{sgn}(\sigma)\,\omega^{\sum_{m\in R} r(m)m \;\; \sum_{m\in C} c(m)m \;\; \sum_{m\in L} l(m)m} \tag{3.3.4}$$

is nonzero if $k = 0$.

Note that in the case $\tilde{e}_j =$ for all $j \in C$ expression (3.3.3) simplifies to

$$\sum_{\substack{\sigma\in S_n:\ \exists \sigma'\in S_C:\\ (\sigma,\sigma')\in S_{k_j}'}} \operatorname{sgn}(\sigma)\,\omega^{\sum_{m\in R} r(m)m \;\; \sum_{m\in C} c(m)m \;\; \sum_{m\in L} l(m)m}. \tag{3.3.5}$$

The set $S_{k'}$, the functions l, r and c and their domains L, R and C are given in Definition B.7 and Lemma B.8.

Theorem 3.37. *Suppose that a final-boundary value problem is specified by equations* (2.1.1) *and* (2.1.3) *and the final condition*

If n is odd and $a = i$,

$$
\begin{aligned}
q(x;t) = {} & \frac{i}{2}\sum_{\substack{k\in K^+\\ \cup K^{D+}\\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} e^{i\lambda_k^n T}\zeta_j(\lambda_k) + \frac{i}{2}\sum_{\substack{k\in K^-\\ \cup K^{D-}}} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\hat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^-} e^{i\lambda_k^n T}\zeta_j(\lambda_k) \\
& + \frac{1}{2\pi}\left\{\int_{\partial E^+} - \int_{E_0^+} - \sum_{k\in K^+\cup K^E_D}\int_{E_k}\right\} P(\lambda)\sum_{j\in J^+}\frac{e^{i\lambda^n T}\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda \\
& + \frac{1}{2\pi}\left\{\int_{\partial E^-} - \int_{E_0^-} - \sum_{k\in K^-\cup K^D_E}\int_{E_k}\right\} \hat{P}(\lambda)\sum_{j\in J^-}\frac{e^{i\lambda^n T}\zeta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}\,d\lambda \\
& - \frac{1}{2\pi}\left\{\sum_{k\in K^{\mathbb{R}}}\int_{E_k} + \int_0 + \int_{\mathbb{R}}\right\} P(\lambda)
\end{aligned}
$$

If n is even and $a = \; i$,

$$q(x;t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K_D^+ \\ \cup K_E^D}} \operatorname*{Res}_{=\;_k} \frac{P(\;)}{_{\mathrm{PDE}}(\;)} \sum_{j \in J^+} e^{\,i\;_k^n T}\;_j(\;_k) + \frac{i}{2} \sum_{\substack{k \in K \\ \cup K^D \\ \cup K_D^E}} \operatorname*{Res}_{=\;_k} \frac{\hat{P}(\;)}{_{\mathrm{PDE}}(\;)} \sum_{j \in J} e^{\,i\;_k^n T}\;_j(\;_k)$$

$$+ \frac{i}{4} \frac{1}{_{\mathrm{PDE}}{}'(0)} \, \mathrm{d} \sum_{j \in J^+ \cup J} \;_j(0)$$

$$+ \frac{1}{2} \begin{cases} \int_{\partial E^+} \quad \int_{E_0^+} \end{cases}$$

Lemma 3.42. *Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$ be such that $a = \ i$ if n is odd and* $\mathrm{Re}(a) > 0$ *if n is even. Let $D = \{ \ \in \mathbb{C}$ such that* $\mathrm{Re}(a \ ^n) < 0\}$ *and let the polynomials c_j be defined by*
c_j

Find $q \in C^\infty([0,1] \times [0,T])$ such that the partial differential equation

$$\partial_t q(x,t) - a(-i\partial_x)^n q(x,t) = 0$$

holds on $[0,1] \times [0,T]$ with boundary conditions

$$A \begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

and initial condition

$$q(x,0) = X(x).$$

The following are equivalent:

(1) *The problems Π and Π' are all well-posed in the sense that they have unique solutions.*
(2) *The problem Π is well-posed and its solution admits a series representation with an integral of the boundary data.*
(3) *The problem Π' is well-posed and its solution admits a series representation with an integral of the boundary data.*
(4) *Assumption 3.2 and Assumption 3.3 both hold.*

If $a = \pm i$ then the following are equivalent to one another and to (1):

(5) *The problems Π and Π'' are all well-posed in the sense that they have unique solutions.*
(6) *The problem Π'' is well-posed and its solution admits a series representation with an integral of the boundary data.*

If n is even then the following are equivalent to one another and to (1):

(7) *The problem Π is well-posed.*
(8) *The problem Π' is well-posed.*
(9) *Assumption 3.2 holds.*
(10) *Assumption 3.3 holds.*

Proof. Corollaries 3.31 and 3.43 and Theorems 3.32 and 3.44 show that (1) is equivalent to (4).

If (4) holds then Π is well-posed so Theorem 3.13 implies (2). If Assumption 3.2 is false then, by Theorem 3.32, (2) is false. If Assumption 3.3 is not true then it is not possible to close the contours of integration in I_2 and I_4, defined in equations (3.1.1), hence there exists no series representation of the solution to Π. Hence (2) implies (4). In the same way, (3) is equivalent to (4).

If $a = \pm i$ then Lemma 3.47 states that Π' and Π'' are equivalent problems. Hence (1) and (5) are equivalent and (3) and (6) are equivalent.

Corollary 3.31 and Theorem 3.32 show that (7) is equivalent to (9). Corollary 3.43 and Theorem 3.44 show that (8) is equivalent to (10).[2] As n

Find $q \in C^\infty([0,1] \times [0,T])$ *such that the partial differential equation*

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0 \tag{3.3.15}$$

holds on $[0,1] \times [0,T]$ *with boundary conditions*

$$A \begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(t) \\ H_2(t) \\ \vdots \\ H_n(t) \end{pmatrix} \tag{3.3.16}$$

and initial condition

$$q(x,0) = X(x). \tag{3.3.17}$$

Let $'$ be the following final-boundary value problem:

Find $q \in C^\infty([0,1] \times [0,T])$ *such that the partial differential equation*

$$\partial_t q(x,t) - a(-i\partial_x)^n q(x,t) = 0 \tag{3.3.18}$$

holds on $[0,1] \times [0,T]$ *with boundary conditions*

$$A \begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(T-t) \\ H_2(T-t) \\ \vdots \\ H_n(T-t) \end{pmatrix} \tag{3.3.19}$$

92 14.TJ/F15 10.9096615 10.90i83a Td [(b) -32 (Td, 070Td [1 (qui.9091 Tf3eR) 16 Td 51

*and final conditi*5.766 0 Td [(()]TJ/F3 8(caRB09eJ 0 -6.982 Td [(@)]TJ/F43 J/F45 10.9091 4RB09eJ 0 -

Proof of Lemma 3.47. Assume is well-posed, in the sense that it has a unique C^1 smooth solution q. We apply the map $t \mapsto T - t$ to the problem $'$. Then $\partial_t q(x;t) \mapsto -\partial_t q(x;T - t)$ hence partial differential equation (3.3.18) becomes

$$\partial_t q(x;T - t) + a(-i\partial_x)^n$$

CHAPTER 4

operator is equal to its principal part. This means we have no need to define operators of the form

$$\sum_{j=0}^{n} a_j(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j.$$

Locker studies the principal part of this operator to yield results about the full operator. He uses perturbation methods to show that the properties of the full operator may be inferred from the properties of the principal part but such deductions about Locker's more general operator are beyond the scope of this work. The partial differential equations studied in Chapters 2 and 3

Definition 4.2 (Classification of boundary conditions). *The boundary conditions* (2.1.3) *of the differential operator T may be written in the matrix form*

$$A \begin{pmatrix} u^{(n-1)}(0) \\ u^{(n-1)}(1) \\ u^{(n-2)}(0) \\ u^{(n-2)}(1) \\ \vdots \\ u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

where A is the boundary coefficient matrix.

If each boundary condition only involves derivatives of the same order then we call the boundary conditions non-Robin*. Otherwise we say that a boundary condition is* of Robin type*.*

Proof. *Formally self-adjoint*: Let $u \in D(T)$, $v \in H^n[0;1]$. Then the inner product may be evaluated

$$\begin{aligned}
\langle Tu, v\rangle &= \int_0^1 (-i)^n u(n)(x) v(x)\,\mathrm{d}x \\
&= (-i)^n \sum_{j=1}^{n} (-i)^{j-1} \left[u^{(n-j)}(1) v^{(j-1)}(1) - u^{(n-j)}(0) v^{(j-1)}(0) \right] \\
&\qquad\qquad\qquad\qquad + (-i)^n (-1)^n \int_0^1 u(x) \overline{v}^{(n)}(x)\,\mathrm{d}x \\
&= (-i)^n \sum_{j=1}^{n} (-i)^{j-1} \left[u^{(n-j)}(1) v^{(j-1)}(1) - u^{(n-j)}(0) v^{(j-1)}(0) \right] + \langle u, -v\rangle. \qquad (4.1.3)
\end{aligned}$$

As $u \in D(T)$ and the boundary coefficient matrix is rank n we have n linear equations in $u^{(j)}$. Hence we may construct another n linear equations in $v^{(j)}$ to ensure that the sum in equation (4.1.3) evaluates to 0, so that an adjoint boundary coefficient matrix may be constructed. Indeed Chapter 3 of [10] gives a method for finding the adjoint boundary coefficients in terms of the boundary coefficients of T using Green's functions. Then one may define the operator

Notation 4.9. For a differential operator T, we define the polynomials Δ_1 and Δ_0 as follows.

$$\Delta_1(\lambda) = \det \begin{array}{c} \\ \left(\begin{array}{ccc} Q_1(i\lambda) & P_1(i\lambda\omega^{k-1}) & Q_1(i\lambda\omega^{k-1}) \\ \vdots & \vdots & \vdots \\ Q_n(i\lambda) & P_n(i\lambda\omega^{k-1}) & Q_n(i\lambda\omega^{k-1}) \end{array}\right) \end{array} \begin{array}{c} \scriptstyle 2 \leq k \leq \nu \quad\quad \nu+1 \leq k \leq n \\ \end{array}, \tag{4.1.7}$$

$$\Delta_0(\lambda) = \det \left(\begin{array}{ccc} P_1(i\lambda) & P_1(i\lambda\omega^{k-1}) & Q_1(i\lambda\omega^{k-1}) \\ \vdots & \vdots & \vdots \\ P_n(i\lambda) & P_n(i\lambda\omega^{k-1}) & Q_n(i\lambda\omega^{k-1}) \end{array}\right). \tag{4.1.8}$$

As each of the polynomials P_k, Q_k are of degree no greater than m_k, the maximum degree of the polynomials Δ_1, Δ_0 is $\sum_{k=1}^{n} m_k$. For this reason w68m

As stated above, we focus upon degenerate irregular di erential operators. It is not clear that Locker's proof of Theorem 4.11 illuminates the study of these di erential operators.

