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D PARTM NT OF MATH MATICS

VARIATIONAL PRINCIPLES AND THE FINITE ELEMENT METHOD FOR CHANNEL FLOWS

by

S. L. WAKELIN

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Ch pter 1

Introduction

The problem of fluid flow over an uneven topography and through constricting channels has been of interest to hydraulic engineers and meteorologists for many years. Variational methods have been widely used in other areas for even longer but have only recently begun to play a significant part in the problems of fluid mechanics. The finite element method is a relatively recent technique, which has advanced the construction of approximate solutions, particularly in relation to elliptic problems. This thesis brings together these three subjects.

More specifically the aim of this thesis is to generate numerical approximations to the solutions of the equations governing the irrotational motion of an homogeneous, incompressible, inviscid fluid over a fixed bed profile. The method implemented here depends on the derivation of variational principles which are satisfied for the solutions of these equations of motion. Approximate solutions to the equations are derived as those functions in a finite dimensional space for which the functionals of the variational principles are stationary with respect to variations in that space.

Luke (1967) showed that a variational principle in which the integrand (the Lagrangian density) is taken to be the fluid pressure, as given by Bernoulli's energy integral, has as its natural conditions the equations governing a free surface flow. The natural conditions of a variational principle are those which make the corresponding functional stationary. The natural conditions of Luke's principle are Laplace's equation, holding in the fluid domain, the no flow condition across the bed and the dynamic and kinematic free surface conditions. Luke assumes that all of the variations vanish on the other space boundaries and at the ends of the time interval.

Hamilton's variational principle in particle mechanics (see, for example, Goldstein (1980)) has, as the Lagrangian, the difference between the kinetic and potential energies of a system. The natural conditions of the variational principle are Lagrange's equations of motion. Salmon (1988) considers applications of classical Hamiltonian theory to fluid mechanics. Many of these applications, such as, for example, in Serrin (1959) and in Seliger and Whitham (1968), have been within the area of gas dynamics. However, Miles and Salmon (1985) derived equations describing the motion of weakly dispersive non-linear gravity waves using Hamilton's principle. In this thesis Hamilton's principle is adapted to give a principle whose natural conditions are the equations governing a free surface flow, and this principle can be rearranged to give Luke's principle, in the case where the variations vanish on all time and space boundaries except the free surface and the domain bed.

There is a point of contact between free surface flows and compressible gas flows if the shallow water approximation to free surface flows is invoked (Stoker

(1957)). Shallow water theory is an approximation to three-dimensional free surface flows in circumstances where the fluid depth is small compared with some characteristic length scale of the motion, such as the radius of curvature of the free surface. In this thesis shallow water theory at its lowest order is considered; this is the basic theory used in hydraulics to model flows in open channels and also gives good approximations to the motion of tides in the oceans and the breaking of waves on shallow beaches. The flow domain over which approximations to shallow water flows are considered here, is a channel of slowly varying breadth, so that, to a first approximation, the flow can be thought of as being quasi one-dimensional.

A substantial part of the thesis — Chapter 3 — is devoted to the derivation of the variational principles corresponding to three-dimensional free surface flows and to shallow water flows. Hamilton's principle and a modified version of Luke's principle are used as the starting points of the investigation. By approximating the variables of three-dimensional flows by their shallow water counterparts it is possible to derive variational principles which are satisfied for solutions of the shallow water equations of motion. It is shown that Hamilton's principle and the modified version of Luke's principle are essentially the same, as are the two variational principles for shallow water which are derived from them. Different representations of the variational principle for shallow water are available, based on the notion of a closed sequence of Legendre transforms introduced by Sewell (1987). The variational principles for shallow water flows are enhanced by the addition of boundary terms so that variations can be allowed which do not necessarily vanish on the boundaries. This is an important step since, in the practical implementation of a variational principle, if the variations are to vanish on the boundaries then it implies that the solution must be known there.

There is, however, an undesirable feature of these principles, that conditions on some of the flow variables must be given at both ends of the time interval. This problem does not arise in steady shallow water flows, which are considered in some detail. The variational principles for these flows are deduced from the principles for time-dependent flows.

Further variational principles are created by making the assumption that the shallow water flow is quasi one-dimensional, yielding variational principles for time-dependent and time-independent quasi one-dimensional flows.

A number of simpler variational principles can be derived by constraining the variations to satisfy one or more of the natural conditions. A selection of these constrained principles is presented, some of which fit with the notion of reciprocal variational principles.

The final section of Chapter 3 deals with the derivation of variational principles for steady discontinuous flows, that is, for flows which contain hydraulic jumps. The differential equations of shallow water flow are valid in regions of the domain excluding the discontinuity while at the discontinuity the equations of motion are replaced by jump conditions, which relate the values of the flow variables on either side of the discontinuity. One of the jump conditions is used in the formulation of the variational principles and the others are derived as natural conditions by making an assumption about the variations in the flow variables at the discontinuity.

The remainder of the thesis is concerned with using the variational principles to generate approximate solutions for flows in channels. The Ritz method (see

conditions are only generated as natural conditions by imposing specific conditions on the variations. It is not clear how these conditions could be implemented in practice and the algorithm used here is based on generating separate approximations to the continuous parts of the solution and coupling the approximations at the discontinuity by using the jump conditions, in the process of which an approximation to the position of the discontinuity is also found.

Chapter 5 deals with finding approximations to steady two-dimensional continuous shallow water flows, by extending the algorithms of Chapter 4.

In Chapter 6 two further applications of the variational principles are investigated. In an attempt to study the accuracy of the shallow water approximation to free surface flows a version of Luke's principle for steady state flows is used to generate approximations in this case. Finally an algorithm to generate approximations to time-dependent quasi one-dimensional shallow water flows is considered.

Ch pter 2

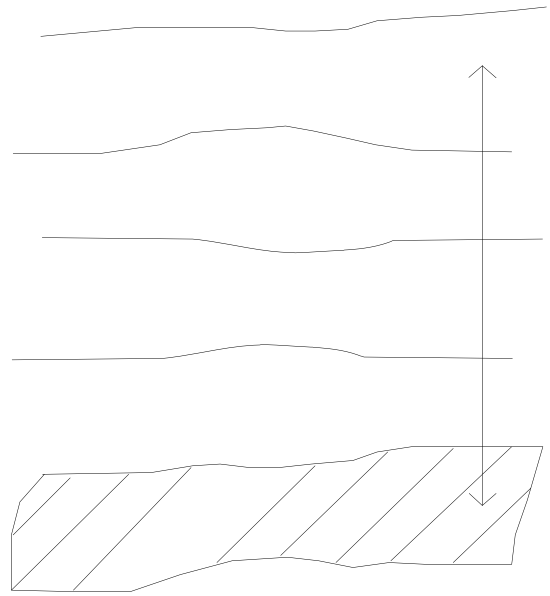
B ckground Fluid Dyn mics

In this chapter the equations governing the three-dimensional motion of an incompressible, homogeneous fluid under a free surface are given and adapted to the various problems which will be considered later. An approximation to such a three-dimensional motion can be devised by assuming that the fluid depth is small compared with a typical horizontal length scale of the motion. This so-called shallow water approximation generates a simplified set of equations by removing the vertical motion, at lowest order. Shallow water theory is often applied in channels and this case only will be considered.

2.1 Free Surface Flows

In this section the equations governing the motion of a fluid under a free surface are given. The fluid is assumed to be incompressible and homogeneous and the motion is assumed to be irrotational.

Let x, y, z be cartesian coordinates, with z measured vertically upwards from the equilibrium position of the free surface, and let t be the time. Consider the



where — is the derivative following the motion and is defined by

$$—— \quad —+ \quad \tilde{} \qquad (2\ 4)$$

The flow is irrotational so that

$$= \tilde{} \qquad (2\ 5)$$

for some = ( ), where is called the velocity potential. Therefore the conservation of mass equation for irrotational flow may be written as

$$\tilde{} \quad = 0 \qquad (2\ 6)$$

The integrated version of the conservation of momentum equation for irrotational flow is

$$\tilde{-} + \quad + \frac{1}{2}\tilde{} \quad \tilde{} \quad + \quad = 0 \qquad (2\ 7)$$

$-$

equation of the boundary, at every point of this boundary the equation

$$\text{——} = 0 \qquad (2\ 9)$$

must be satisfied (Lamb (1932)).

The equation of the free surface is given by $\quad = 0$. Thus the kinematic free surface condition is

$$\text{——}(\qquad) = 0$$

which may be rewritten as

$$+ \quad + \qquad = 0 \qquad \text{on} \quad = \qquad (2\ 10)$$

The equation of the fixed bed is given by $\ + \ = 0$. This gives the condition of zero flow through the bed as

$$\text{——}(\ +\ ) = 0$$

or

$$+ \quad + \quad = 0 \qquad \text{on} \quad = \qquad (2\ 11)$$

quation (2.9) also applies at any lateral boundary across which there is no flow.

quations (2.1), (2.3), (2.5), (2.8), (2.10) and (2.11) constitute the set of equations governing three-dimensional flow in an arbitrary domain Ω.

Shallow water theory offers an approximation to free surface flows in circumstances where the water depth is much less than some characteristic length scale an averaging process in which the fluid motion is replaced by a representative motion in the horizontal spatial coordinates. ach particle can be thought of as the aggregate of all the actual fluid particles lying in the same vertical line.

2.2.1 Derivation of the Shallow Water Equations

To lowest order, shallow water theory can be generated by assuming that the fluid pressure is hydrostatic (Stoker (1957)). That is,

$$\tilde{p}(x, y, z, t) = \rho g\,(\eta - z)\,, \tag{2.12}$$

where the constant surface pressure has been set to zero for convenience.

quation (2.12) can be used to determine a vertically averaged replacement for the pressure, $p = p(x, y, t)$, defined by

$$p = \frac{1}{\rho}\int_{-h}^{\eta} \tilde{p}\,dz,$$

from which it follows that

$$p = \frac{1}{2} g d^2, \tag{2.13}$$

where $d(x, y, t)$ is the fluid depth at location (x, y) and at time t, that is $d = h + \eta$.

quation (2.12) implies that $\tilde{\boldsymbol{\nabla}}\tilde{p}$ is independent of z and so, from $(2.3)_{1,2}$, the acceleration of the water particles in the x and y directions is also independent of z. Thus, if the horizontal components of velocity, u and v, are independent of z at any time, they will remain independent of z throughout the motion. Substituting (2.12) into $(2.3)_3$ gives the result that, in lowest order shallow water theory, the vertical acceleration of the fluid particles is zero, that is, negligible compared with g. It is also negligible compared with u and v. These results can be summarised

where E is an energy per unit mass, referred to as an energy for short, defined by

$$E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v}. \tag{2.19}$$

quation (2.18) is the equation of conservation of momentum in shallow water. The integrated version of (2.18) is more common in the variational principles which follow later and, for irrotational flow, this is given by

$$\phi_t + E = gh, \tag{2.20}$$

where an arbitrary function of t has been absorbed into ϕ_t. quation (2.20) is the Bernoulli equation for shallow water.

For the particular case where the equilibrium depth, h, is a constant, that is, the bed is horizontal, the conservation of momentum equation is given by

$$\mathbf{v}_t + \boldsymbol{\nabla} E = 0. \tag{2.21}$$

Integrating (2.21) with respect to x and y and using (2.15) yields

$$\phi_t + E = 0, \tag{2.22}$$

where an arbitrary function of t has again been absorbed into ϕ_t. quation (2.22) is consistent with (2.20) for the case h = constant. In (2.22) the constant term gh has been absorbed into ϕ_t which is equivalent to moving the reference level for potential energy in the coordinate system from $z = 0$ to $z = -h$.