4.2. Eigenvalues of T

Find $q \in C^\infty([0,1] \times [0,T])$ *that satisfies the partial differential equation* (4.2.1) *on* $[0,1] \times [0,T]$, *subject to the final condition*

$$q(x,T) = q_T(x) \quad \text{for } x \in [0,1] \tag{4.2.4}$$

and the boundary conditions (4.2.3) *where* $f_j(t) = \partial_x^j q(0,t$

The boundary data do not affect A hence the inhomogeneous / homogeneous boundary value problems associated with a have the same boundary coefficient matrix. Finally, note that Lemma 2.17 holds for final-boundary value problems as well as initial-boundary value problems.

4.2.1. Non-Robin with a symmetry condition

In this section we assume that the boundary conditions of the differential operator are non-Robin and obey Condition 4.14 below.

Condition 4.14. Let the boundary coefficients of a differential operator, an initial-boundary value problem or a final-boundary value problem be such that

$$\begin{aligned} r \in \widehat{J}^{+} &\iff n-1-r \in \widetilde{J}^{-} \text{ and} \\ r \in \widehat{J}^{-} &\iff n-1-r \in \widetilde{J}^{+}. \end{aligned} \tag{4.2.5}$$

Also, for all $j \in \widetilde{J}^{-} \cap \widehat{J}^{+}$,

$$\widehat{J}^{+}_{j}\; j = \widehat{J}^{+}_{n-1-j}\; n-1-j.$$

We note that the index sets $\widehat{J}^{\pm}$, $\widetilde{J}^{\pm}$ are defined in Notation 2.12 and Notation 4.3 in terms of the boundary coefficients (and implicitly of n) only, not in terms of a or the data of the initial- or final- boundary value problems. Hence, by Lemma 4.13, Condition 4.14 holds for a particular differential operator T if and only if it holds for any particular boundary value problem associated with T.

Although Condition 4.14 imposes a strong symmetry on the boundary conditions it turns out to be quite a natural condition. Almost all of our earlier examples of initial- and final-boundary value problems have boundary coefficients that obey this condition, as do boundary coefficient matrices such as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & 0 & 0 \end{pmatrix}$$

Theorem 4.15. *Let T be a differential operator of the form described in Definition 4.1 with non-Robin boundary conditions obeying Condition 4.14. Then the determinant function Δ has the same zeros as the determinant functions Δ_{PDE} from each of the associated boundary value problems.*

It is known that Condition 4.14 is not sharp. It is an open and interesting question whether it may be discarded entirely.

Proof. Fixing a particular, permissible value of a, and showing that the nonzero zeros of Δ and of Δ_{PDE} from the homogeneous initial-boundary value problem associated with $(T; a)$ are equal is sufficient proof as we may extend this to the full result using Lemma 4.13. We choose $a = i$ as it is one of the two values permissible for both odd and even n.

The function Δ_{PDE} is defined by equation (2.3.3) as the determinant of the reduced global relation matrix A, which is defined by equation (2.2.5) for homogeneous, non-Robin boundary conditions. Indeed, for each $j \in \mathcal{J}^-$, there is a column of A given by

$$\omega^{(n-1-j)(k-1)} c_j(\lambda); \quad k \in \{1, 2, \dots, n\} \tag{4.2.6}$$

and for each $j \in \mathcal{J}^+$, there is a column of A given by

$$\omega^{(n-1-j)(k-1)} c_j(\lambda)\left(e^{-i\omega^{k-1}\lambda} + \mathfrak{b}^+_{\mathcal{J}^+_j\, j}\right); \quad k \in \{1, 2, \dots, n\}. \tag{4.2.7}$$

The monomials are defined by

$$c_j(\lambda) = -i\lambda^n (i\lambda)^{-(j+1)}.$$

The exponential $e^{i\ \sum_{k=1}^{1} !^{k\ 1}}$ is entire and nonzero on $\mathbb{C}$ so the zeros of are the same as the zeros of det M^0.

We now study the columns of $M^0(\ )$. Note rst that, as the boundary conditions are non-Robin, the polynomials P_j and Q_j are each monomials of order m_j, hence

$$M^0_{jk}(\ ) = !^{m_j(k\ \ 1)}(i\ )^{m_j}\quad {}_{j\,m_j}e^{\ i!^{k\ \ 1}} + {}_{j\,m_j}\ :$$

By the de nition of m_j, for each $j\ 2\ f1;2;:::;ng$, m_j lies in at least one of J^+ and J

of A

Then

$$\begin{aligned}
(\ )e^{i\sum_{k=1}\omega^{k-1}} &= \det M'(\ )\\
&= (i\ )^3 \det\begin{pmatrix} (e^{i\ } + {}_1) & (e^{i\ }+{}_2) & (e^{i\ }+{}_3)\\ \omega^2(e^{i\omega}+{}_1) & \omega(e^{i\omega}+{}_2) & (e^{i\omega}+{}_3)\\ \omega(e^{i\omega^2}+{}_1) & \omega^2(e^{i\omega^2}+{}_2) & (e^{i\omega^2}+{}_3)\end{pmatrix}\\
&= (i\ )^3\det\left(\begin{matrix} (e^{i\ }+{}_1) & (e^{i\ }+{}_2) & (e^{i\ }+{}_3) & \\ (e^{i\omega}+{}_1) & \omega^2(e^{i\omega}+{}_2) & \omega(e^{i\omega}+{}_3) & {}_{i\omega^2}\ \ {}_2)\ {}^2(e^{i\omega}\end{matrix}\right.
\end{aligned}$$

From the partial differential equation, we obtain a relation between the left hand sides of equations (4.3.9) and (4.3.11), indeed

$$a\sum_{k\in\mathbb{N}} {}^{n}_{k}\,{}_{k}(x)\,{}_{k}(t) + a\sum_{k\in\mathbb{N}} T({}_{k})(x)\,{}_{k}(t) = 0.$$

Hence

$$\sum_{k\in\mathbb{N}} (T \quad {}^{n}_{k})\,{}_{k}(x)\,{}_{k}(t) = 0,$$

hence, by the minimality of the ${}_k$, each ${}_k$ is an eigenfunction of T with eigenvalue ${}^{n}_{k}$.

Theorem 4.19. *If the boundary conditions of an initial-boundary value problem are such that the problem is well-posed, its solution has a series representation and all zeros of* ${}_{\mathrm{PDE}}$ *are simple then the eigenfunctions of the associated ordinary differential operator T form a complete system in $L_2[0,1]$.*

Proof. Choose some $q_0 \in C^\infty[0,1]$, to specify a particular initial-boundary value problem. Solving that problem and expressing its solution as a discrete series, we know from Theorem 4.18 that the series expansion (3.0.6)–(3.0.9) is in terms of the eigenfunctions of T. Evaluating both sides of this equation at $t = 0$ we obtain an expansion of the initial datum in terms of the eigenfunctions. Hence the eigenfunctions form a complete system in $C^\infty[0,1]$. As $C^\infty[0,1]$ is dense in $L^2[0,1]$, the result is proven.

Another immediate corollary to Theorem 4.18 is

Corollary 4.20. *The PDE discrete spectrum of a well-posed initial-boundary value problem that admits a series representation and for which all zeros of* ${}_{\mathrm{PDE}}$ *are simple is a subset of the discrete spectrum of the ordinary differential operator with which it is associated.*

4.4. The failure of the system of eigenfunctions to be a basis

In this section we develop some of the theory of biorthogonal sequences as presented in Section 3.3 of [15], giving expanded versions of the proofs Davies presents. These definitions are also given in the survey [56]. Sedletskii's survey and its references also give an extensive treatment of the exponential systems we investigate. Biorthogonal sequences are essential to the study of our differential operators as they are non-self-adjoint. This means that their eigenfunctions, together with the eigenfunctions of the adjoint operator, form a biorthogonal pair of sequences. We use the following notational convention:

Notation 4.21. Let B be a Banach space with dual space $B^\star$, the space of linear functionals defined on B. Let $f \in B$ and let $\quad \in B^\star$. We define the use of angled brackets,

$$\langle f, \quad \rangle = \quad (f),$$

to mean the functional acting upon the element f of the Banach space.

Lemma 4.24. *Let B be a Banach space with basis $(f_n)_{n\in\mathbb{N}}$. Then there exists a sequence $(\ _n)_{n\in\mathbb{N}}$ in B^* such that the Fourier coefficients of f with respect to $(f_n)_{n\in\mathbb{N}}$ are given by $\ _n = \langle f, \ _n\rangle$. Furthermore, $((f_n)_{n\in\mathbb{N}}, (\ _n)_{n\in\mathbb{N}})$ is a biorthogonal pair.*

Proof. Equip $\mathbb{N}$ with the discrete topology and let $K = \mathbb{N} \cup \{\infty\}$ be the Alexandrov one-point compactification of $\mathbb{N}$. Let

$$C = \{s : K \to B \text{ such that } s \text{ is continuous, } s(1) \in \mathbb{C}f_1 \text{ and } \forall\, n > 2, (s(n) - s(n-1)) \in \mathbb{C}f_n\}.$$

Define a norm on C by

$$\|s\|_C = \sup_{n\in K} \|s(n)\|_B = \max_{n\in K} \|s(n)\|_B,$$

the latter equality is justified by the compactness of K and the continuity of s. Then $(C, \|\ \|_C)$ is a Banach space. Let the operator $X : C \to B$ be defined by $Xs = s(\infty)$. Then X is a bounded, linear operator with norm 1. We show in the next two paragraphs that X is a bijection.

As B has a basis, for any $g \in B$ there exist Fourier coefficients $\ _r$ such that $g = \sum_{r=1}^{\infty} \ _r f_r$. Let $s_g : K \to B$ denote the function defined by

$$s_g(n) = \sum_{r=1}^{n} \ _r f_r.$$

Certainly $s_g(1) \in \mathbb{C}f_1$ and $(s_g(n) - s_g(n-1)) \in \mathbb{C}f_n$. Any open ball in B contains either no $s_g(n)$, finitely many $s_g(n)$ or finitely many plus all $s_g(n)$ for n greater than some N. Each of these are open sets in the topology on K so s_g is continuous. This establishes $s_g \in C$, the domain of X. But $Xs_g = g$ and g may be any point in B, so X is onto.

Let $s, t \in C$ be such that $Xs = Xt$, that is $s(\infty) = t(\infty)$. By the definition of C, there exist sequences $(\ _n)_{n\in\mathbb{N}}$ and (uTJ -249.362 -1114 -1.636 Td [(g.9701 Tf 4.242.9091 T8 0 Td

C ($\in C$

In Definition 4.23 it is not required that the sequence $(f_n)_{n\in\mathbb{N}}$ be complete for the pair of sequences $((f_n)_{n\in\mathbb{N}}, (\ _n)_{n\in\mathbb{N}})$ to be biorthogonal. Lemma 4.25 concerns the existence of a biorthogonal pair in the case that one sequence is known to be complete.

Lemma 4.25. *Let $(f_n)_{n\in\mathbb{N}}$ be a complete sequence in a Banach space B. There exists a sequence $(\ _n)_{n\in\mathbb{N}}$ in B^**

I, as $n \to \infty$. If P_n are uniformly bounded in norm and $(f_n)_{n\in\mathbb{N}}$ is complete then $(f_n)_{n\in\mathbb{N}}$ is a basis.

Remark 4.27. Before giving a formal proof of the above Lemma we give a heuristic idea of the reason that these projection operators must be uniformly bounded in norm.

Let $((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}})$ and $((c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}})$ be pairs of sequences in a Banach space B such that $\langle a_j, b_k\rangle = 0$ and $\langle c_j, d_k\rangle = 0$ for all $j \neq k$ and $\langle a_k, b_k\rangle \neq 0$, $\langle c_k, d_k\rangle \neq 0$ for all k. Then each pair can be normalised into a biorthogonal pair in the following way.

Define new sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$ by

$$A_n = \frac{a_n}{\sqrt{\langle a_n, b_n\rangle}}, \qquad B_n = \frac{b_n}{\sqrt{\langle a_n, b_n\rangle}},$$
$$C_n = \frac{c_n}{\sqrt{\langle c_n, d_n\rangle}}, \qquad D_n = \frac{d_n}{\sqrt{\langle c_n, d_n\rangle}}.$$

Our pairs of systems, $((A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}})$ and $((C_n)_{n\in\mathbb{N}}, (D_n)_{n\in\mathbb{N}})$, are both biorthogonal systems; we have performed a biorthonormalisation on the original pairs. But this does not mean

7 f

hence P_n is of finite rank. This also shows Q_n is finite rank. We show that P_n satisfies the definition of a projection operator:

$$P_n^2 f = \sum_{r=1}^{n}$$

exists a sequence $(N_n)_{n\in\mathbb{N}}$ such that $P_r g_n = g_n$ for all $r > N_n$. Then

$$\begin{aligned}\lim_{k\to\infty}\|(P_k - I)f\| &= \lim_{k\to\infty}\|(P_k - I)\lim_{n\to\infty} g_n\| \\ &= \lim_{k,n\to\infty}\|(P_k - I)g_n\| \\ &= \lim_{k,n\to\infty}\|g_k - g_n\| = 0;\end{aligned}$$

as $(g_n)_{n\in\mathbb{N}}$ is Cauchy. Hence $(f_n)_{n\in\mathbb{N}}$ is a basis and the Fourier coefficients are given by

$$\alpha_n = P_n f.$$

Lemma 4.28. *Let* $((f_n)_{n\in\mathbb{N}}; (\alpha_n)_{n\in\mathbb{N}})$

CHAPTER 5

Two interesting examples

In this chapter we present the detailed analysis of two examples, both for $q_t = q_{xxx}$ with boundary conditions

$$
\begin{aligned}
q_x(0,t) + \beta q_x(1,t) = 0 \text{ or } q_x(0,t) &= 0, \\
q(0,t) &= 0, \\
q(1,t) &= 0.
\end{aligned}
$$

The second of these may be considered as the limit of the first as the coupling constant β approaches 0. For each example we investigate both the homogeneous initial-boundary value problem and the associated the differential operator.

For each example we break the analysis into major themes by section. In Section 5.1 we adapt the standard notation used throughout the thesis to include a superscript β or 0 to distinguish between the two examples. In Sections 5.2 and 5.3 each example has its own subsection so no additional notational identification is necessary. At the end of each section we present a third subsection comparing and contrasting the two cases. In the final subsection we also indicate how the arguments presented in that section may (or may not) be generalised to higher order and to other kinds of boundary conditions.

We conclude the chapter by comparing and contrasting all four of the calculations presented, discussing their relative usefulness and complexity. In the next chapter we discuss some directions for further work, informed by the results of this chapter.

5.1. The problems and regularity

In this section we set up the boundary conditions to be investigated in this chapter. We define the differential operators and the initial-boundary value problems we wish to discuss and calculate some of the simple quantities associated with each. One set of boundary conditions is *coupled* and the other *uncoupled*; we use these words to distinguish between the two problems in this chapter but this does not imply our conclusions are true for all coupled or uncoupled boundary conditions.

5.1.1. The differential operator

Let T^β, respectively T^0, be the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix

$$
A^\beta = \begin{pmatrix} 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ respectively } A^0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (5.1.1)
$$

where $\beta \in \mathbb{R} \setminus \{-1, 0, 1\}$.

The corresponding values of $\ $, ω defined by Notation 4.6 are

$$
\begin{aligned}
\ &= 2; \\
\omega &= e^{2\ i}
\end{aligned}
\tag{5.1.2}
$$

The polynomials $_1;\ _0$ associated with T , respectively

Then there exists a minimal $Y \; 2 \; \mathbb{N}$ such that $((\;\; _k)_{k=Y}^{1};(\;\; _k)_{k=Y}^{1})$ is a biorthogonal sequence in $L^2[0;1]$. *Moreover*

$$h \;\; _k; \;\; _k i = 2\cos\left(\frac{p\overline{3}}{2}(\;\; i \;\; _k)\right)\left[\cosh\left(\frac{3}{2}(\;\; i \;\; _k)\right) + 2\cos\left(\frac{p\overline{3}}{2}(\;\; i \;\; _k)\right)\right] \quad 8 \tag{5.2.7}$$

$$= (\;\; 1)^k \frac{p\overline{3}}{2} e^{p\overline{3}\;\;(k+\frac{1}{6})} + O(1) \text{ as } k \; ! \;\; 1: \tag{5.2.8}$$

Lemma 5.6. *Let $\;\; _k$, $\;\; _k$ and $\;\; _k$ be the eigenvalues and eigenfunctions from Lemma 5.4. Then*

$$k \;\; _k k^2 = k \;\; _k k^2 \tag{5.2.9}$$

$$\begin{aligned} &= \frac{1}{i \;\; _k}\Big(\sinh(\;\; i \;\; _k)[\cos(p\overline{3}(\;\; i \;\; _k)) \quad 6] + p\overline{3}\cosh(\;\; i \;\; _k)\sin(p\overline{3}(\;\; i \;\; _k)) \\ &\quad + 3\sinh(2(\;\; i \;\; _k)) \quad 3e^{\frac{1}{2}(\;\; i \;\; _k)}\left[\cos\left(\tfrac{p\overline{3}}{2}(\;\; i \;\; _k)\right) + p\overline{3}\sin\left(\tfrac{p\overline{3}}{2}(\;\; i \;\; _k)\right)\right] \\ &\quad + 3e^{-\frac{1}{2}(\;\; i \;\; _k)}\left[\cos\left(\tfrac{p\overline{3}}{2}(\;\; i \;\; _k)\right) \quad p\overline{3}\sin\left(\tfrac{p\overline{3}}{2}(\;\; i \;\; _k)\right)\right]\Big) \end{aligned} \tag{5.2.10}$$

$$= \frac{3}{6}\, p\overline{3} e^{\frac{4}{p\overline{3}}(k+\frac{1}{6})}$$

But if $(\ ) = 0$ also then

$$
\begin{aligned}
& 0 = e^{i\ } + \omega e^{i\omega\ } + \omega^2 e^{i\omega^2\ };\\
\Rightarrow\ & 0 = (\omega\ \ \omega^2)\ e^{i\omega\ }\ \ e^{i\omega^2\ }\\
\Rightarrow\ & \ = 2k\ i\ \exists\, k \in \mathbb{Z}.
\end{aligned}
$$

But

$$
(2k\ i) = \begin{cases} \sqrt{3}e^{\ \sqrt{3}k} & k \text{ even,}\\ 2 + \sqrt{3}e^{\ \sqrt{3}k} & k \text{ odd,}\end{cases}
$$

which is strictly positive, hence $2k\ i$ is not a zero of for any $k \in \mathbb{Z}$ and every zero of is simple.

Proof of Lemma 5.4. The adjoint boundary coefficient matrix $A^\star$ may be constructed using the method presented in Section 3 of Chapter 11 of [10], particularly Theorem 3.1, but in this case a direct calculation easily shows that $T^\star$ is adjoint to T.

The matrix $A^\star$ is in reduced row echelon form so we may calculate * using Definition 4.7:

$$
\begin{aligned}
{}^\star(\ ) &= i\ e^{i\ }(\omega^2\ \ \omega)\sum_{r=0}^{2}\omega^r e^{\ i\omega^r\ }\\
&= e^{2i\ }\ (\ ); \qquad (5.2.13)
\end{aligned}
$$

where is the characteristic determinant of T. The argument in the proof of Lemma 5.2 may be applied to establish that the set of eigenvalues of $T^\star$ is $\{\ {}^3_k : k \in \mathbb{N}\}$.

The final statement may be proved using the argument in the proof of Lemma 5.3.

Proof of Lemma 5.5. For any $j, k \in \mathbb{N}$,

$$
{}^3_k\langle\ _k,\ _j\rangle = \langle T\ _k,\ _j\rangle = \langle\ _k, T^\star\ _j\rangle = {}^3_j\langle\ _k,\ _j\rangle;
$$

hence if $j \neq k$ then $\langle\ _k,\ _j\rangle = 0$. Hence, provided there does not exist $k \in \mathbb{N}$ such that $\langle\ _k,\ _k\rangle = 0$, the eigenfunctions of T and $T^\star$ form a biorthogonal sequence. Indeed, by the following asymptotic calculation there must exist some $Y > 1$ such that $\langle\ _k,\ _k\rangle \neq 0$ for all $k > Y$.