The conservation of mass equation for free surface flow (2.1) may be written as

$$u_x + v_y + w_z = 0.$$

Integrating through the fluid depth at a point ( ) gives

$$(\quad + \quad + \quad) \quad = 0 \tag{2 23}$$

quations (2 14) imply that and are independent of so (2.23) may be rewritten as

$$(\quad + \quad) \quad + [\quad] \quad = 0$$

Then, substituting for and using the kinematic boundary conditions of zero flow through the free surface and the bed, (2.10) and (2.11), (2.23) becomes

$$+ \quad + \quad + \quad = 0$$

Using 0, the conservation of mass equation for shallow water can be written as

$$+ \quad = 0 \tag{2 24}$$

where

$$= \tag{2 25}$$

$$d_t + \nabla.(d\mathbf{v}) = 0, \tag{2.29}$$

respectively, where the pressure, p, is given by (2.13). quations (2.27)—(2.29) may be regarded as the equations governing a two-dimensional gas flow in which d plays the part of density and the term $g\nabla h$ in (2.28) is thought of as a body force or as a heat source. For the special case where the equilibrium depth, h, is a constant, the forcing term $g\nabla h$ in (2.28) is zero and (2.28) is the usual momentum conservation equation for gas flow. Also, in gas dynamics terminology, equation (2.13), which defines p as a function of the 'density' d, is an 'adiabatic' relation (Courant and Friedrichs (1948)).

In Chapter 3 variational principles for shallow water flows are derived and subsequently used, in Chapter 4, to generate numerical approximations to channel flows. Variational principles for compressible gas flows have been developed previously by, for example, Bateman (1929), Sewell (1963) and Wixcey (1990). The analogy of shallow water theory with gas dynamics provides a connection between those principles and variational principles for shallow water.

2.3 The Quasi One-dimensional Shallow Water Approximation

For certain flow domains the motion can be approximated by making the assumption that it is dependent on one space dimension and time only.

Consider a channel which extends over the interval $[x_e, x_o]$ of the x-axis. Let $B(x)$ be the breadth of the channel, defined at each point x in $[x_e, x_o]$. Assume that the channel is of rectangular cross-section and that it is symmetric about

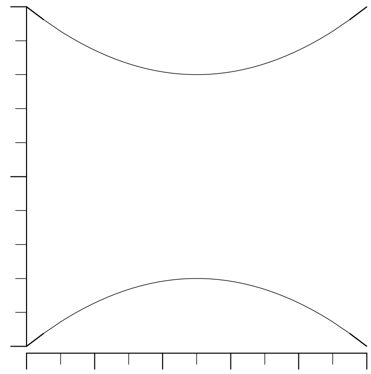


Figure 2.2: for $x_e = 0, x_o = 10$ and $B(x) = 6 + 4\left(\frac{x}{5} - 1\right)^2$.

the x-axis so that the domain, , of the problem is given by

$$= \left\{ (x, y) : x \in [x_e, x_o]\, ; y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2} \right] \right\}. \qquad (2.30)$$

Then, provided that the breadth is a slowly varying function of x, the flow is quasi one-dimensional in the x-direction, to a first approximation.

Figure 2.2 shows for the example $x_e = 0, x_o = 10$ and $B(x) = 6 + 4\left(\frac{x}{5} - 1\right)^2$.

The equations of quasi one-dimensional shallow water motion can be derived from the full shallow water equations of Section 2.2.1 by assuming that the flow variables are functions of x and t only. Let the flow variables be of the form $d = d(x,t)$, $\phi = \phi(x,t)$, $E = E(x,t)$, $Q = Q(x,t)$ and $v = v(x,t)$, where Q is the one-dimensional mass flow and v is the velocity in the x-direction, the other variables being depth, velocity potential and energy, as before. Although the same symbols are used for depth, velocity potential and energy in one and two dimensions, the context will always make clear whether the flow being studied is quasi one-dimensional or two-dimensional. The operator $\boldsymbol{\nabla}$ in (2.15) and (2.18) is replaced by $\frac{\partial}{\partial x}$. The term $\boldsymbol{\nabla}.\mathbf{Q}$ in (2.24) is replaced by $\frac{1}{B}\frac{\partial}{\partial x}(QB)$.

Thus, the quasi one-dimensional shallow water equations of motion are given by

$$v = \phi_x \qquad \text{irrotationality condition,} \tag{2.31}$$

$$v_t + E_x = gh_x \qquad \text{conservation of momentum,} \tag{2.32}$$

$$d_t + \frac{1}{B}(BQ)_x = 0 \qquad \text{conservation of mass,} \tag{2.33}$$

where the mass flow, Q, is given by

$$Q = dv, \tag{2.34}$$

and the energy, E, is given by

$$E = gd + \frac{1}{2}v^2. \tag{2.35}$$

The integrated version of the conservation of momentum equation (2.32) is

$$\phi_t + E = gh, \tag{2.36}$$

where ϕ is related to v by (2.31).

Let the equilibrium depth, h, be constant. Then the conservation of momentum equation is given by

$$v_t + E_x = 0. \tag{2.37}$$

Integrating (2.37) with respect to x gives

$$\phi_t + E = 0, \tag{2.38}$$

where an arbitrary function of t has been absorbed into ϕ_t. quation (2.38) is consistent with (2.36) when the reference level for potential energy in the vertical is moved from $z = 0$ to $z = -h$ by redefining the velocity potential to be

$$\phi := \phi - ght. \tag{2.39}$$

The boundary conditions for quasi one-dimensional flow are given, for example, by

$$Q = C_e \qquad \text{at } x = x_e, \tag{2.40}$$

$$Q = C_o \qquad \text{at } x = x_o, \tag{2.41}$$

for known functions $C_e(t)$ and $C_o(t)$. quations (2.40) and (2.41) can be derived from the two-dimensional shallow water boundary condition (2.26) using the fact that there is zero flow through the channel sides and that Q varies only with x and t and is constant across the channel breadth by assumption.

quations (2.31)—(2.33) govern the motion of quasi one-dimensional shallow water. Notice that the irrotationality condition (2.31) has become essentially redundant.

2.4 Equations for Steady State Shallow Water Flows

The equations of motion for steady state shallow water can be derived from the time-dependent equations of Sections 2.2 and 2.3. In this section the steady state equations are derived by assuming that all of the flow variables are independent of time. The velocity potential ϕ, however, is not a physical flow variable and cannot be assumed time-independent, although its form can be deduced using the equations of motion.

Differentiating with respect to t the irrotationality condition (2.31) and the integrated conservation of momentum equation (2.36) gives

$$v \quad = \quad \phi \tag{2.57}$$

$$\text{and} \qquad \phi \quad + E \quad = \quad 0 \tag{2.58}$$

respectively. Thus, using 0 and 0, the velocity potential must satisfy

$$= 0 \;\; \text{and} \qquad = 0$$

Therefore must be of the form

$$(\quad) = \; (\;) \; + \; \tilde{} (\;) \qquad (2\ 59)$$

where $\tilde{}$ is an arbitrary function and $= 0$.

Thus, for steady quasi one-dimensional flow,

$$= \; \tilde{} \qquad (2\ 60)$$

The value of the constant function in (2.59) can be deduced using the integrated conservation of momentum equation (2.36). Substituting for in (2.36) using (2.59) gives

$$= \qquad +$$

where $+$ =constant from (2.55). Thus is given by

$$(\quad) = (\quad (\;) + \quad (\;)) \;\; + \; \tilde{} (\;)$$

where $\tilde{}(\;)$ is identified as the velocity potential for one-dimensional steady flow.

Let the equilibrium depth, , be constant. Then the conservation of momentum equation for steady quasi one-dimensional motion is given by

$$= 0$$

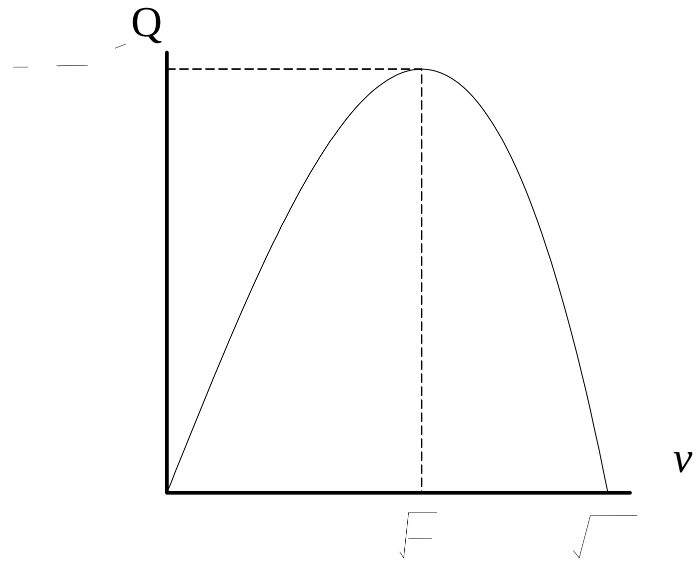


Figure 2.3: Q as a function of v for constant E.

The definitions of mass flow, Q, and energy, E, (2.34) and (2.35) can be used to express each of the variables Q, E, d and v as functions of one or more of the other variables. Furthermore, taking $Q = |\mathbf{Q}|$ and $v = |\mathbf{v}|$, the relationships also hold for two-dimensional flows.

Rearranging the definition of mass flow gives $d = \frac{Q}{v}$. Substituting this into the definition of energy and rearranging gives

$$Q = \frac{v}{g}\left(E - \frac{1}{2}v^2\right). \tag{2.61}$$

The graph of the variation of Q with v for a constant E is given in Figure 2.3. Only the portion of the curve described by (2.61) which lies in the sector $Q \geq 0$ and $v \geq 0$ is considered relevant since the motion is assumed to be always in the positive x direction.

Notice that the velocity v has the range $0 \leq v \leq \sqrt{2E}$. If v exceeded $\sqrt{2E}$ then, from (2.61), the mass flow, Q, would be negative. This would contradict the assumption of positive flow and correspond to a non-physical negative depth as can be seen directly from (2.35). The value

$$v_L = \sqrt{2E}$$

is known as the limit velocity and is the maximum velocity attainable by a flow with energy E.

Notice also that, for each E, the mass flow lies in the range $0 \leq Q \leq \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}}$. The value

$$Q_* = \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}} \tag{2.62}$$

of Q is known as the critical mass flow. The value of the velocity at which the critical mass flow occurs is the critical velocity and is given by

$$c_* = \sqrt{\frac{2E}{3}}. \tag{2.63}$$

A flow is termed subcritical or supercritical depending on whether v is less than or greater than the critical velocity.

A similar graph is constructed by substituting $v = \frac{Q}{d}$ into the definition of E, (2.35), to give an expression relating Q, E and d. This may be rearranged to give

$$Q = \left(2\left(E - gd\right)\right)^{\frac{1}{2}} d. \tag{2.64}$$

The graph of the variation of Q with d for a fixed E is given in Figure 2.4. The portion of the line shown is such that $Q \geq 0$ and $d \geq 0$, the remainder of the line having no physical meaning.