As $i\ _k \in \mathbb{R}^+$,

$$
\begin{aligned}
\overline{\ _k} &= \ _k\\
\overline{\ _k} &= \ _k.
\end{aligned} \qquad (5.2.14)
$$

Using equations (5.2.14), we calculate

$$
\begin{aligned}
\langle \phi_k, \psi_k\rangle &= \int_0^1 \phi_k(x)\,\psi_k(x)\,\mathrm{d}x \\
&= \int_0^1 \sum_{r,l=0}^{2} \Big(e^{i(\omega^r-\omega^l)\lambda_k x}\; e^{i(\omega^{r+1}-\omega^{l+1})\lambda_k} + e^{i(\omega^{r+2}-\omega^{l+2})\lambda_k} \\
&\qquad\qquad - e^{i(\omega^{r+1}\omega^{l+2})\lambda_k} - e^{i(\omega^{r+2}-\omega^{l+1})\lambda_k}\Big)\,\mathrm{d}x \\
&= \sum_{r=0}^{2} \left(2 - e^{i\omega^r(1-\omega)\lambda_k} - e^{-i\omega^r(1-\omega)\lambda_k}\right) \\
&\quad - \sum_{r=0}^{2} \Big[\int_0^1 e^{i\omega^r(1-\omega)\lambda_k x}\,\mathrm{d}x \left(e^{i\omega^{r+1}(1-\omega)\lambda_k} + e^{i\omega^{r+2}(1-\omega)\lambda_k} - e^{i\omega^{r+1}(1-\omega^2)\lambda_k} - 1\right) \\
&\quad + \int_0^1 e^{i\omega^r(1-\omega^2)\lambda_k x}\,\mathrm{d}x \left(e^{i\omega^{r+1}(1-\omega^2)\lambda_k} + e^{i\omega^{r+2}(1-\omega^2)\lambda_k} - e^{i\omega^{r+2}(1-\omega)\lambda_k} - 1\right)\Big].
\end{aligned}
\tag{5.2.15}
$$

The first sum in equation (5.2.15) evaluates to the right hand side of equation (5.2.7). The second sum evaluates to 0.

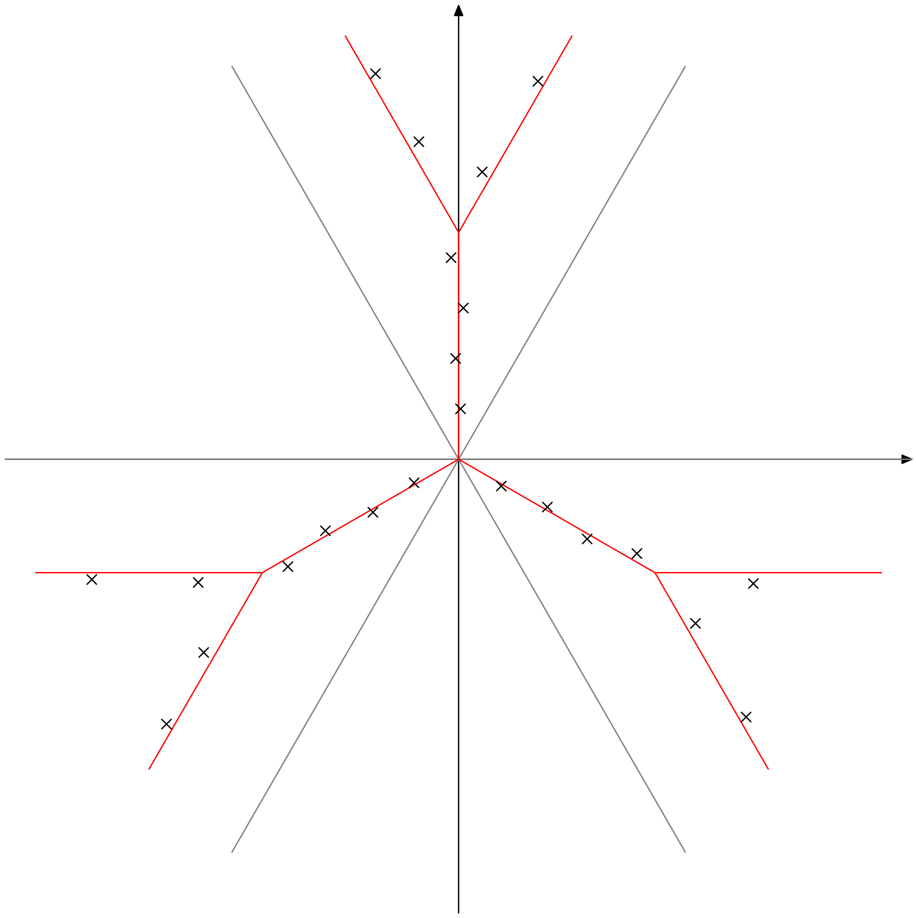
The asymptotic expression (5.2.8) follows from

$$\cos\left(\frac{\sqrt{3}}{2}(\;\; i\lambda_k)\right) = (-1)^k\frac{\sqrt{3}}{\quad}$$

$k\,p$

5.2.3. The limit ! 0

We may wish to consider the calculations in Subsection 5.2.2 as the limit ! 0 of such calculations for the coupled operator. For concreteness, we specify 2 (1; 0) and consider the one-sided limit ! 0 . Indeed, we may show that if $_k$ is a zero of $_{\text{PDE}}$ then $!^r$ $_k$ is also a zero of $_{\text{PDE}}$ and $_k$ is a zero of $^?_{\text{PDE}}$. We may express the zeros of $_{\text{PDE}}$ as the complex



We consider one particular generalisation. Let $n > 3$ be an odd number with $n = 2 \quad 1$

We also define I_1 to be the identity transformation on $\{1\}$, the only element of S_1. For Condition 3.22, $k = 1$ so we must check that

$$\sum_{\substack{\sigma\in S_3:\\ (\sigma;1)\in S_1 \,{}_j\, I_1}} \operatorname{sgn}(\sigma)\,\omega^{\ 2\ \sigma(3)} \tag{5.3.1}$$

is nonzero. However $(\sigma;1) \in S_1 \,{}_j\, I_1$ if and only if $\sigma(2) \in \{1\ \ j; 2\ \ j\}$ so expression 5.3.1 evaluates to

$$\omega^{2j}(\omega^{\ 6}\ \ \omega^{\ 6}\ \ \omega^{\ 1} + \omega^{\ 2}) = \omega^{2j}(\omega\ \ \omega^{2}) \neq 0.$$

For Condition 3.36, $k = 0$ so we check

$$\sum_{\substack{\sigma\in S_3:\\ \sigma(2)=\ j(2)}} \operatorname{sgn}(\sigma)\,\omega^{\ 2\ \sigma(3)} = \omega^{2j}(1\ \ \omega^{2}) \neq 0.$$

As n is odd, the boundary conditions are homogeneous and non-Robin, Conditions 3.19 and 3.22 hold and Conditions 3.35 and 3.36 hold, Theorem 3.50 guarantees the initial-boundary value problem is well-posed and that its solution admits a series representation.

5.3.2. Uncoupled

Theorem 5.8. *The initial-boundary value problem associated with* $(T^0; i)$ *is well-posed but its solution does not admit a series representation.*

Proof. Using Notation 3.18, $L = 1$, $R = 2$ and $C = 0$. Hence Condition 3.19 holds. To check Condition 3.22 we write

$$R = \{3; 2\}; \qquad L = \{3\};$$

$$r : 3 \mapsto 2; \qquad l : 3$$

hence its determinant,

$$\Delta_{\mathrm{PDE}}(\lambda) = i\lambda(\omega^2 - \omega)\sum_{r=0}^{2} \omega^r e^{i\lambda\omega^r};$$

and the functions

$$\eta_1(\lambda) = i\lambda(\omega^2 - \omega)\sum_{r=0}^{2} \omega^r \hat{q}_0(\omega^r\lambda) e^{i\lambda\omega^r};$$

$$\eta_2(\lambda) = i\lambda\sum_{r=0}^{2} \hat{q}_0(\omega^r\lambda)\left(\omega^{r+1}e^{-i\lambda\omega^{r+1}} - \omega^{r+2}e^{-i\lambda\omega^{r+2}}\right);$$

$$\eta_3(\lambda) = i\lambda\sum_{r=0}^{2} \hat{q}_0(\omega^r\lambda)\left(e^{-i\lambda\omega^{r+2}} - e^{-i\lambda\omega^{r+1}}\right);$$

$$\eta_4(\lambda) = \eta_5(\lambda) = \eta_6(\lambda) = 0.$$

As $a = i$, the regions of interest in Assumption 3.3 are

$$\tilde{E}_j \subset E_j = \left\{\lambda \in \mathbb{C} : \frac{(2j-1)\pi}{3} < \arg(\lambda) < \frac{2j\pi}{3}\right\}.$$

We consider the particular ratio

$$\frac{\eta_3(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}, \quad \lambda \in \tilde{E}_2. \tag{5.3.2}$$

For $\lambda \in \tilde{E}_2$, $\mathrm{Re}(i\lambda\omega^r) < 0$ if and only if $r = 2$ so we may approximate ratio (5.3.2) by its dominant terms as $\lambda \to \infty$ from within $\tilde{E}_2$,

$$\frac{(\hat{q}_0(\lambda) - \hat{q}_0(\omega\lambda))e^{-i\lambda\omega^2} + \hat{q}_0(\omega^2\lambda)(e^{-i\lambda\omega} - e^{-i\lambda}) + o(1)}{(\omega^2 - \omega)e^{i\lambda} + (1 - \omega^2)e^{i\lambda\omega} + o(1)}.$$

We expand the integrals from the $\hat{q}_0$ in the numerator and multiply the numerator and denominator by $e^{-i\lambda\omega}$ to obtain

$$\frac{i\int_0^1 \left(e^{i\lambda(1-x)} - e^{i\lambda(1-\omega x)} - e^{i\lambda\omega^2(1-x)} + e^{-i\lambda(2\omega-\omega^2 x)}\right)q_0(x)\,\mathrm{d}x + o\left(e^{\mathrm{Im}(\omega\lambda)}\right)}{\sqrt{3}(e^{i\lambda(1-\omega)} + \omega) + o\left(e^{\mathrm{Im}(\omega\lambda)}\right)}. \tag{5.3.3}$$

Let $(R_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\lambda_j = R_j e^{i\frac{7\pi}{6}} \in \tilde{E}_2$, R_j is bounded away from $\{\frac{2\pi}{\sqrt{3}}(k + \frac{1}{6}) : k \in \mathbb{N}\}$[4] and $R_j \to \infty$ as $j \to \infty$. Then $\lambda_j \to \infty$ from within $\tilde{E}_2$. We evaluate ratio (5.3.3) at $\lambda = \lambda_j$,

$$\frac{i\int_0^1 \left(2ie^{\frac{R_j}{2}(1-x) - \frac{\sqrt{3}R_j}{2}i}\sin\left(\frac{\sqrt{3}R_j x}{2}\right) - e^{-R_j(1-x)}\left(1 - e^{-\sqrt{3}R_j i}\right)\right)q_0(x)\,\mathrm{d}x + o\left(e^{\frac{R_j}{2}}\right)}{\sqrt{3}(e^{-\sqrt{3}R_j i} + \omega) + o\left(e^{\frac{R_j}{2}}\right)}. \tag{5.3.4}$$

The denominator of ratio (5.3.4) is bounded away from 0 by the definition of R_j and the numerator tends to ∞ for any nonzero initial datum. This establishes that Assumption 3.3 does not hold which imples that there is no series representation of the solution.