As in Figure 2.3, the mass flow in Figure 2.4 lies in the range $0 \leq Q \leq \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}}$. The depth of flow is always in the range $0 \leq d \leq \frac{E}{g}$. If d exceeds $\frac{E}{g}$ then, from (2.64), Q is undefined. The value

$$d_L = \frac{E}{g}$$

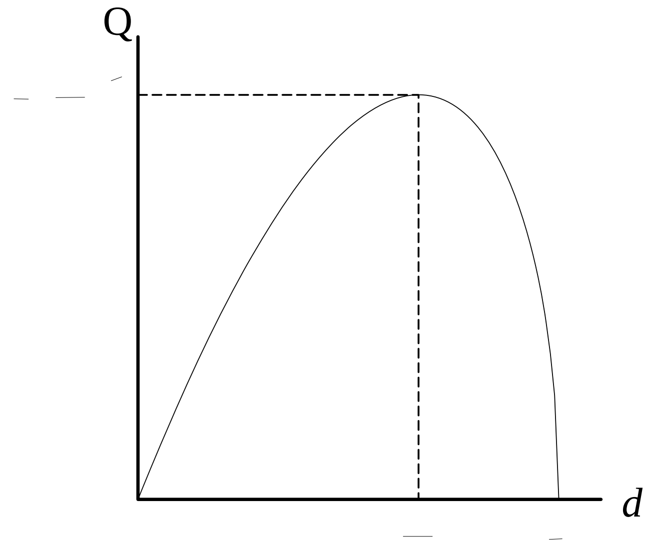


Figure 2.4: Q as a function of d for constant E.

is known as the limit depth and is the maximum attainable depth for a flow with energy E. The value of the depth which corresponds to the critical mass flow Q , defined by (2.62), is called the critical depth and is given by

$$d \ = \frac{2E}{3g}. \tag{2.65}$$

The critical values of depth and velocity satisfy the relationship

$$gd \ = \frac{2E}{3} = c \ . \tag{2.66}$$

In fact, a flow is said to be critical if the velocity, , and the depth, , satisfy

$$= \quad \text{—} \qquad (2\ 67)$$

Substituting (2.67) into (2.35) yields the definitions of critical velocity and critical depth, as given by (2.63) and (2.65).

From (2.35) a depth in the range corresponds to a subcritical flow; otherwise, if the depth lies in the range 0 , the flow is supercritical.

From Figures 2.3 and 2.4, when the mass flow, , has attained its critical value there is only one possible depth and one velocity — those which correspond to critical flow. If is in the range 0 there are two possible values of and — one corresponding to a supercritical flow and one to a subcritical flow.

In the case of steady quasi one-dimensional shallow water motion the flow variable graphs, Figures 2.3 and 2.4, can be used to deduce information about the variations of depth and velocity in a channel of slowly varying breadth.

The equations of motion for steady quasi one-dimensional flow are given by

$$=$$

$$(\quad) = 0$$

which are equations (2.55) and (2.56), that is, conservation of momentum and conservation of mass. These equations may be integrated to give

$$= \hat{}$$

$$\text{and} \qquad =$$

where $\hat{}$ and are constants, to be defined, and $= (\quad)$.

Thus, if ( ) and ( ) are given for in [ ], the values of energy and mass flow are known at each point in the interval [ ]. That is,

$$= \hat{} + \tag{2.68}$$

$$= \frac{}{} \tag{2.69}$$

Solution values for the velocity lie on the surface of which Figure 2.3 is a cross-section for constant and for the depth lie on the surface of which Figure 2.4 is

If the equilibrium depth of the fluid is constant the energy E given by (2.68) is also constant and the solutions of v and d lie on the curves in Figures 2.3 and 2.4 for fixed E. Consider a channel whose breadth decreases to a minimum, B say, such that $B(x) = B$ for some x (x, x). An example of such a channel is given in Figure 2.2. Moving along the channel, from the inlet at = , as decreases increases (using (2.69)) and so, from Figure 2.3, a subcritical will increase and a supercritical will decrease in value. Once the point of minimum breadth has been passed, decreases as increases so that the subcritical decreases and the supercritical increases in value. Similarly, using Figure 2.4, moving along the channel from = a subcritical will decrease then increase and a supercritical will increase then decrease, in step with the increase then decrease of mass flow .

When 0, less information about the flow can be obtained from the curves given by (2.61) and (2.64). Consider the curve in Figure 2.3 to be a cross-section for constant , through the surface created by taking to be function of and in equation (2.61). The solution lies on the surface and, as and vary in accordance with equations (2.68) and (2.69), the values that takes during the motion can be traced on the surface. A similar surface representing as a function of and , as defined by equation (2.64), enables the variation of to be traced as and vary during the motion. As varies in response to a changing channel breadth it is not possible, in general, to determine whether the velocity and depth of flow will increase or decrease, since this depends also on the change in as determined by the variation in .

For the energy $= \hat{\ } +$, assumed known, the mass flow in a channel of breadth $(\ )$ cannot exceed the value of the critical mass flow , given by (2.62). This bound on the maximum possible value of the mass flow imposes a lower bound on the minimum breadth of the channel. From (2.69) the minimum breadth, $\hat{\ }(\ )$ at each point , for which a continuous flow is possible is

$$\hat{\ }(\ ) = \frac{\ }{\ } = \frac{2\ \hat{\ } + \ (\ )}{3}$$

If $(\ )$ $\hat{\ }(\ )$ for any in [ ] then the flow becomes blocked. If $(\ )$ $\hat{\ }(\ )$ for all in [ ] then the flow remains wholly subcritical or wholly supercritical throughout the channel. If $(\ ) = \hat{\ }(\ )$ at a particular point in [ ] then the flow is critical at that point and there is the possibility of transitional flow.

It can be shown, using (2.68) and (2.69), that a flow with constant energy may be critical at a point, say, only if the breadth at that point is stationary with respect to , that is $(\ ) = 0$. Using the definition of mass flow (2.34) and quation (2.71) implies that when a flow is critical, that is, equation (2.67) is satisfied, the breadth and equilibrium depth must be such that

$$gh' + \frac{B'}{B}v^2 = 0.$$

Thus for a domain with $h' \equiv 0$ critical flow occurs only at a point where $B' = 0$. Conversely, if $h' = 0$ and $B' = 0$ at a point then either $v^2 = gd$ and the flow is critical or $d' = 0$, that is, the fluid depth has reached either a minimum or a maximum at that point. For a domain where $h' \not\equiv 0$ equation (2.71) provides less information; for example, it is not possible in general to determine without knowing the solution in advance where stationary points of the solution might lie or whether, given appropriate conditions, the flow becomes critical.

One further flow variable graph is considered here. This involves a quantity which is of particular use when considering discontinuous motions, namely, the flow stress P. The reason for this utility is that the value of flow stress varies continuously even when the flow variables are discontinuous (in the sense of hydraulic jumps) — as is described in Section 2.6. The flow stress is defined by

$$P = p + dv^2 \tag{2.72}$$

for quasi one-dimensional flow, where p is the pressure given by (2.13). Using the definition of energy E, (2.35), to substitute for $d = \frac{1}{g}\left(E - \frac{1}{2}v^2\right)$ and rearranging gives

$$P = \frac{1}{2g}\left(E - \frac{1}{2}v^2\right)\left(E + \frac{3}{2}v^2\right). \tag{2.73}$$

quation (2.73) can be used to draw a flow variable graph of P as a function of E and v, but a more interesting relationship is that between P, Q and E.

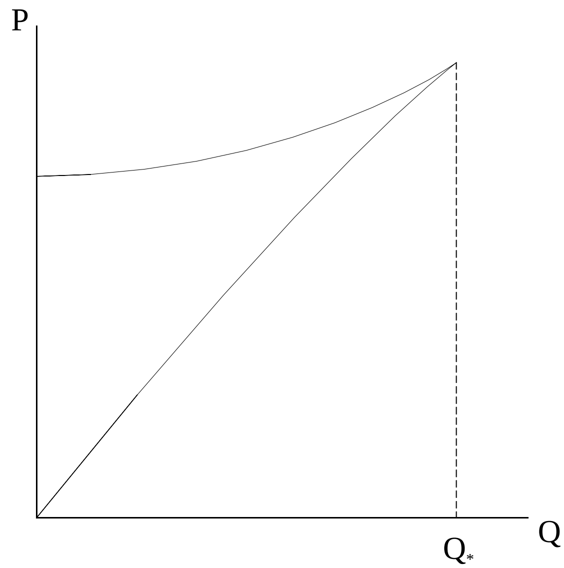


Figure 2.5: as a function of for constant .

The flow variable graph in Figure 2.5 is created by regarding as a parameter in equations (2.61) and (2.73). In practice it is plotted by taking, for each fixed , 2 1 values of in the permitted range, that is,

$$= \frac{1}{1} \sqrt{\frac{2}{3}} \qquad = 1 \; \ldots$$

$$= \frac{}{1} \sqrt{2} \quad \sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}} \qquad = \; + 1 \; \ldots \; 2 \quad 1$$

Then the 2 1 points ( ), given by

$$= - \quad \frac{1}{2}$$

$$= \frac{1}{2} \quad \frac{1}{2} \quad + \frac{3}{2}$$

for = 1 ... 2 1, trace the curve for as a function of for fixed. The

2.6 Discontinuous Flows

This chapter has, so far, dealt with the equations of motion for continuous flows. In this section equations of motion for discontinuous flows in shallow water are considered.

It is possible to control the depth and velocity, and therefore also the mass flow and energy, of a flow at the inlet and outlet positions of a channel using, for example, weirs or sluice gates. Thus a situation might occur where the imposed inlet and outlet conditions cannot be achieved by a continuous flow in the channel. In such circumstances a discontinuity may occur.

The differential equations of Sections 2.2, 2.3 and 2.4, which model the flow in shallow water, are only valid for continuous solutions. At points of discontinuity the differential equations no longer apply and other equations are needed to govern the motion. These equations, known as jump conditions, relate the values of the flow variables on one side of the discontinuity to their values on the other side.

In this thesis only time-independent discontinuous flows are considered, the stationary discontinuity being known as a hydraulic jump. The jump conditions for quasi one-dimensional and two-dimensional flows are given in Sections 2.6.1 and 2.6.2.

2.6.1 Discontinuous Flows in One Dimension

In quasi one-dimensional motion a hydraulic jump consists of a point (a value of x) where the depth and velocity of the shallow water flow are discontinuous.

Consider the channel which extends over the interval $[x_e, x_o]$ of the x-axis and has slowly varying breadth $B(x)$. Let $x_s \in (x_e, x_o)$ be the position of the hydraulic jump. Then the equations of motion for continuous flow hold in the two intervals (x_e, x_s) and (x_s, x_o). Thus in $(x_e, x_s) \cup (x_s, x_o)$ the flow variables satisfy

$$\begin{aligned} E' &= gh' \qquad \text{conservation of momentum,} &(2.74)\\ (BQ)' &= 0 \qquad \text{conservation of mass,} &(2.75) \end{aligned}$$

where Q and E are defined by (2.34) and (2.35), as before.

At the position of the hydraulic jump, x_s, the flow variables must satisfy the jump conditions which are alternative statements of conservation of mass and momentum, valid at a discontinuity. The jump conditions are given by

$$[P]_{x_s} = 0 \tag{2.76}$$

$$\text{and} \qquad [BQ]_{x_s} = 0, \tag{2.77}$$

from Stoker (1957), where P is the flow stress defined by (2.72). The brackets $[\,\cdot\,]_{x_s}$ denote the jump in the value of the quantity at the point x_s. That is, for example, $[P]_{x_s} = P|_{x_s+} - P|_{x_s-}$, where + denotes the x_o side of x_s and − the x_e side of x_s. The third jump condition is given by

$$[E]_{x_s} \neq 0,$$

which states that the energy E is not conserved at a jump. Discounting the possibility that there is an energy source at x_s gives the inequality

$$[E]_{x_s} < 0, \tag{2.78}$$

which is justified by the fact that, in reality, mechanical energy may be converted into heat energy through turbulence at the jump.

quations (2.74)—(2.78) govern the motion of a quasi one-dimensional shallow water flow with a discontinuity at the point x_s. In certain cases the jump conditions (2.76)—(2.78) can be used to uniquely determine the position of the hydraulic jump. This may be illustrated using the graph in Figure 2.5 which relates flow stress, mass flow and energy.

The usefulness of the flow variable graph lies in the fact that equations (2.74) and (2.75) can be solved for E and Q, given a particular domain. Applying the jump conditions (2.77) and (2.78) to the solutions of (2.74) and (2.75) gives the variations of Q and E throughout the channel, as follows.

Using equations (2.75) and (2.77) gives the variation of Q in the channel as

$$Q(x) = \frac{CB_e}{B(x)} \qquad x \in [x_e, x_o]\,. \tag{2.79}$$

From equation (2.74)

$$E(x) - gh(x) = \text{ constant} \qquad x \in (x_e, x_s) \cup (x_s, x_o).$$

Assuming that $[h]_{x_s} = 0$ equation (2.78) gives

$$[E - gh]_{x_s} < 0.$$

Thus let

$$\begin{aligned} E - gh &= E_e \qquad x \in [x_e, x_s) \\ \text{and} \qquad E - gh &= E_o \qquad x \in (x_s, x_o]\,, \end{aligned}$$

where E_e and E_o are constants such that $E_e > E_o$.

In Figure 2.6, which shows the variation of P with Q for two distinct values of E, let $E_1 = E_e + gh(x_s)$ and $E_2 = E_o + gh(x_s)$. Then the point of intersection,

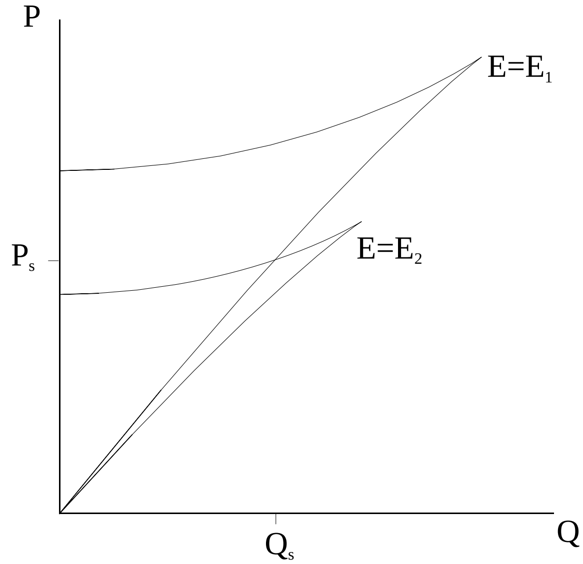


Figure 2.6: P as a function of Q for two distinct values of E.

where $P = P_s$ and $Q = Q_s$, is the point of discontinuity of a flow with energy $E = E_e + gh(x_e)$ at inlet and energy $E = E_o + gh(x_o)$ at outlet. Notice that the point of intersection occurs on the supercritical branch of the line corresponding to E_1 and on the subcritical branch of the line corresponding to E_2. The jump condition (2.78) ensures that this is always true, that is, the flow on the inlet side of a discontinuity is always supercritical and the flow immediately on the other side of the discontinuity is always subcritical.

Let the undisturbed fluid depth, h, be constant. Then the discontinuous flow, determined by $E = E_e + gh$ at inlet and $E = E_o + gh$ at outlet, can be traced on the curves corresponding to $E_1 = E_e + gh$ and $E_2 = E_o + gh$ in Figure 2.6. In particular, given just E_e and E_o, the mass flow at the discontinuity, Q_s, can be deduced. In this way the position of the discontinuity in the interval $[x_e, x_o]$ may be found. From (2.79) the position of the discontinuity, x_s, satisfies

$$B(x_s) = \frac{CB_e}{Q_s} \tag{2.80}$$

and, since the breadth function $B(x)$ is known, the value of x_s can be calculated.

There are three possible situations arising.

1. in ( ) is uniquely determined by inverting equation (2.80).

2. There is no value of in ( ) which satisfies (2.80).

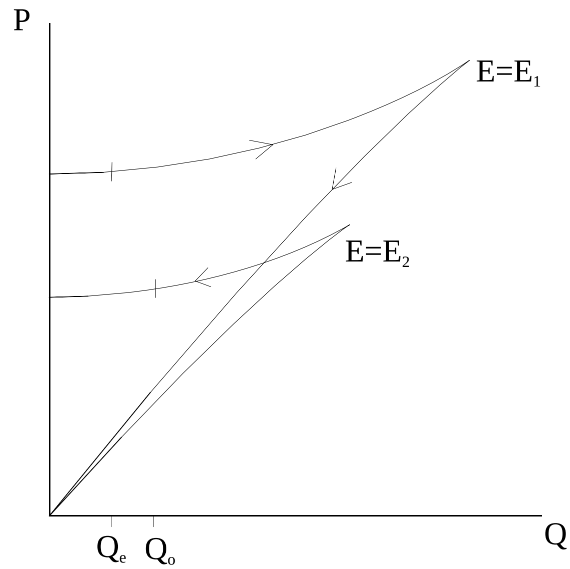


Figure 2.7: Flow path for a discontinuous critical flow.

critical point where $Q = Q_*$. The flow must become supercritical here for the discontinuity to occur so the solution tracks along the supercritical branch until it reaches the position of the discontinuity. The solution point then switches to the subcritical branch of the E_2 curve until the mass flow equals the mass flow Q_o at the outlet of the channel.

For the example with constant undisturbed fluid depth Figure 2.7 can be used to define a range of possible outlet conditions, given an inlet condition for the flow. For the case of a critical flow a hydraulic jump may occur anywhere in the range $(\hat{x}, x_o)$, where $\hat{x}$ is the position of the channel throat. Let E_1 be the energy of the flow at inlet and let E_2 be the energy at outlet. A discontinuity at the channel throat requires that the curves for E_1 and E_2 intersect at the cusp of the E_1 curve. This requires $E_2 = E_1$ so that the discontinuity is of zero strength. The value $E = E_1$ is the maximum energy at outlet that a discontinuous flow can achieve. The minimum value of E at outlet is obtained when the discontinuity lies right at the channel outlet. In Figure 2.7 let the curve corresponding to

concave bend, only hydraulic jumps which extend across the whole width of the channel are studied here.

Consider a steady discontinuous shallow water flow in a channel . Let Σ be

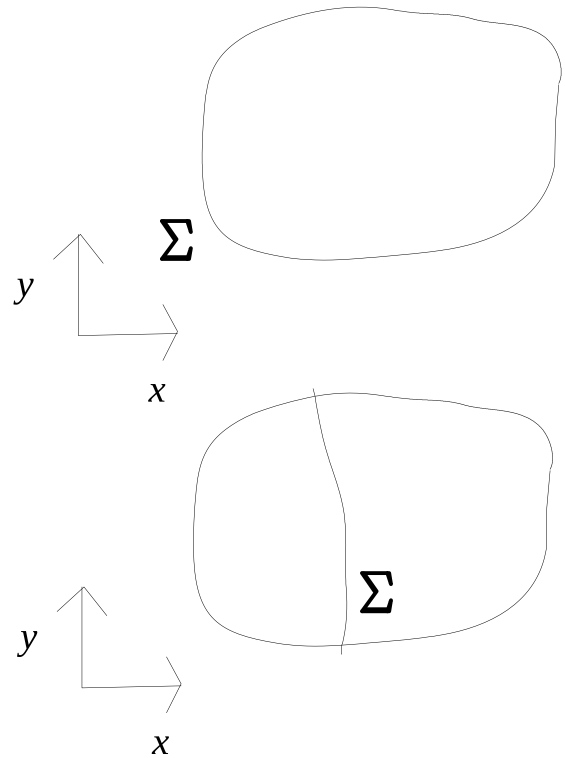


Figure 2.9: Domain for two-dimensional discontinuous flows.

where E is a constant. The equation of conservation of momentum in is satisfied by

$$E = E \ + gh,$$

where is a constant such that .

As in the one-dimensional case the flow is supercritical before the hydraulic jump, that is in , and subcritical after the hydraulic jump, that is in .

in fact, closely related.

Let be cartesian coordinates, defined as in Chapter 2, and let be the time. Let $\Omega = \quad (\quad)$ be the spatial domain to be considered, where is a fixed region of the plane, is the undisturbed fluid depth and is the height of the free surface above the equilibrium position, as shown in Figure 2 1.

The Bernoulli equation (2.7) for free surface flows gives an expression for the fluid pressure ˜ as a function of the velocity potential , that is,

$$\tilde{} = \qquad + \quad + \frac{1}{2}\tilde{} \qquad \tilde{} \qquad\qquad (3\ 1)$$

where ˜ is defined by (2.2).

Luke (1967) uses the expression (3.1), for ˜, as the Lagrangian density (the integrand of the Lagrangian) in a variational principle. Luke's principle was stated for a constant equilibrium depth but can be generalised to allow for a non-constant depth and, for the given three-dimensional domain Ω, the modified variational principle is

$$\tilde{}\,(\quad) = \qquad\qquad + \quad + \frac{1}{2}\tilde{} \qquad \tilde{} \qquad\qquad = 0 \qquad (3\ 2)$$

Let the variations in and be such that $= 0$ and $= 0$ on the lateral

$$\tilde{} \quad = \qquad\qquad + \quad + \frac{1}{2}\,\tilde{} \quad \tilde{}$$

$$(\quad (\quad + \qquad + \qquad)) \qquad (\quad (\qquad + \qquad + \quad))$$

$$\tilde{} \qquad\qquad = 0$$

which yields the natural conditions

$$\tilde{} \qquad = \; 0 \quad \text{in } \Omega \tag{3.3}$$

$$+ \quad + \frac{1}{2}\,\tilde{} \quad \tilde{} \qquad = \; 0 \quad \text{on} \quad = \tag{3.4}$$

$$+ \qquad + \qquad\qquad = \; 0 \quad \text{on} \quad = \tag{3.5}$$

$$+ \qquad + \qquad = \; 0 \quad \text{on} \quad = \tag{3.6}$$

for ( ). quation (3.3) is the equation of conservation of mass for irrotational flow, that is, (2.6). quations (3.4)—(3.6) are equivalent to (2.8), (2.10) and (2.11), where the irrotationality condition (2.5) has been assumed, and are therefore the dynamic free surface condition, the kinematic free surface condition and the condition of zero flow through the bed, respectively. Thus the variational principle (3.2) generates the governing equations of a free surface flow. No at-'J4ahe(3]he2r')the

][+ΣD]+])dynf6F˜0'96'u)NoF˜06+9J[]++F˜]˜˜

tion. The revised variational principle which achieves this is

$-$

$-$

integrated over Ω and (t_1, t_2), and

$$\lambda \left(\eta_t + u\eta_x + v\eta_y - w\right)\big|_{z=\eta} + \mu \left(uh_x + vh_y + w\right)\big|_{z= \ h},$$

integrated over   and (t_1, t_2). The variational principle based on Hamilton's principle is therefore

$$\delta J_2(\eta, \ , \lambda, \mu, \nu) = \delta\left\{\int_{t}^{t}\ \iint_D \left(\int_{\ h}^{\eta} \left(\rho\left(\frac{1}{2}\ . \ - gz\right) + \nu\tilde{\boldsymbol{\nabla}}. \ \right) dz\right.\right. \tag{3.16}$$
$$\left.\left. + \lambda \left(\eta_t + u\eta_x + v\eta_y - w\right)\big|_{z=\eta} + \ \mu \left(uh_x + vh_y + w\right)\big|_{z= \ h}\right) dx\, dy\, dt\right\} = 0.$$

As in the case of Luke's principle (3.2) the variations are assumed to vanish on the lateral space boundaries and on the time boundaries. The natural conditions of (3.16) are given by

$$\left.\begin{aligned} \rho \ \ - \tilde{\boldsymbol{\nabla}}\nu \ &= \\ \tilde{\boldsymbol{\nabla}}. \ \ &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{3.17}$$

$$\lambda_t + \tilde{\boldsymbol{\nabla}}.(\lambda \ \ ) - \rho\left(\frac{1}{2}\ . \ - gz\right) - \nu\tilde{\boldsymbol{\nabla}}. \ \ = 0 \quad \text{on } z = \eta, \tag{3.18}$$

$$\eta_t + u\eta_x + v\eta_y - w = 0 \quad \text{on } z = \eta, \tag{3.19}$$

$$(\nu - \lambda)\,\tilde{\boldsymbol{\nabla}}(z - \eta) = 0 \quad \text{on } z = \eta, \tag{3.20}$$

$$uh_x + vh_y + w = 0 \quad \text{on } z = -h, \tag{3.21}$$

$$(\nu - \mu)\,\tilde{\boldsymbol{\nabla}}(z + h) = 0 \quad \text{on } z = -h, \tag{3.22}$$

for $t \in (t_1, t_2)$. The fluid is homogeneous by hypothesis. Therefore, identifying $\frac{\nu}{\rho}$ as the velocity potential, equations (3.17)—(3.22) together are equivalent to (3.8)—(3.11) and hence to (3.3)— (3.6). For consistency of notation the Lagrange multiplier ν is relabelled as $\nu = \rho\chi$, so that $(3.17)_1$ gives the usual irrotationality condition (2.5). Then, using (3.20) and (3.22), the Lagrange multipliers λ and μ may be identified as $\lambda = \rho\chi|_{z=\eta}$ and $\mu = \rho\chi|_{z= \ h}$.