---

[4] Of course this is guaranteed by $\lambda_j \in \tilde{E}_2$ and the asymptotic expression (5.2.2) for λ_k but by adding this condition explicitly we avoid having to resort to Lemma 5.2

Remark 5.9. In the proof of Theorem 5.8 we use the example of the ratio $\frac{\zeta_3(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ being unbounded as $\lambda \to \infty$ from within $\widehat{E}_2$. It may be shown using the same argument that $\frac{\zeta_2(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ is unbounded in the same region and that both these ratios are unbounded for $\lambda \in \widehat{E}_3$ using $\lambda_j = R_j e^{i\frac{11\pi}{6}}$.

CHAPTER 6

Conclusion and further work

6.1. Conclusion

In this work, we have discussed the mutual interaction of two conceptually separate approaches to the study of linear di erential operators and linear partial di erential equations.

Conjecture 6.1. *Equation* (6.2.1) *holds for arbitrary boundary conditions of any order.*

In the third order this conjecture has been established for all non-Robin boundary conditions but it was necessary to investigate many sets of boundary conditions individually. A general argument that does not require symmetry conditions has thus-far been elusive.

It has been shown that equation (6.2.1) holds for general Robin boundary conditions but the symmetry condition has not been removed. For Robin boundary conditions, the symmetry condition becomes very complex to express. Completely removing the symmetry condition is a topic of current research. The zeros of the two characteristic determinants are of great interest in their respective problems and to show that the zeros are the same and of the same orders in general would be of great utility.

6.2.2. Regularity conditions for well-posedness

In Example 3.24 we derive necessary and su cient conditions for well-posedness of the initial-boundary value problems associated with $(T; i)$ and $(T; \ i)$ where T is the third order operator with pseudo-periodic boundary conditions. In the second row of Table 1 on page 134 we note that these conditions correspond to the polynomials $_1$ and $_0$ from Notation 4.9 not being identically zero. The same result holds for simple boundary conditions, as is shown in Table 1. It would be interesting to know if this correspondence extends to arbitrary boundary conditions:

Conjecture 6.2. *For any di erential operator T, $_1 = 0$ only if the initial-boundary value problem associated with $(T; i)$ is ill-posed and $_0 = 0$ only if the initial-boundary value problem associated with $(T; \ i)$ is ill-posed.*

If this conjecture is true then it gives even simpler conditions for well-posedness of an initial-boundary value problem. Even if it holds only for odd order initial-boundary value problems with non-Robin boundary conditions, problems whose well-posedness may already be checked by Conditions 3.19 and 3.22, it is still gives a much easier check for well-posedness than the existing conditions. Indeed, a related conjecture is:

Conjecture 6.3. *Let T be an odd order di erential operator with non-Robin boundary conditions. Then*

deg($_1$) = p_0 *if and only if Conditions 3.19 and 3.22 hold for the initial-boundary value problem associated with* $(T; i)$.

deg($_0$) = p_0 *if and only if Conditions 3.19 and 3.22 hold for the initial-boundary value problem associated with* $(T; \ i)$.

6.2.3. Rates of blow-up

There is a further suggestion of a link in the comparison of the two calculations for the uncoupled case in Chapter 5. In the proof of Theorem 5.8 we require that the R_j are bounded away from the set $f \frac{2}{3}(k + \frac{1}{6}) : k \ 2 \ \mathbb{N}g$. Consider what happens in the limit where this bound

The essential missing link in the proof of this Conjecture 6.6 is Conjecture 6.1. We assume this is the case and proceed with a proof.

Proof of Conjecture 6.6 under an assumption. Under Conjecture 6.1, the nonzero zeros of are precisely the nonzero zeros of $_{PDE}$. Further, if $\neq 0$ is a zero of $_{PDE}$, then representation exists and Case 1 holds if and only if precisely one of the initial-boundary value problems associated with $(T; i)$ and $(T; \ \ i)$ is well-posed.

If both the initial-boundary value problems are ill-posed, for example all boundary conditions at the same end, then it is possible that neither of these cases hold but we are not really interested in that situation. Indeed, Examples 10.1 and 10.2 of [48] show that the discrete spectra of such problems may be empty or the entirity of $\mathbb{C}$.

Finally, we formally state our original conjecture.

Conjecture 6.9. *The rate of blow-up depends upon the location of the eigenvalues.*

Case 1: *Let*

$$_k(\ ; \ ^0) = (Xk + Y + \ )e^{i \ 0} + z: \tag{6.2.8}$$

Then there exists some $W \in \mathbb{C} \setminus \{0\}$ *such that*

$$\lim_{\to 0} \left[\frac{\frac{\| \ _k \| \| \ _k \|}{\langle \ _k, \ _k \rangle}}{\sup_{\substack{j \in \{1,2,\dots,2n\} \\ {}^0 \in V}} \frac{|_j[\ _k(\ ; \ ^0)]|}{\mathrm{PDE}[\ _k(\ ; \ ^0)] \ _k(\ ; \ ^0)}} \right] = W + o(1) \quad \text{as} \ \to \infty; \tag{6.2.9}$$

where V is the interval $[0; \frac{}{n})$ or $[\frac{}{n}; \frac{2}{n})$ which contains ^{0} and $_j = \ _j$ or $_j = \ _j$, whichever is appropriate for the given V (this will depend upon a).

Case 2: *Then*

$$\frac{\| \ _k \| \| \ _k \|}{\langle \ _k, \ _k \rangle} = O(1) \quad \text{as } k \to \infty$$

APPENDIX A

Results

Boundary Conditions	Boundary Coe cient	Boundary Conditions	Locker's

Boundary Conditions Name	Boundary Conditions For BVP	Locker's Regularity	Well-posed IBVP on $\partial_t q + i(-i\partial_x)^n q = 0$	Well-posed IBVP on $\partial_t q - i(-i\partial_x)^n q = 0$	Can Be Solved Using Separation of Variables
Quasiperiodic Periodic , = 1	$\partial_x^j q(0,t) + \ \partial_x^j q(1,t) = 0$ $j \in \{0,1,\dots,n-1\}$	Regular	X	X	I = 1
Pseudoperiodic	$\partial_x^j q(0,t) + \ _j \partial_x^j q(1,t) = 0$ $j \in \{0,1,\dots,n-1\}$	Regular , two identities on $_j$ do not hold. Otherwise degenerate.	See Example 3.26	See Example 3.26	
Coupled in order $\frac{n+1}{2}$	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0,1,\dots,\frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(0,t) \quad \partial_x^{\frac{n-1}{2}} q(1,t) = 0$	Regular	See Example B.6 Conjecture: true for $n > 7$	Conjecture: true for $n > 7$	
Simplest for Well-posed IBVP if a=i	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0,1,\dots,\frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(0,t) = 0$	Degenerate as $_0 = 1$	X		
Simplest for Well-posed FBVP if a=-i	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0,1,\dots,\frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(1,t) = 0$	Degenerate as $_1 = 1$		X	
Simplest for Well-posed FBVP	$q(0,t) = q(1,t) = 0$ $q_x(1,t) = 0$	Degenerate as $_0 = 0$		X	
Simple		Degenerate as $_0 = 0$ or $_1 = 0$	See Example 3.27 and [53]	See Example 3.27 and [53]	
General Uncoupled		Degenerate as $_0 = 0$ or $_1 = 0$	Unknown. Conjecture: As above.	Unknown. Conjecture: As above.	

Table 2. Results: Odd order $n > 3$

APPENDIX B

Some proofs

B.1. Standard theorems

Here we list some standard mathematical results. The first three are well-known and the fourth is not obscure. They have in common that they do not fit into the areas of mathematics covered by this thesis but are necessary fundamentals for those topics. We list them, without proof, to remind the reader of the results.

Theorem B.1 (Green). *Let Ω be a simply connected open set in $\mathbb{R}^2$ whose boundary, $\partial\Omega$, is a positively oriented piecewise smooth, simple closed curve. Let $F;G:\overline{\Omega}\to\mathbb{C}$ be continuous functions with continuous partial derivatives. Then*

$$\int_{\partial\Omega}(F\,\mathrm{d}x+G\,\mathrm{d}y)=\iint_{\Omega}\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

This theorem originally appears appears in [34]. It is used in Section 2.1 to obtain the `global relation', an essential step in Fokas' unified transform method

Theorem B.2 (Cramer's rule). *If the square matrix A is full rank then the equation*

$$Ax=y$$

has solution

$$x_j=\frac{\det A_j}{\det A}$$

where A_j is the matrix A with the j^{th} column replaced by the vector y.

A proof is given in [50] but it may be found either as a result or an exercise in any first textbook on linear algebra. This theorem is used in Section 2.3 to solve a full rank system of linear equations which is obtained in Section 2.2.