Thus Hamilton's principle can be adapted to give (3.16) which, on relabelling $\nu = \rho\chi$ and using (3.20) and (3.22) to identify the Lagrange multipliers λ and μ, has as its natural conditions the irrotationality condition and the conservation of mass equation in the domain Ω and boundary conditions on the free surface and the bed.

The adaptation of Hamilton's principle to fluid flow is given by Seliger and Whitham (1968) for compressible flows. In that case the conservation of mass and two other constraints on the principle are necessary, the other constraints being related to energy balance, in the form of entropy conservation for a particle, and conservation of particle identities. In the current problem of irrotational free surface flows, the entropy does not appear and conservation of particle labels is apparently not required.

3.2 Shallow Water Flows

Variational principles for shallow water flows can be considered from two points of view. The principles in Section 3.1 for free surface flows can be modified, by applying the shallow water approximation to the variables, or Hamilton's principle can be applied directly to the variables of shallow water theory, using the gas dynamics analogy of Section 2.2.2. This section deals with the first method.

Now consider (3.16) – the 'Hamilton' principle. Under the conditions of the shallow water approximation the integral over z can be carried out and the terms evaluated at $z = -h$ and $z = \eta$ can be combined since, from (2.14), u and v take the same values at these levels. The result is the functional $J \ = J\ (d, \ \ , \phi)$ given by

$$J \ = \int_t^t \iint_D \rho \left(\frac{1}{2} d \ \ . \ \ - \frac{1}{2} g d \ \ + g d h + \phi \left(d_t + \ \ . \left(d \ \ \right) \right) \right) dx\, dy\, dt. \qquad (3.25)$$

Assuming that the variations vanish on the space and time boundaries, the natural conditions of $\delta J \ = 0$ are

$$\left.\begin{array}{rcl} \phi_{tD} + g\eta + \ \ . \ \ \phi - - \ . & = & 0 \\ - \ \ \phi & = & \\ d \ + \ \ . \left(d \ \ \right) & = & 0 \end{array}\right\} \quad \text{in } \hat{\ } {}^{D}, \qquad (3.26)$$

which together are equivalent to (3.24).

Thus the 'pressure' and 'Hamilton' free surface principles reduce to 'shallow water' principles when the variables are approximated using shallow water theory.

By using the divergence theorem and integration by parts, the functional (3.23) can be rearranged to give

$$\begin{aligned} J \ = \int \iint \rho \left(\left(\ \ - \frac{1}{2} d \ \ \right) . \ \ - \frac{1}{2} g d \ \ + g d h + \phi \left(d \ + \ \ . \ \ \right) \right) dx\, dy\, dt \\ - \int \ \int \rho\phi \ \ . \ \ d\Sigma - \iint \ [\rho d \phi] \ \ dx\, dy, \end{aligned}$$

that is, for $\ \ = d$ the two functionals (3.23) and (3.25) are the same, to within boundary terms.

3.2.2 Further Functionals

Before considering variational principles with boundary conditions as natural conditions the 'pressure' and 'Hamilton' functionals for unsteady shallow water motion are rewritten in different variables. The variational principles with boundary terms, generated using the modified functionals, can then be related to two further variational principles.

First consider (3.23), which was derived from the free surface principle based on an expression for the pressure. As the fluid is assumed homogeneous, the density ρ can be set equal to unity without losing generality. As a further simplification of notation, the term

$$\phi_t + \frac{1}{2}\mathbf{v}.\mathbf{v} + gd - gh$$

may be written as

$$\phi_t + E - gh.$$

This suggests the use of E as a new variable. From the definition of E, (2.19), we have

$$d = \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right),$$

which allows for the definition of a new function $p(\mathbf{v}, E)$ obtained by substituting for d in the 'pressure' $\frac{1}{2}gd^2$. Thus

$$p(\mathbf{v}, E) = \frac{1}{2g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right)^2. \qquad (3.27)$$

The integrand of the functional being constructed is now

$$p(\mathbf{v}, E) - d\left(\phi_t + E - gh\right) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right), \qquad (3.28)$$

which is the integrand of (3.23) after making the substitutions outlined above. A functional with integrand (3.28) will be referred to as a 'p' functional for unsteady shallow water flow.

The 'Hamilton' functional (3.25) is rewritten similarly. The density ρ is taken to be unity as before, the change of variable made is from $\mathbf{v}$ to $\mathbf{Q}$ using a rearrangement of (2.25), that is $\mathbf{v} = \frac{\mathbf{Q}}{d}$, and the function $r(\mathbf{Q}, d)$ is defined to be

$$r(\mathbf{Q}, d) = \frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d} - \frac{1}{2}gd^2. \tag{3.29}$$

Making these substitutions in the integrand of (3.25) yields the expression

$$r(\mathbf{Q}, d) + gdh + \phi\left(d_t + \boldsymbol{\nabla}.\mathbf{Q}\right). \tag{3.30}$$

A functional with integrand (3.30) will be referred to as an 'r' functional for unsteady shallow water flow.

The structures of the integrands of the 'p' and 'r' functionals are similar in that they may both be expressed in the form

$$\text{function} + \text{multiplier} \times \text{conservation law}.$$

For the 'p' functional (3.28) is

$$p + \text{multiplier} \times \begin{array}{c}\text{conservation of}\\ \\ \text{momentum}\end{array} + \text{multiplier} \times \begin{array}{c}\text{irrotationality}\\ \\ \text{condition}\end{array},$$

and for the 'r' functional (3.30) is

$$r + gdh + \text{multiplier} \times \text{conservation of mass}.$$

These forms for the integrands of the 'p' and 'r' functionals suggest obvious ways of constraining the variational principles based on these functionals. For example, if a 'p' variational principle is constrained to satisfy the conservation of momentum equation, by setting $E = gh - \phi_t$, and irrotationality, by setting $\mathbf{v} = \boldsymbol{\nabla}\phi$, the expression (3.28) becomes

$$p\left(\boldsymbol{\nabla}\phi, gh - \phi_t\right),$$

which depends on the variable ϕ alone. Constrained variational principles are dealt with in more detail in Sections 3.5, 3.6 and 3.7.

In Section 3.5 boundary terms will be added to the 'p' and 'r' functionals so that variations which do not necessarily vanish at the space and time boundaries are allowed. Two further functionals, whose integrands are related to (3.28) and (3.30) by a closed quartet of Legendre transforms will also be constructed.

It is evident that there exists a number of constrained and unconstrained variational principles related to time-dependent shallow water flows. To avoid deriving separately the natural conditions of each variational principle a functional of the general form of the time-dependent shallow water functional can be used to generate the natural conditions of a general shallow water variational principle. Then the natural conditions for each different case can be obtained immediately. This general variational principle, along with those for steady state and quasi one-dimensional shallow water flows, is considered in Section 3.3. The natural conditions of general variational principles for time-independent discontinuous flows are derived in Section 3.4.

In this section the natural conditions of a general form of variational principle which includes the cases of shallow water flows are derived. All of the variables

Using Taylor series and integrating by parts, the first variation of (3.31) may be written as

$$= \quad (\quad) \quad (\quad)$$

$$+ \quad (\quad + \quad) \quad + \quad (\quad + \quad)$$

Thus the natural conditions of $= 0$ are

$$: \quad (\quad) \quad (\quad) = 0 \quad (\quad); \quad (\quad) \qquad (3.32)$$

$$: \quad (\quad + \quad) = 0 \quad = 0 \; 1; \quad (\quad) \qquad (3.33)$$

$$: \quad (\quad + \quad) = 0 \quad = 0 \; 1; \quad (\quad) \qquad (3.34)$$

for $= 1 \ldots$. quations (3.32) are the uler equations of the variational

—

The first variation of (3.35) may be written as

= — + +

using Taylor series and integration by parts. The natural conditions of = 0 are therefore

: — = 0 ( ) (3.36)

: + = 0 = 0 1 (3.37)

for = 1 ... .

The domain in two-dimensional flow is a simply connected open set in the plane. For the general time-dependent case a third coordinate is added, where lies in the interval [ ]. Let ( ) ... ( ) be scalar variables and let ( ) ... ( ) be vector variables, where = ( ) for = 1 ... . The variables and are assumed to be differentiable functions of , and .

Consider a functional of the form

( ) = ( )

\+ ( ) Σ + [( )] (3.38)

Using Taylor series, the divergence theorem and integration by parts the first variation of the functional (3.38) may be written as

$$= \quad (\quad)$$

$$+$$

$$+ \quad + \quad + \quad + \quad \Sigma$$

$$+ \quad (\quad + \quad) \quad + \quad +$$

where the unit vector is the outward normal to the boundary Σ and the following notation is used.

$$\text{—} \qquad \text{—} \qquad \text{—} \quad \text{—}$$

$$\text{—} \quad \text{—} \qquad \text{—} \quad \text{—} \qquad \frac{}{(\quad)}$$

The natural conditions of $= 0$ are given by

$$: \quad (\quad) = 0 \quad (\quad) \quad ; \quad (\quad) \tag{3.39}$$

$$: \quad = \quad (\quad) \quad ; \quad (\quad) \tag{3.40}$$

$$: \quad + \quad = 0 \quad (\quad) \quad \Sigma ; \quad (\quad) \tag{3.41}$$

$$: \quad + \quad = \quad (\quad) \quad \Sigma ; \quad (\quad) \tag{3.42}$$

$$: \quad (\quad + \quad) \quad = 0 \quad = 0 \;\; 1 ; (\quad) \tag{3.43}$$

$$: \quad + \quad = \quad = 0 \;\; 1 ; (\quad) \tag{3.44}$$

(3.39)–(3.44). In Sections 3.5.1 and 3.5.2 variational principles are derived whose

3.4 Discontinuous Variables

In this section general versions of the shallow water variational principles allowing for discontinuous variables are studied. Only time-independent discontinuous flows will be considered so the extra coordinate of Section 3.3, which is identified with the time, is no longer used.

3.4.1 General Variational Principles for One-dimensional Flow

Let the interval $[x_0, x_1]$ of the x-axis be the domain over which the integrand of the general functional is integrated and let x_s be a point in the interior of this interval. Let $u_i(x)$ $i = 1, \ldots, m$ be a set of functions defined on $[x_0, x_1]$, as before. Assume that all of the u_i are continuous in $[x_0, x_s) \cup (x_s, x_1]$. This allows for one or more of the u_i to be discontinuous at the point x_s.