Lemma B.3 (Jordan). *If the only singularities of $f:\mathbb{C}\to\mathbb{C}$ are poles then, for any $a>0$,*

$$\lim_{R\to\infty}\int_0 \exp(iaRe$$

$s_k > 0$ *such that all zeros of* f *outside* $B(0; r)$ *lie in* N *semi-strips,* S_k*, perpendicular to* L_k *and of width* s_k*. Further, as* $R \to \infty$*, the number of zeros of* f *within* $B(0; R) \cap S_k$ *is asymptotically given by*

$$\frac{2\pi R}{l_k}.$$

This result appears as Theorem 8 of [42] and is proved in the preceeding Section 9 using as we integrate along the diameter that divides D from E once in each direction so the contribution from this section of the contours Γ_k^D and Γ_k^E cancels out. We conclude for $k \in K$

$$\int_{\Gamma_k^E} \frac{\hat{P}(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda + \int_{\Gamma_k^D} \frac{\hat{P}(\lambda)e^{a\lambda nT}}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda = \int_{\Gamma_k} \frac{\hat{P}(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda. \tag{B.2.2}$$

2. $k \in K^+$

A similar calculation may be performed in the upper half-plane; for $k \in K^+$

$$\int_{\Gamma_k^E} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda + \int_{\Gamma_k^D} \frac{P(\lambda)e^{a\lambda nT}}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda = \int_{\Gamma_k} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda. \tag{B.2.3}$$

3. $k \in K_E^D$

We now turn our attention to the integrals whose contours touch the real axis but do not pass through 0. Using fact (B.2.1) once again,

$$\int_{\Gamma_k^D} \frac{P(\lambda)e^{a\lambda nT}}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda = \int_{\Gamma_k^D} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda;$$

but for the contours in the lower half-plane we must be more careful. We rewrite

$$\begin{aligned}\int_{\Gamma_k^E} \frac{\hat{P}(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda &= \int_{\Gamma_k^E} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda \\ &\quad + \int_{\Gamma_k^E} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \left[e^{-i\lambda} \sum_{j \in J} \rho_j(\lambda) - \sum_{j \in J^+} \rho_j(\lambda) \right] d\lambda.\end{aligned}$$

Hence for $k \in K_E^D$

$$\begin{aligned}&\int_{\Gamma_k^E} \frac{\hat{P}(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda + \int_{\Gamma_k^D} \frac{P(\lambda)e^{a\lambda nT}}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda \\ &= \int_{\Gamma_k} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J^+} \rho_j(\lambda)\,d\lambda + \int_{\Gamma_k^E} \frac{P(\lambda)}{\mathrm{PDE}(\lambda)} \left[e^{-i\lambda} \sum_{j \in J} \rho_j(\lambda) - \sum_{j \in J^+} \rho_j(\lambda) \right] d\lambda,\end{aligned} \tag{B.2.4}$$

cancelling the integrals in each direction along the real interval $(\lambda_k - \varepsilon_k, \lambda_k + \varepsilon_k)$.

4. $k \in K_D^E$

For the contours in the lower half-plane

$$\int_{\Gamma_k^D} \frac{\hat{P}(\lambda)e^{a\lambda nT}}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda = \int_{\Gamma_k^D} \frac{\hat{P}(\lambda)}{\mathrm{PDE}(\lambda)} \sum_{j \in J} \rho_j(\lambda)\,d\lambda$$

and for the contours in the upper half-plane

$$\int_{\;k}^{E} \frac{P(\;)}{\mathrm{PDE}(\;)} \sum_{j \in J^+} {}_j(\;)\,d \;=\; \int_{\;k}^{E} \frac{\hat{P}(\;)}{\mathrm{PDE}(\;)} \sum_{j \in J} {}_j(\;)\,d$$
$$+ \int_{\;k}^{E} \frac{P(\;)}{\mathrm{PDE}(\;)} \left[\sum_{j \in J^+} {}_j(\;) \quad e^{\;i\;} \sum_{j \in J} {}_j(\;) \right] d\;.$$

Hence for $k \in K_D^E$

$$\int_{\;k}^{E} \frac{P(\;)}{\mathrm{PDE}(\;)} \sum_{j \in J^+} {}_j(\;)\,d \;+\; \int_{\;k}^{D} \frac{\hat{P}(\;)e^{a\;\;nT}}{\mathrm{PDE}(\;)} \sum_{j \in J} {}_j(\;)\,d$$
$$= \int_{\;k} \frac{\hat{P}(\;)}{\mathrm{PDE}(\;)} \sum_{j \in J} {}_j(\;)\,d \;+\; \int_{\;k}^{E} \frac{P(\;)}{\mathrm{PDE}(\;)} \left[\sum_{j \in J^+} {}_j(\;) \quad e^{\;i\;} \sum_{j \in J} {}_j(\;) \right] d\;. \tag{B.2.5}$$

5. $k \in K_E^E$

Similarly to the above,

$$\int_{\;k}^{+} \frac{P(\;)}{\mathrm{PDE}(\;)} \sum_{j \in J^+} {}_j(\;)\,d \;+\; \int_{\;k} \frac{\hat{P}(\;)e^{a\;\;nT}}{\mathrm{PDE}(\;)} \sum_{j \in J} {}_j(\;)\,d$$
$$= \int_{\;k} P(\;)$$

1. n odd, $a = i$. Then

$$\int_{\Gamma_0^E} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_0^D} \frac{\widehat{P}(\lambda)e^{a\lambda^n T}}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda$$
$$= \int_{\not\Gamma_0} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \int_{\not\Gamma_0} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \left[e^{i\lambda^n T} \sum_{j\in J} \zeta_j(\lambda) - \sum_{j\in J^+} \zeta_j(\lambda) \right] d\lambda. \tag{B.2.9}$$

From equations (B.2.7) and (B.2.9) we obtain

$$\int_{\Gamma_0^{E^+}} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_0^{D^+}} \frac{P(\lambda)e^{i\lambda^n T}}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda$$
$$+ \int_{\Gamma_0^{E}} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_0^{D}} \frac{\widehat{P}(\lambda)e^{i\lambda^n T}}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda$$
$$= \int_{\Gamma_0} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_0} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \left[e^{i\lambda^n T} \sum_{j\in J} \zeta_j(\lambda) - \sum_{j\in J^+} \zeta_j(\lambda) \right] d\lambda. \tag{B.2.10}$$

We now use equations (B.2.2), (B.2.3), (B.2.4) and (B.2.10) together with equations (3.1.9)–(3.1.12) to write

$$\sum_{l=2}^{5} I_l = \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \sum_{\substack{k\in K \\ \cup K^{D}\cup K^{E}}} \int_{\Gamma_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda$$
$$+ \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0} \right\} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \left[e^{i\lambda^n T} \sum_{j\in J} \zeta_j(\lambda) - \sum_{j\in J^+} \zeta_j(\lambda) \right] d\lambda$$
$$+ \int_{\Gamma_{\mathbb{R}}} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_{\mathbb{R}}} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda. \tag{B.2.11}$$

We rewrite the last term as

$$\int_{\Gamma_{\mathbb{R}}} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \int_{\Gamma_{\mathbb{R}}} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \left[e^{i\lambda^n T} \sum_{j\in J} \zeta_j(\lambda) - \sum_{j\in J^+} \zeta_j(\lambda) \right] d\lambda,$$

hence equation (B.2.11) as

$$\sum_{l=2}^{5} I_l = \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J^+} \zeta_j(\lambda)\,d\lambda + \sum_{\substack{k\in K \\ \cup K^{D}\cup K^{E}}} \int_{\Gamma_k} \frac{\widehat{P}(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \sum_{j\in J} \zeta_j(\lambda)\,d\lambda$$
$$+ \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0} + \int_{\Gamma_{\mathbb{R}}} \right\} \frac{P(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} \left[e^{i\lambda^n T} \sum_{j\in J} \zeta_j(\lambda) - \sum_{j\in J^+} \zeta_j(\lambda) \right] d\lambda. \tag{B.2.12}$$

2. n odd, a = i. Then

$$\begin{aligned}
&\int_{E_0^+} \frac{P(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d + \int_{D_0^+} \frac{P(\ )e^{i\ nT}}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d \\
&\quad + \int_{E_0} \frac{\hat{P}(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d + \int_{D_0} \frac{\hat{P}(\ )e^{i\ nT}}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d \\
&= \int_{0} \frac{\hat{P}(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d + \int_{0}^{+} \frac{P(\ )}{\mathrm{PDE}(\ )} \left[\sum_{j\in J^+} {}_j(\ ) \quad e^{i} \sum_{j\in J} {}_j(\ ) \right] d \, . \qquad \text{(B.2.13)}
\end{aligned}$$

Using a similar argument to that above we write

$$\begin{aligned}
\sum_{l=2}^{5} I_l &= \sum_{\substack{k\in K^+\\ \cup K^{D^+}\cup K^{E^+}}} \int_{k} \frac{P(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d + \sum_{\substack{k\in K\\ \cup K^{D}\cup K^{E}\\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{k} \frac{\hat{P}(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d \\
&\quad + \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{E_k} + \int_{0}^{+} \int_{\mathbb{R}} \right\} \frac{P(\ )}{\mathrm{PDE}(\ )} \left[\sum_{j\in J^+} {}_j(\ ) \quad e^{i} \sum_{j\in J} {}_j(\ ) \right] d \, . \qquad \text{(B.2.14)}
\end{aligned}$$

3. n even, a = i. Then

$$\begin{aligned}
&\int_{E_0^+} \frac{P(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d + \int_{D_0^+} \frac{P(\ )e^{a\ nT}}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d \\
&\quad + \int_{E_0} \frac{\hat{P}(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d + \int_{D_0} \frac{\hat{P}(\ )e^{a\ nT}}{\mathrm{PDE}(\ )} \sum_{j\in J} {}_j(\ )\,d \\
&= \frac{1}{2} \int_{0} \frac{P(\ )}{\mathrm{PDE}(\ )} \sum_{j\in J^+} {}_j(\ )\,d + \frac{1}{2} \int \quad \hat{P}(\ )
\end{aligned}$$

We may now write

$$\sum_{l=2}^{5} I_l = \left\{ \sum_{k\in K^+ \cup K^{D+} \cup K^{E+} \cup K^D_E} \int_{k}^{+\frac{1}{2}} \int_{0} \right\} \frac{P(\ )}{P_{DE}(\ )} \sum_{j\in J^+} {}_j(\ )\, d$$

$$+ \left\{ \sum_{k\in K \cup K^D \cup K^E \cup K^E_D} \int_{k}^{+\frac{1}{2}} \int_{0} \right\} \frac{\hat{P}(\ )}{P_{DE}(\ )} \sum_{j\in J} {}_j(\ )\, d$$

$$+ \left\{ \sum_{k\in K^D_E} \int_{k}^{E} {}^{+\frac{1}{2}} \int_{0} \int_{0}^{1} \sum_{k\in K^E_D} \int_{k}^{E} {}^{\frac{1}{2}} \int_{0}^{+} \int_{0}^{1} \right\}$$

$$\frac{P(\ )}{P_{DE}(\ )} \left[e^{i} \sum_{j\in J} {}_j(\ ) \sum_{j\in J^+} {}_j(\ ) \right] d\ . \qquad \text{(B.2.16)}$$

4. n even, $a = e^{i}$, $\in (\ \frac{}{2};$

equation (2.2.18)) and rearrange the result. To this end we define the matrix-valued function $X^{lj} : \mathbb{C} \to \mathbb{C}^{(n-1)\times(n-1)}$ entrywise by

$$(X^{lj})_{sr}(\lambda) = A_{(s+l)(r+j)}(\lambda);$$

where $(s+l)$ is taken to be the least positive integer equal to $s+l$ modulo n and $(r+j)$ is taken to be the least positive integer equal to $r+j$ modulo n. So the matrix X^{lj} is the $(n-1)\times(n-1)$ submatrix of