Consider a functional of the form

$$\hat{I}_2(u_1, \ldots, u_m, x_s) = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1}\right) f(x, u_1, \ldots, u_m, u_1', \ldots, u_m')\, dx \\ + [g(x, u_1, \ldots, u_m)]_{x_0}^{x_1}. \qquad (3.50)$$

Using Taylor series the first variation of $\hat{I}_2$ is given by

$$\delta \hat{I}_2 = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1}\right) \left\{\sum_{i=1}^{m} f_{u_i} \delta u_i + f_{u_i'} \delta u_i'\right\} dx \\ + \left[\sum_{i=1}^{m} g_{u_i} \delta u_i\right]_{x_0}^{x_1} + \delta x_s\, f|_{x_s-} - \delta x_s\, f|_{x_s+}, \qquad (3.51)$$

where the superscript $-$ denotes the x_0 side of x_s and $+$ denotes the x_1 side of x_s. The first three terms are due to the variations of the u_i and the last two are due to the variation of the position of the discontinuity, x_s.

Applying integration by parts to the terms of (3.51) yields

$$: \quad + \quad = 0 \quad = 0\ 1 \tag{3.55}$$

$$\hat{} \quad : \quad = 0 \tag{3.56}$$

$$: \quad = 0 \tag{3.57}$$

for $= 1 \ldots$. quations (3.54) are the uler equations and (3.55) are boundary conditions at and . quations (3.54) and (3.55) are identical to the corresponding natural conditions for continuous variables, derived in Section 3.3. quations (3.56) and (3.57) are due to the discontinuities in the at the point . It is possible that there is a number of such points of discontinuity in the interval ( ). If so, then there are equations of the form (3.56) and (3.57) corresponding to each of these points. In this case the uler equations (3.54) are derived for the interval ( ) excluding all points of discontinuity.

The natural conditions of a variational principle with discontinuous variables, defined in two dimensions, are derived in this section. The method is similar to that of the one-dimensional case in Section 3.4.1. The difference is that in two dimensions the variables are discontinuous across a curve in the domain of integration instead of at an isolated point.

Let be the domain in the plane over which the integrand of the functional is to be integrated and let Σ be the boundary of , as before. Let Σ be a smooth,

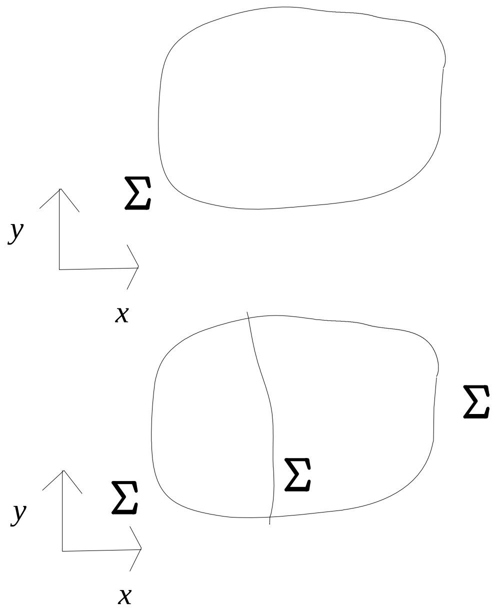


Figure 3.1: The two-dimensional domain.

Let $u \;\; = u \;\; (x, y)$ for $i = 1, \ldots, m$ and $\;\; = (v \;\; (x, y), v \;\; (x, y))$ for $i = 1, \ldots, n$ be the scalar and vector variables. The u and are assumed to be continuous in the domains and but may be discontinuous across the curve Σ .

Consider a functional of the form

$$\hat{I} \; (u \;, \quad , \Sigma \;) = \qquad + \qquad F(x, y, u \;, \quad u \;, \quad , \quad . \quad) \, dx \, dy$$

$$+ \qquad + \qquad G(x, y, u \;, \quad) \, d\Sigma, \tag{3.58}$$

where $i = 1, \ldots, m$, $k = 1, \ldots, n$, Σ is the part of Σ which bounds and Σ is the part which bounds .

The first variation of $\hat{I}\phi RMf/dn.l3rPSK/k/NP\psi\psi S/kNNP\psi\psi Fkm\tau P\psi\psi KN/Nkk/NP\psi\psi S/\psi/N$ is given by

$$\hat{} = \qquad + \qquad + \qquad (\quad)$$

$$+ \qquad + \qquad (\quad)$$

$$+ \quad + \qquad + \qquad \Sigma$$

$$+ \qquad + \qquad \Sigma \qquad \Sigma \tag{3.59}$$

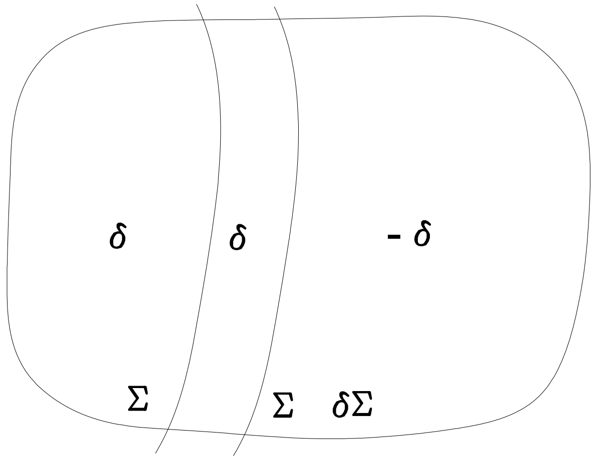


Figure 3.2: Variation of Σ_s.

where δ is the region of enclosed by Σ_s and its variation $\Sigma_s + \delta\Sigma_s$ — see Figure 3.2 — and $\delta\Sigma$ is the change in the length of Σ caused by the variation of Σ_s. The superscripts $-$ and $+$ indicate that the functions are evaluated on the or $^+$ sides of Σ_s, respectively.

Applying the divergence theorem to the $\boldsymbol{\nabla}(\delta u_i)$ and $\boldsymbol{\nabla}.(\delta \ _k)$ terms in (3.59) yields

$$
\begin{aligned}
\delta \hat{I}_4 = \left(\iint_{D} + \iint_{D}\right) & \left\{ \sum_{i=1}^{m} \left(F_{u} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u}\right) \delta u_i \right. \\
& \left. + \sum_{k=1}^{n} \left(F \quad - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.} \quad \right).\delta \ _k \right\} dx\, dy \\
+ \left(\int_{\Sigma} + \int_{\Sigma}\right) & \left\{ \sum_{i=1}^{m} \left(G_{u} + F_{\boldsymbol{\nabla} u}. \quad \right)\delta u_i + \sum_{k=1}^{n}\left(G \quad + F_{\boldsymbol{\nabla}.} \quad \right).\delta \ _k \right\} d\Sigma \\
+ \iint_{\delta D} F \quad dx\, dy & - \iint_{\delta D} F^{+}\, dx\, dy + \int_{\delta\Sigma} G \quad d\Sigma - \int_{\delta\Sigma} G^{+}\, d\Sigma \\
+ \int_{\Sigma} & \left\{ \sum_{i=1}^{m} F_{\boldsymbol{\nabla} u}. \quad \delta u_i + \sum_{k=1}^{n} F_{\boldsymbol{\nabla}.} \quad .\delta \ _k \right. \\
& \left. - \sum_{i=1}^{m} F^{+}_{\boldsymbol{\nabla} u}. \quad \delta u_i - \sum_{k=1}^{n} F^{+}_{\boldsymbol{\nabla}.} \quad .\delta \ _k \right\} d\Sigma.
\end{aligned}
$$

Consider a point on the curve Σ_s. Let δn be the displacement of this point, under the variation, in the direction of , the unit normal on the surface Σ_s with direction out of the subdomain . Then the integral over the domain δ can be written in terms of an integral along Σ and the variation , that is,

$$= \quad \Sigma$$

As in the one-dimensional case equations relating the values of the variables on either side of the discontinuity are obtained by using the total variations. Let be the displacement of a point on Σ, under the variation, in the direction of , the unit tangent vector at the point. Then the total variations of the flow variables on the curve Σ are given by

$$\hat{} \quad = \quad + \text{—} \quad + \text{—} \quad = 1 \ldots$$

$$\hat{} \quad = \quad + \text{—} \quad + \text{—} \quad = 1 \ldots$$

where — denotes differentiation in the direction of and — differentiation in the direction of . As in the one-dimensional case, only variations whose total

—— ——

—— ——

The previous two sections have dealt with generating general expressions for the natural conditions of certain types of variational principles. These are used in this section and the remaining sections of this chapter to deduce the natural conditions of variational principles for shallow water flows.

The derivation of variational principles for time-dependent motion in shallow water is continued here using the functionals created in Section 3.2.2.

The variational principles for shallow water flows considered so far have all been such that the variations vanish at the boundaries of the domains of integration.

for ( ). Similarly the domain is divided into two parts, say = + , where $\hat{} = \quad (\quad),$

$$\text{——} = \; - \qquad \frac{1}{2} \qquad \text{and} \qquad \text{——} = \frac{1}{-} \qquad \frac{1}{2}$$

Thus the first two conditions in $\hat{}$ are

$$- \qquad \frac{1}{2} \qquad + \quad = \quad \text{and} \; \frac{1}{-} \qquad \frac{1}{2} \qquad\qquad = 0$$

which together give

$$= \qquad \text{and} \qquad = \quad + \frac{1}{2}$$

so that the last three natural conditions in $\hat{}$ are the conservation laws and the irrotationality condition. The last four natural conditions are space and time boundary conditions. The first of these is a condition on the normal component of mass flow on Σ and the second is a condition on the velocity potential on Σ , both for ( ). The remaining conditions are for depth and velocity potential, in and respectively, at the initial and final times. These last conditions are not desirable in a practical sense since they require knowledge of the solution at the final time.

Consider now the 'r' functional — that with integrand (3.30). The domain and domain boundary are again divided into two, as for the 'p' principle, to provide a choice of boundary conditions. Using the same functions, = ( ),1p1whic0tJM'+])

un'M+[]V++[)6+]J9[ndF˜[J3fr6'6D

The natural conditions of the 'r' principle $\delta I_2 = 0$ may be deduced using (3.39)–(3.44) and are given by

$$\left.\begin{array}{rcl} r_d + gh - \phi_t &=& 0 \\ r_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \hat{\ } ,$$

$$\begin{array}{rcll} C - \mathbf{Q.n} &=& 0 & \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\ f - \phi &=& 0 & \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\ d|_{t_i} - g_i &=& 0 & \text{in } \ _d \text{ for } i = 1, 2, \\ \phi|_{t_i} - h_i &=& 0 & \text{in } \ _\phi \text{ for } i = 1, 2. \end{array}$$

The first two natural conditions in the domain may be rewritten as

$$-\frac{1}{2}\frac{\mathbf{Q.Q}}{d^2} - gd + gh - \phi_t = 0 \ , \ \frac{\mathbf{Q}}{d} - \boldsymbol{\nabla}\phi = \mathbf{0}$$

so that, using (2.25), the equations of motion and the irrotationality condition have again been derived. The boundary conditions are identical to those of the 'p' principle.

Thus there exist two functionals, (3.68) and (3.69), whose natural conditions of the first variation are the equations of motion in the domain of the problem together with prescribed conditions on mass flow and velocity potential on the boundary of the domain, and conditions on the depth and velocity potential over regions of the domain at the initial and final times.

3.5.2 A Quartet of Functionals

A sequence of Legendre transforms can be used to generate a quartet of functionals which have as natural conditions of their first variations the equations of time-dependent motion in shallow water. Two such functionals — based on the and functions — have already been described and were independently derived from the ‘pressure’ and ‘Hamilton’ functionals for three-dimensional free surface flows. Two further functionals are now sought.

By applying the divergence theorem and integration by parts, the ‘p’ functional (3.68) can be expressed in the form

$$= \quad ((\quad) \quad + \quad + \quad + (\; + \quad))$$

$$+ \quad (\quad) \; \Sigma \qquad \Sigma$$

$$((\quad)) \quad ((\quad))$$

(3.70)

Comparing this with , as given by (3.69), suggests that there is a relationship between the two functions ( ) and ( ) such that

$$(\quad) = (\quad) \quad +$$

$$- \quad -$$

Notice that is equal to the total energy of a fluid particle.

The function is also a Legendre transform of ( ), with active and passive, in that, using (2.25), we may write

$$(\quad) = \qquad (\quad) \qquad (3\ 73)$$

with the first derivatives

$$= \qquad =$$

quations (3.71) and (3.73) imply the required connection, that

$$(\quad) = \qquad (\quad) = (\quad) \qquad +$$

in value.

The intermediate function can be bypassed and and may be connected directly by a Legendre transform. Since = and = , then if and are both active variables, the transformation of is

$$(\quad) = \qquad + (\quad) \qquad (3\ 74)$$

and

$$= \qquad =$$

A fourth function ( ) completes a closed quartet of functions related by Legendre transforms and is derivable from , and by using appropriate active variables. cannot be given explicitly, but is defined by eliminating and from

$$(\quad) = - \qquad -$$

$$r(\quad , d) - P(\quad , E) \quad = \quad -Ed. \tag{3.77}$$

The functions P and R can be used as bases for functionals, the natural conditions of the first variations of which include the equations of motion in shallow water. The functionals may be generated by substituting for p and r in the integrands of (3.68) and (3.69). The process is to use (3.76) to substitute for p in the integrand of (3.68) and (3.73) to substitute for r in the integrand of (3.69) by what is essentially a change of variables using the definitions of E and $\quad$, (2.19) and (2.25). Although (3.71) could be used to substitute for p in (3.68) and (3.77) could be used to substitute for r in (3.69) it would not change the nature of the functionals being generated. For instance integration by parts and the divergence theorem can be used on the 'P' functional generated by substituting (3.77) into (3.69) to give the functional formed by substituting (3.76) into (3.68).

Let the functional $I \, (E, \quad , d, \phi)$ be defined by

$$\begin{aligned} I \; = & \quad (P(\quad , E) - \quad . \quad \phi - d\,(\phi \; + E - gh)) \; dx \, dy \, dt \\ + & \quad C\phi \, d\Sigma \, dt + \quad (\phi - f) \quad . \quad d\Sigma \, dt \\ + & \quad (d\,(\phi - h \;))| \quad - (d\,(\phi - h \;))| \quad dx \, dy \\ + & \quad \phi| \quad g \; - \phi| \quad g \quad dx \, dy. \end{aligned} \tag{3.78}$$

Then the natural conditions of the 'P' principle $\delta I \; = 0$ are

$$\begin{aligned} P \; - \quad \phi \; &= \\ P \; - d \; &= \; 0 \\ \phi \; + E - gh \; &= \; 0 \\ d \; + \quad . \quad &= \; 0 \end{aligned} \qquad \text{in } \hat{\ } ,$$

$$
\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\
f - \phi &= 0 \qquad \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\
d|_{t_i} - g_i &= 0 \qquad \text{in } {}_d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 \qquad \text{in } {}_\phi \text{ for } i = 1, 2.
\end{aligned}
$$

The first condition in $\hat{}$ is

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0}.$$

Thus if equations (2.19) and (2.25) are assumed, the 'P' principle yields the conservation laws and the irrotationality condition as natural conditions in $\hat{}$, and gives the same boundary conditions on ϕ and $\mathbf{Q}$ at space boundaries and on d and ϕ at time boundaries as are obtained from the 'p' and 'r' principles.

The 'R' Principle

Now consider a principle based on the function R. Let the functional $I_4(\mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$
\begin{aligned}
I_4 = & \int_{t_1}^{t_2} \iint_D \left(-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + gdh + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\right)\, dx\, dy\, dt \\
& + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\; d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\
& - \iint_{D_d} \left((\phi\,(d - g_2))|_{t_2} - (\phi\,(d - g_1))|_{t_1} \right)\, dx\, dy \\
& - \iint_{D_\phi} \left(d|_{t_2} h_2 - d|_{t_1} h_1 \right)\, dx\, dy. \qquad (3.79)
\end{aligned}
$$

The natural conditions of the 'R' principle $\delta I_4 = 0$ are

$$
\left.
\begin{aligned}
-R_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\
-R_d + gh - \phi_t &= 0 \\
\mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\
d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0
\end{aligned}
\right\} \quad \text{in } \hat{}\,,
$$

$$C - \quad . \quad = \quad 0 \qquad \text{on } \Sigma_Q \text{ for } t \in (t\ \ , t\ \ ),$$

$$f - \phi \quad = \quad 0 \qquad \text{on } \Sigma_\phi \text{ for } t \in (t\ \ , t\ \ ),$$

$$d|_t - g_i \quad = \quad 0 \qquad \text{in } \ _d \text{ for } i = 1, 2,$$

$$\phi|_t - h_i \quad = \quad 0 \qquad \text{in } \ _\phi \text{ for } i = 1, 2.$$

The first two conditions in $\hat{\ }$ may be written as

$$-d \ + \quad = \quad \text{and} \quad - gd - \frac{1}{2}\ . \ + gh - \phi \ = 0.$$

Thus the natural conditions of the 'R' principle include the equations of motion in shallow water and the same boundary conditions as obtained previously from the 'p', 'r' and 'P' principles.

So there exists a quartet of functionals (3.68), (3.69), (3.78) and (3.79), based on the four functions p, r, P and R, from which the shallow water equations can be derived as the natural conditions of the first variations. Figure 3.3 shows the relations between the p, r, P and R functions.

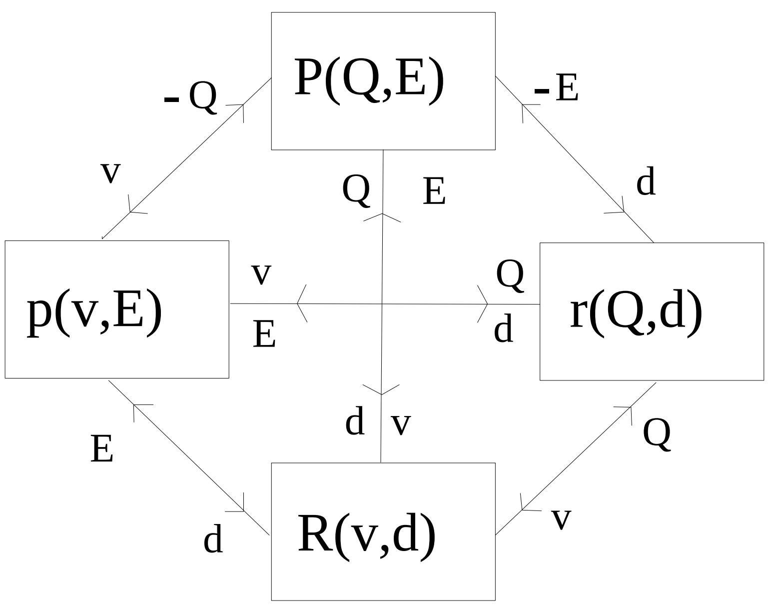


Figure 3.3: A quartet of Legendre transforms.

3.5.3 Constrained and Reciprocal Principles

Variational principles can be constrained by allowing only variations which satisfy one or more of the natural conditions. The principles constrained in this way will have the remaining natural conditions as natural conditions (Courant and Hilbert (1953)). There are several ways in which the variations of the 'p', 'r', 'P' and 'R' principles of Sections 3.5.1 and 3.5.2 can be constrained; only those constraints which produce variational principles which are reciprocal, in a sense to be defined shortly, are studied here.

Reciprocal 'p' and 'r' Principles

The functional (3.68), used in the 'p' principle, has an integrand which contains the integrated conservation of momentum equation and the irrotationality condition explicitly. It seems natural to constrain the 'p' principle to satisfy these two conditions. This can be done by specifying

$$\left.\begin{aligned} E &= -\phi_t + gh \\ \mathbf{v} &= \boldsymbol{\nabla}\phi \end{aligned}\right\}, \tag{3.80}$$

which results in the functional I_1 reducing to a functional $I_1^c(\mathbf{Q}, d, \phi)$. The constrained principle is given by

$$\begin{aligned} \delta I_1^c = \delta \Bigg\{ &\int_{t_1}^{t_2} \iint_D \hat{p}(\phi)\, dx\, dy\, dt + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt \\ &+ \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt + \iint_{D_\phi} \Big((d\,(\phi - h_2))|_{t_2} - (d\,(\phi - h_1))|_{t_1} \Big)\, dx\, dy \\ &+ \iint_{D_d} \Big(\phi|_{t_2} g_2 - \phi|_{t_1} g_1 \Big)\, dx\, dy \Bigg\} = 0, \end{aligned} \tag{3.81}$$

where

$$\hat{p}(\phi) = p(\boldsymbol{\nabla}\phi, -\phi_t + gh) = \frac{1}{2g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)^2.$$

The natural conditions are

$$\frac{1}{\ } \qquad + \frac{1}{2} \qquad + \qquad \frac{1}{\ } \qquad + \frac{1}{2} \qquad = 0$$

$$\text{in } \hat{\ }$$

$$\frac{1}{\ } \qquad + \frac{1}{2} \qquad + \qquad = 0 \qquad \text{on } \Sigma \quad \text{for} \quad (\qquad)$$

$$\frac{1}{\ } \qquad + \frac{1}{2} \qquad + \qquad = 0 \qquad \text{on } \Sigma \quad \text{for} \quad (\qquad)$$

$$= 0 \qquad \text{on } \Sigma \quad \text{for} \quad (\qquad)$$

$$\frac{1}{\ } \qquad + \frac{1}{2} \qquad + \qquad = 0 \qquad \text{in} \qquad \text{for} \quad = 1 \ \ 2$$

$$= 0 \qquad \text{in} \qquad \text{for} \quad = 1 \ \ 2$$

$$\frac{1}{\ } \qquad + \frac{1}{2} \qquad + \qquad = 0 \qquad \text{in} \qquad \text{for} \quad = 1 \ \ 2$$

the first of which may be r[6MV4˙t˜+99˙t)2FtV+V[D'4+MDM4isedtV+V[D'4˜06VstV+V[D'4Σts

$$(\hat{\ }(\ \ ) + \qquad) \qquad + \qquad (\ \ + \qquad) \ \Sigma$$

$$+ \qquad \Sigma \qquad (\ (\qquad)) \qquad (\ (\qquad))$$

$$= 0 \tag{3.84}$$

where

$$\hat{\ }(\ \ ) = \ (\qquad) = \frac{1}{2}\overline{\qquad} \quad \frac{1}{2} \ (\qquad)$$

The natural conditions are

$$+ \qquad + \frac{1}{2} \qquad = \qquad \text{in } \hat{\ }$$

$$\begin{aligned}
+ \quad &= 0 \quad \text{on } \Sigma \quad \text{for} \quad (\qquad) \\
\frac{1}{2} \qquad + \quad &= 0 \quad \text{on } \Sigma \quad \text{for} \quad (\qquad) \\
\frac{1}{2} \qquad + \quad &= 0 \quad \text{on } \Sigma \quad \text{for} \quad (\qquad) \\
&= 0 \quad \text{in} \quad \text{for} \quad = 1 \ \ 2 \\
+ \quad &= \qquad \text{in} \quad \text{for} \quad = 1 \ \ 2 \\
+ \quad &= \qquad \text{in} \quad \text{for} \quad = 1 \ \ 2 \\
&= 0 \quad \text{on } \Sigma \quad \Sigma \quad \text{for} \quad = 1 \ \ 2 \\
&= 0 \quad \text{on } \Sigma \quad \Sigma \quad \text{for} \quad = 1 \ \ 2 \\
&= 0 \quad \text{on } \Sigma \quad \Sigma \quad \text{for} \quad = 1 \ \ 2 \\
&= 0 \quad \text{on } \Sigma \quad \Sigma \quad \text{for} \quad = 1 \ \ 2
\end{aligned}$$

condition can be derived as a consequence of the conservation of momentum and the boundary conditions which specify that the flow is irrotational at = .