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

whose $(1,1)$ entry is the $(l+1, r+j)$ entry. Then

$$\widehat{\Delta}_j(\lambda) = \sum_{l=1}^{n} u(\lambda; l) \det X^{lj}(\lambda). \tag{B.2.19}$$

Using equations (2.3.1) of Definition 2.19,

$$\begin{aligned}
&\sum_{j\in J^+} \Delta_j(\lambda) - e^{-i\alpha}\sum_{j\in J^-}\Delta_j(\lambda) \\
&= \left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\lambda)\widehat{\Delta}_j(\lambda) + \sum_{j:J^0_j \text{ odd}} c_{(J^0_j-1)/2}(\lambda)\widehat{\Delta}_{j+n}(\lambda)\right) \\
&\quad - e^{-i\alpha}\left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\lambda)\widehat{\Delta}_j(\lambda) + \sum_{j:J^0_j \text{ even}} c_{J^0_j/2}(\lambda)\widehat{\Delta}_{j+n}(\lambda)\right) \\
&= \left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\lambda)\widehat{\Delta}_j(\lambda) + \sum_{j:J^0_j \text{ odd}} c_{(J^0_j-1)/2}(\lambda)\left[\bar{\zeta}_j(\lambda) - \sum_{k=1}^{n}\widehat{A}_{jk}\widehat{\Delta}_k(\lambda)\right]\right) \\
&\quad - e^{-i\alpha}\left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\lambda)\widehat{\Delta}_j(\lambda) + \sum_{j:J^0_j \text{ even}} c_{J^0_j/2}(\lambda)\left[\bar{\zeta}_j(\lambda) - \sum_{k=1}^{n}\widehat{A}_{jk}\widehat{\Delta}_k(\lambda)\right]\right);
\end{aligned} \tag{B.2.20}$$

and, by equation (B.2.19), the right hand side of equation (B.2.20) equals

$$\sum_{l=1}^{n} u(\lambda; l)\left[\left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\lambda)\det X^{lj} - \sum_{j:J^0_j \text{ odd}} c_{(J^0_j-1)/2}(\lambda)\sum_{k=1}^{n}\widehat{A}_{jk}\det X^{lj}\right)\right.$$

$$- e^{-i\alpha}\left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\lambda)\det X^{lj} - \sum_{j:J^0_j \text{ odd}} \ldots\right.$$

j:J'_j 79 d5t X 30dd)]TJ/F43 10.9091 Tf -0.667 -2.879 Td [( )]TJ/F44 7.9701 T

c_{J^0}

$j:J^{00}_j$ (J

equals

$$
\begin{aligned}
= \sum_{l=1} u(\lambda;l) &\left[\left(\sum_{j:J_j\text{ odd}} c_{(J_j-1)/2}(\lambda)\det X^{lj} - \sum_{j:J'_j\text{ odd}} c_{(J'_j-1)/2}(\lambda) \sum_{\substack{k:J_k\text{ odd}\\ k:J_k\text{ even}}} \hat{A}_{jk}\det X^{lj}\right)\right.\\
&\left. - e^{-i\lambda}\left(\sum_{j:J_j\text{ even}} c_{J_j/2}(\lambda)\det X^{lj} - \sum_{j:J'_j\text{ even}} c_{J'_j/2}(\lambda) \sum_{\substack{k:J_k\text{ odd}\\ k:J_k\text{ even}}} \hat{A}_{jk}\det X^{lj}\right)\right] + H(\lambda).
\end{aligned}
$$

In the line above we have split the index set $\{1,2,\dots,n\}$ for the sums over k into two sets. We now seperate the sums for these sets, in the process interchanging the dummy variables j and k in the final two sums on each line, rewriting the square bracket as

$$
\begin{aligned}
&\left[\sum_{j:J_j\text{ odd}}\left(c_{(J_j-1)/2}(\lambda) - \sum_{k:J'_k\text{ odd}} c_{(J'_k-1)/2}(\lambda)\hat{A}_{kj} + e^{-i\lambda}\sum_{k:J'_k\text{ even}} c_{J'_k/2}(\lambda)\hat{A}_{kj}\right)\det X^{lj}\right.\\
&\left. + \sum_{j:J_j\text{ even}}\left(c_{J_j/2}(\lambda)e^{-i\lambda} - \sum_{k:J'_k\text{ odd}} c_{(J'_k-1)/2}(\lambda)\hat{A}_{kj} + e^{-i\lambda}\sum_{k:J'_k\text{ even}} c_{J'_k/2}(\lambda)\hat{A}_{kj}\right)\det X^{lj}\right].
\end{aligned}
\tag{B.2.21}
$$

We now change the dummy variable k in the sums of expression (B.2.21). For k such that J'_k is odd we may define $r \in \hat{J}^+$ such that $r = (J'_k-1)/2$ and $k = \hat{J}^+_r$. Then, by the definition of the reduced boundary coefficient matrix (2.2.20),

$$
\hat{A}_{kj} = \begin{cases} \hat{J}^+_{r\,(J_j-1)/2} & J_j \text{ odd,}\\ \hat{J}^+_{r\,J_j/2} & J_j \text{ even.}\end{cases}
$$

Similarly, for k such that J'_k is even we may define $r \in \hat{J}^-$ such that $r = J'_k/2$ and $k = \hat{J}^-_r$. Then

$$
\hat{A}_{kj} = \begin{cases} \hat{J}^-_{r\,(J_j-1)/2} & J_j \text{ odd,}\\ \hat{J}^-_{r\,J_j/2} & J_j \text{ even.}\end{cases}
$$

This and the definition of the reduced global relation matrix (2.2.19) allow us to write

$$
\begin{aligned}
&c_{(J_j-1)/2}(\lambda) - \sum_{k:J'_k\text{ odd}} c_{(J'_k-1)/2}(\lambda)\hat{A}_{kj} + e^{-i\lambda}\sum_{k:J'_k\text{ even}} c_{J'_k/2}(\lambda)\hat{A}_{kj}\\
&\quad = c_{(J_j-1)/2}(\lambda)\left(1 - \sum_{r\in\hat{J}^+}(i\lambda)^{(J_j-1)/2-r}\,\hat{J}^+_{r\,(J_j-1)/2} + e^{-i\lambda}\sum_{r\in\hat{J}^-}(i\lambda)^{(J_j-1)/2-r}\,\hat{J}^-_{r\,(J_j-1)/2}\right)\\
&\quad = A_{1j}(\lambda)
\end{aligned}
\tag{B.2.22}
$$

We define the functions l, r and c that map the indices of each boundary datum to the position of that boundary datum within V as follows:

$$l : 5 \mapsto 1; \qquad l : 4 \mapsto 3;$$
$$r : 5 \mapsto 2; \qquad r : 4 \mapsto 4;$$
$$c : 3 \mapsto 5.$$

These functions are defined such that

$$\mathrm{Domain}(l) = \{j+1 : \tilde{f}_j(\lambda) \text{ is an entry of } V\}$$

$$\mathrm{Domain}(r) = \{j+1 : \tilde{g}_j(\lambda) \text{ is an entry of } V \text{ which corresponds to a BC which does NOT couple the ends of the interval}\}$$

$$\mathrm{Domain}(c) = \{j+1 : \tilde{g}_j(\lambda) \text{ is an entry of } V \text{ which corresponds to a BC which couples the ends of the interval}\}.$$

They are also injective, and their ranges are all $\{1;2;3;4;5\}$ but their codomains are disjoint. This is guaranteed by defining the functions so that for j in the relevant domains

$$A_{k\,l(j)} = \omega^{k(n-j)} c_{j-1}(\lambda)$$
$$A_{k\,r(j)} = -\omega^{k(n-j)} e^{-i\omega^k\lambda} c_{j-1}(\lambda)$$
$$A_{k\,c(j)} = -\omega^{k(n-j)}\left(e^{-i\omega^k\lambda} + \beta_{p\,j-1}\right) c_{j-1}(\lambda)$$

where p is the index of the unique boundary condition that couples f_{j-1} and g_{j-1}. We may now write

$$\begin{aligned}
\Delta_{\mathrm{PDE}}(\lambda) = {} & c_4^2(\lambda)c_3^2(\lambda)c_2(\lambda)\sum_{\sigma\in S_5}\mathrm{sgn}(\sigma)\,\omega^{\sum_{j\in\{4,5\}} l(j)(n-j)} \\
& (-1)^2\omega^{\sum_{j\in\{4,5\}} r(j)(n-j)} e^{-i\sum_{j\in\{4,5\}}\omega^{r(j)}\lambda} \\
& (-1)\,\omega^{\sum_{j\in\{3\}} c(j)(n-j)}\prod_{j\in\{3\}}\left(e^{-i\omega^{c(j)}\lambda} + \beta_{12}\right) c_4^2(\lambda)c_3^2(\lambda)c_2(\lambda) \\
= {} & \lambda^4\beta_{12}\sum_{\sigma\in S_5}\mathrm{sgn}(\sigma)\,\omega^{(4[\sigma(3)+\sigma(4)]+3\sigma(5))} e^{-i\left(\omega^{\sigma(2)}+\omega^{\sigma(4)}\right)\lambda} \\
& + \lambda^4\sum_{\sigma\in S_5}\mathrm{sgn}(\sigma)\,\omega^{(4[\sigma(3)+\sigma(4)]+3\sigma(5))} e^{-i\left(\omega^{\sigma(2)}+\omega^{\sigma(4)}+\omega^{\sigma(5)}\right)\lambda}. \qquad \text{(B.3.1)}
\end{aligned}$$

For any given $\sigma \in S_5$ we can now calculate coefficients of $e^{-i\sum_{j=1}^{2}\omega^{\sigma(j)}\lambda}$ and $e^{-i\sum_{j=1}^{3}\omega^{\sigma(j)}\lambda}$ in $\Delta_{\mathrm{PDE}}(\lambda)$; the first coming from the first sum in the right hand side of equation (B.3.1) and the second from the second sum.

We look first at $e^{-i\sum_{j=1}^{2}\omega^{\sigma(j)}\lambda}$. If we choose some $\sigma \in S_5$ such that

$$\sum_{j\in\{2,4\}}\omega^{\sigma(j)} = \sum_{j=1}^{2}\omega^{\sigma(j)}$$

$$\Leftrightarrow \quad \{\sigma(j) : j \in \{2,4\}\} = \{\sigma(j) : j \in \{1,2\}\}$$

then the coefficient of $e^{i\sum_{j=1}^{2}\omega^{(j)}}$ in $_{PDE}$ is given by

$$4^{12}\sum_{\substack{\in S_5:\\ \{(2),(4)\}\\ =\{(2),(4)\}}} \mathrm{sgn}(\ )\,\omega^{(4[(3)+(4)]+3(5))}. \tag{B.3.2}$$

Similarly, we may choose some $\in S_5$ such that

$$\sum_{j\in\{4,5\}}\omega^{r(j)}+\omega^{(5)}=\sum_{j=1}^{3}\omega^{(j)}$$

$$\Leftrightarrow\quad \{(j): j\in\{2,4,5\}\}=\{(j): j\in\{1,2,3\}\}$$

then the coefficient of $e^{i\sum_{j=1}^{3}\omega^{(j)}}$ in $_{PDE}$ is given by sgn( )(4[(3)+16 Td [(f)]TJ/F44 7.9701 Tf 4.234 0 Td [( )]TJ/F41 7.9

$$4\sum_{\in S_5:\ \in{}^{3};(\ 5\}\}=;\{(2)\}}$$

Definition B.7. *Let us define the index sets $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ and, as the boundary conditions are non-Robin, simplify the notation for the coupling coefficients so that only a single index is used:*

$\mathcal{L} = \{j+1 : \;_{rj} = 0 \;\forall r\}$ *so that* $L = |\mathcal{L}|$

$\mathcal{R} = \{j+1 : \;_{rj} = 0 \;\forall r\}$ *so that* $R = |\mathcal{R}|$

$\mathcal{C} = \{j+1 : \exists r : \;_{rj}, \;_{rj} \neq 0\}$ *so that* $C = |\mathcal{C}|$

$\tilde{\;}_j = \;_{pj-1}$, *where p is the index of the (unique, as the boundary conditions are non-Robin) boundary condition that couples f_{j-1} and g_{j-1}.*

Lemma B.8. *Let $c' : \{1, 2, \dots, C\} \to \mathcal{C}$*

Proof. Note that for any valid choice of l, r and c their codomains have disjoint union $\{1, 2, \ldots, n\}$. The uncertainty in the sign in equations (B.3.6), (B.3.7) and (B.3.8) comes from different choices of the functions l, r and c. For concreteness, we require that l, r and c

$(j) = \; cc' \; '(s)$. This establishes that there is no $' \in S_C$ such that $(\; ; \; ') \in S_{k \; '}$. Hence every $\in S_n$ for which there exists some $' \in S_C$ such that $(\; ; \; ') \in S_{k \; '}$ has the property

$$\forall j \in \{r(p) : p \in R\} \cup \{cc' \; '(p) : p \in \{1, 2, \ldots, k\}\}$$
$$\exists q \in \{r(p) : p \in R\} \cup \{cc' \; '(p) : p \in \{1, 2, \ldots, k\}\} \text{ such that } \; (j) = \; (q). \tag{B.3.10}$$

We define

$$X = \{j | j \in R \text{ such that } \forall p \in R \quad r(j) \neq \; r(p)\} \text{ and}$$
$$Y = \{j | j \in \{1, 2, \ldots, k\} \text{ such that } \forall p \in R \quad cc' \; '(j) \neq \; r(p)\}.$$

It is immediate that, for any $\in S_n$

This yields a complete characterisation of $S_{k\tau'}$ for $k > 1$.

If $k = 0$ the definition (B.3.4) of $S_{k\tau'}$ simplifies to

$$\begin{aligned} S_{0\tau'} &= \left\{ (\sigma, \tau') \in S_n \times S_C : \sum_{j \in R} \omega^{r(j)} = \sum_{j \in R} \omega^{r(\sigma(j))} \right\} \\ &= \{ \sigma \in S_n : \forall j \in R\ \exists p \in R : r(j) = r(\sigma(p)) \} \times S_C. \end{aligned} \tag{B.3.13}$$

In this paragraph we argue that each $S_{k\tau'}$ gives rise to a single unique exponential. The relation $\sim_k$ defined by

$$(\sigma, \tau') \sim_k (\sigma, \tau'') \iff (\sigma, \tau'') \in S_{k\tau'}$$

is an equivalence. From the definition of $S_{k\tau'}$ it is clear that the $\sim_k$-classes are of equal size. Let $(\sigma, \tau'') \in S_{k\tau'}$. Then, by the definition of $S_{k\tau'}$,

$$e^{i \sum_{j \in R} \omega^{r(j)}} e^{i \sum_{j=1}^{k} \omega^{cc' \tau'(j)}} = e^{i \sum_{j \in R} \omega^{r(j)}} e^{i \sum_{j=1}^{k} \omega^{cc' \tau''(j)}}.$$

Hence the equivalence classes of $\sim_k$ each identify a unique sum of powers of ω in the exponent and for each possible sum of $R + k$ powers of ω there exists an equivalence class of $\sim_k$. Hence for a given exponential $e^{i \sum_{r=1}^{R+k} \omega^{\alpha(r)}}$ we may find its coefficient in equation (B.3.9) by choosing some pair $(\sigma, \tau') \in S_n \times S_C$ such that equation (B.3.5) holds and evaluating expression (B.3.6) if $k > 1$ or expression (B.3.7) if $k = 0$.

If $k > 1$ and $\tilde{e}_j = \epsilon$ for all $j \in C$ then expression (B.3.6) simplifies to

$$P_k = (-1)^{R+C} (-a)^n (i\epsilon)^{\left(\sum_{j \in R} j + \sum_{j \in C} j + \sum_{j \in L} j\right)} \sum_{(\sigma, \tau'') \in S_{k\tau'}} \operatorname{sgn}(\sigma\tau'') \omega^{\sum_{j \in R} |r(j) - j| - \sum_{j \in C} |c(j) - j| - \sum_{j \in L} |l(j) - j| - Ck}.$$

If $\sigma \in S_n$ is such that there exists $\tau' \in S_C$ such that $(\sigma, \tau') \in S_{k\tau'}$ then, as above, for $\tau'' \in S_C$, $(\sigma, \tau'') \in S_{k\tau'}$ if and only if

$$\forall j \in \{1, 2, \ldots, k\}\ \exists p \in \{1, 2, \ldots, k\} \text{ such that } \tau''(j) = \tau'(p).$$

Hence, for a given σ, provided there exists some $\tau' \in S_C$ such that $(\sigma, \tau') \in S_{k\tau'}$ there exist $k!(C - k)!$ such τ'. Hence expression (B.3.6) simplifies to expression (B.3.8).

B.4. Admissible functions

This section gives the proof of Lemma 3.30. For convenience we reproduce Definition 1.3 of [27 *be a set of* $2n$ C^∞ *functions on* $[0,T]$ *such that* $\partial_x^j q_0(0) = f_j(0)$ *and* $\partial_x^j q_0(1) = g_j(0)$ *for each* $j \in \{0,1,\dots,n-1\}$. *Let*

$$\tilde{F}(\lambda) = \sum_{j=1}^{n-1} c_j(\lambda)\tilde{f}_j(\lambda), \qquad \text{(B.4.1)}$$

$$\tilde{G}(\lambda) = \sum_{j=1}^{n-1} c_j(\lambda)\tilde{g}_j(\lambda), \qquad \text{(B.4.2)}$$

where $\tilde{f}_j, \tilde{g}_j$ *are defined in Lemma 3.30.*

The set of functions $\{f_j, g_j : j \in \{0,1,\dots,n-1\}\}$ *is called* admissible with respect to q_0 *if and only if* $q_T \in C^\infty[0,1]$ *and the functions* $\tilde{F}, \tilde{G}$ *satisfy the following relation:*

$$\tilde{F}(\lambda) - e^{-i\lambda}\tilde{G}(\lambda) = -\hat{q}_0(\lambda) + e^{a\lambda^n T}\hat{q}_T(\lambda). \qquad \text{(B.4.3)}$$

Proof of Lemma 3.30. By the definition of $\tilde{f}_j, \tilde{g}_j$ in the statement of Lemma 3.30 and the definition of the index sets $J^\pm$ in Definition 2.19 we may write equations (B.4.1) and (B.4.2) as

$$\tilde{F}(\lambda) = \sum_{j\in J^+} \frac{\zeta_j(\lambda) - e^{a\lambda^n T}\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}, \qquad \text{(B.4.4)}$$

$$\tilde{G}(\lambda) = \sum_{j\in J^-} \frac{\zeta_j(\lambda) - e^{a\lambda^n T}\eta_j(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}. \qquad \text{(B.4.5)}$$

Now by Cramer's rule and the calculations in the proof of Lemma 2.17 equation (B.4.3) is satisfied. The following remains to be shown:

(1) $q_T \in C^\infty[0,1]$.

(2) $f_j, g_j \in C^\infty[0,T]$ for each $j \in \{0,1,\dots,n\}$.

(3) $\partial_x^j q_0(0) = f_j(0)$ and $\partial_x^j q_0(1) = g_j(0)$ for each $j \in \{0,1,\dots,n-1\}$.

(1) By Assumption 3.2, ζ_j is entire. Hence $\hat{q}_T$ is entire so, by the standard results on the inverse Fourier transform, $q_T : [0,1] \to \mathbb{C}$, defined by

$$q_T(x) = \frac{1}{2\pi}$$

Hence, by equations (B.4.4) and (B.4.5), $\widetilde{F}(\ ); \widetilde{G}(\ ) \to 0$ as $\ \to \infty$ within $\widetilde{D}$. By Lemma B.10 we have the direct definitions (B.4.6) and (B.4.7) of f_j and g_j in terms of $\widetilde{F}$ and $\widetilde{G}$ and, because $\widetilde{F}(\ ); \widetilde{G}(\ ) \to 0$ as $\ \to \infty$ within $\widetilde{D}$, these definitions guarantee that f_j and g_j are C^∞ smooth.

(3) Equation (2.1.4) guarantees equations (B.4.4) and (B.4.6), imply that the compatibility condition $\partial_x^j q_0(0) = f_j(0)$ is satisfied by construction. Equations (B.4.5) and (B.4.7), imply that the compatibility condition $\partial_x^j q_0(1) = g_j(0)$ is satisfied.

Lemma B.10. *Let $\widetilde{F}$ and $\widetilde{G}$ be defined* $f;$

Combining equations (B.4.8), (B.4.11) and (B.4.12) and equating coefficients of ^{j} we obtain equations (B.4.6).

We have shown that the transforms (B.4.6) are the inverse of the transforms (3.2.30), hence the pair $((\tilde{f}_j)_{j=0}^{n-1}; (f_j)_{j=0}^{n-1})$ satisfies (B.4.6) if and only if it satisfies (3.2.30).

List of theorems, de nitions etc.

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