If Σ = Σ and = , (3.84) reduces to a variational principle involving a functional of alone, namely,

$$(\hat{}(\;) + \quad) \quad + \quad \Sigma$$

$$= 0 \qquad (3.85)$$

constraining the 'P' and 'R' principles and the functionals cannot be reduced to depend on one variable. However, the following structure can be deduced.

Consider the 'P' functional (3.78). Let $\Sigma_Q = \Sigma$ and $\ _d = \ $, and constrain the variables to satisfy conservation of momentum using the first of (3.80). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \left(P(\mathbf{Q}, -\phi_t + gh) - \mathbf{Q}.\boldsymbol{\nabla}\phi\right) dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma C\phi\,d\Sigma\,dt + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1\right) dx\,dy\right\} = 0, \qquad (3.86)$$

where the variables are $\mathbf{Q}$ and ϕ. The natural conditions are given by

$$\left.\begin{array}{rcl} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ (P_{-\phi_t+gh})_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \hat{\ },$$

$$\begin{array}{rcll} C - \mathbf{Q}.\mathbf{n} &=& 0 & \text{on } \Sigma \text{ for } t \in (t_1, t_2), \\ g_i - P_{-\phi_t+gh}|_{t_i} &=& 0 & \text{in } \quad \text{for } i = 1, 2. \end{array}$$

The first two conditions may be rewritten as

$$\left.\begin{array}{rcl} \mathbf{v} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \hat{\ },$$

which are the irrotationality condition and the conservation of mass equation.

In the 'R' functional (3.79) let $\Sigma_\phi = \Sigma$ and $\ _\phi = \ $ and constrain the variations to satisfy conservation of mass by imposing (3.83). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \left(-R(\mathbf{v}, \boldsymbol{\nabla}.\boldsymbol{\psi}) - \boldsymbol{\psi}_t.\mathbf{v} + g\boldsymbol{\nabla}.\boldsymbol{\psi}h\right) dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma f\boldsymbol{\psi}_t.\mathbf{n}\,d\Sigma\,dt - \iint_D \left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right) dx\,dy\right\} = 0, \quad (3.87)$$

which involves a functional of and . The natural conditions are given by

$$= \quad \text{in } \hat{} $$
$$+ \quad =$$

$$+ \quad = 0 \quad \text{on } \Sigma \text{ for } \quad (\quad)$$

$$= \quad \text{in} \quad \text{for} \quad = 1 \; 2$$

$$= 0 \quad \text{on } \Sigma \text{ for} \quad = 1 \; 2$$

The first two equations are

$$(\quad) \quad = \quad \text{in } \hat{}$$
$$+ \quad + - \quad =$$

the second of which is conservation of momentum. This, together with the natural condition in for , implies the irrotationality condition in for [ ].

The constrained 'P' and 'R' principles (3.86) and (3.87) are reciprocal since

In Chapter 2 the equations of motion for time-independent shallow water flows are and give time boundary conditions on in , and from (3.88) we know that

$$= \tilde{}(\quad) = \tilde{}$$

where $=$. Therefore and must be specified so that $= \tilde{}$ for consistency.

First consider the 'p' principle. The functional , given by (3.68), under steady state conditions becomes

$$= \quad (\quad) + \quad \tilde{} \quad + \quad \hat{} \quad \Sigma$$

$$+ \quad \tilde{} \quad \Sigma \quad \hat{}\ \tilde{}$$

where $= (\quad)$. To simplify this define

$$\hat{}(\quad) = \tilde{}(\quad) \quad \frac{1}{2} \tilde{}(\quad + \quad)$$

so that

$$\tilde{}(\quad) = \tilde{}(\quad) \quad \frac{1}{2} \tilde{} \quad = \quad \tilde{}(\quad) \quad \frac{1}{2} \tilde{}(\quad + \quad) = \hat{}(\quad)$$

Also, let

$$\hat{}(\quad) = \tilde{}(\quad) \quad \frac{1}{2} \tilde{}(\quad + \quad)$$

so that

$$(\quad) = \tilde{}(\quad) \quad \frac{1}{2} \tilde{} \quad = \quad \tilde{}(\quad) \quad \frac{1}{2} \tilde{}(\quad + \quad) = \hat{}(\quad)$$

and $\hat{} = \tilde{}$. Then

$$= \quad (\quad) + \quad \hat{} \quad + \quad \hat{}\ \hat{} \quad \Sigma$$

Notice that the final term in (3.89) is a constant and so it may be discarded. Also, throughout the functional there is a constant non-zero multiplier T which may be set equal to unity. Finally, for neatness, the ˆ notation is suppressed and the ‘p’ functional for use in the steady state variational principle is written as

$$= \quad ((\quad) + \quad (\quad)) \quad + \quad \Sigma + \quad (\quad) \quad \Sigma \qquad (3\ 90)$$

where $= \ (\quad)$ and is the known function $= \tilde{\ } +$, for consistency with conservation of momentum (2.42).

By the same process steady state forms of (3.69), (3.78) and (3.79) can be generated, and using the method by which (3.90) was deduced from (3.89) the steady state ‘r’, ‘P’ and ‘R’ functionals may be written

$$= \quad ((\quad) + \quad + \quad)$$

$$+ \quad (\quad) \ \Sigma \qquad \Sigma \qquad (3.91)$$

$$= \quad (\ (\quad) + \quad)$$

$$+ \quad (\quad) \ \Sigma \qquad \Sigma \qquad (3.92)$$

$$= \quad (\quad (\quad) + \quad + \quad + \quad)$$

$$+ \quad (\quad) \ \Sigma \qquad \Sigma \qquad (3.93)$$

where $= \ (\quad)$, $= \ (\quad)$ and $= \ (\quad)$.

The natural conditions of the steady state variational principles $= 0$, $= 0$, $= 0$ and $= 0$ are expected to include the shallow water equations of motion (2.43) and (2.48) and possibly (2.25) or (2.19). quation (2.42) is satisfied exactly since the energy is regarded as a known function $= \tilde{\ } +$, where $\tilde{\ }$ is a given constant.

The natural conditions of $\delta L \quad = 0$, the ‘p’ principle, are

$$\left.\begin{aligned} p \; + \quad\; &= \\ - \quad \phi \; &= \\ . \quad &= \; 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$\begin{aligned} C - \quad . \quad &= \; 0 \qquad \text{on } \Sigma_Q, \\ f - \phi \; &= \; 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$-\frac{}{g}\left(E - \frac{1}{2} \quad . \quad\right) + \quad = \qquad \text{in} \quad .$$

The natural conditions of $\delta L \quad = 0$, the ‘r’ principle, are

$$\left.\begin{aligned} r \quad - \quad \phi \; &= \\ r_d + E \; &= \; 0 \\ . \quad &= \; 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$\begin{aligned} C - \quad . \quad &= \; 0 \qquad \text{on } \Sigma_Q, \\ f - \phi \; &= \; 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} \frac{}{d} - \quad \phi \; &= \\ -\frac{}{} \frac{}{} - gd + E \; &= \; 0 \end{aligned}\right\} \qquad \text{in} \quad .$$

The natural conditions of $\delta L \quad = 0$, the ‘P’ principle, are

$$\left.\begin{aligned} P \quad - \quad \phi \; &= \\ . \quad &= \; 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$\begin{aligned} C - \quad . \quad &= \; 0 \qquad \text{on } \Sigma \quad , \\ f - \phi \; &= \; 0 \qquad \text{on } \Sigma \quad , \end{aligned}$$

the first equation being

$$= \qquad \text{in}$$

where is a function of and using (2.25) and (2.19).

The natural conditions of = 0, the 'R' principle, are

$$\begin{aligned} + \quad &= \\ + \quad &= 0 \\ &= \\ &= 0 \end{aligned} \qquad \text{in}$$

$$= 0 \qquad \text{on } \Sigma$$

$$= 0 \qquad \text{on } \Sigma$$

the first two equations being

$$\begin{aligned} + \quad &= \\ - \quad + \quad &= 0 \end{aligned} \qquad \text{in}$$

Thus the natural conditions of the variational principles for steady state motion, derived from the principles for unsteady motion, include the steady state equations in shallow water — (2.43) and (2.48). In order that the equations are expressed in the forms of (2.43) and (2.48) it is necessary to assume in the 'p' principle that = -(-), in the 'r' principle that = — and in the 'P' principle that = + - and = .

is dependent on the addition of appropriate boundary terms to the free surface functionals for unsteady flow in the same way that boundary terms were added to the functionals for unsteady flow in shallow water in Section 3.5.

3.6.2 Constrained and Reciprocal Principles

The variational principles for steady motion can be constrained in the same way as the ones for unsteady motion were in Section 3.5. The variational principles for time-dependent motion have three natural conditions which can be used as constraints — singly or in pairs — conservation of mass, conservation of momentum and the irrotationality condition. For the variational principles for steady flow there are just two — conservation of mass and the irrotationality condition — since conservation of momentum is satisfied implicitly.

Reciprocal 'p' and 'P' Principles

Consider the integrands of the functionals (3.90)–(3.93). In Section 3.5 emphasis was placed on the structure of the integrands of the 'p' and 'r' functionals — they were expressed as a function of $(\mathbf{Q}, d)$ or $(\mathbf{v}, E)$ plus a multiple of a conservation law or the irrotationality condition. For steady flows the 'p' and 'P' functionals exhibit a similar property, that is, the integrands can be expressed as functions of $\mathbf{Q}$ or $\mathbf{v}$ plus a multiple of conservation of mass or the irrotationality condition, since E is a known function. Thus the 'p' principle will be constrained by irrotationality and the 'P' principle by conservation of mass.

Let $\Sigma_Q = \Sigma$. Then the 'p' principle constrained by irrotationality is a functional of ϕ alone and is given by

$$\delta\left\{\iint_D p(\boldsymbol{\nabla}\phi, E)\,dx\,dy + \int_\Sigma C\phi\,d\Sigma\right\} = 0, \tag{3.94}$$

with the natural conditions

$$\begin{aligned}\nabla.\left(\frac{1}{g}\left(E-\frac{1}{2}\nabla\phi.\nabla\phi\right)\nabla\phi\right) &= 0 \qquad \text{in} \quad , \\ \frac{1}{g}\left(E-\frac{1}{2}\nabla\phi.\nabla\phi\right)\nabla\phi.\mathbf{n}-C &= 0 \qquad \text{on } \Sigma,\end{aligned}$$

the first of which is conservation of mass in .

Let $\Sigma_Q = \Sigma$ and $\mathbf{Q}.\mathbf{n} = C$ on Σ. Then the constrained 'P' principle is a functional of $\mathbf{Q}$ alone and is given by

$$\delta\left\{\iint_D P(\mathbf{Q},E)\,dx\,dy\right\} = 0, \tag{3.95}$$

where $\mathbf{Q}$ is constrained by $\nabla.\mathbf{Q} = 0$ in and by $\mathbf{Q}.\mathbf{n} = C$ on Σ. The variational principle (3.95) has as its only natural condition the irrotationality condition.

Thus the 'p' and 'P' steady principles display the same relationship as the 'p' and 'r' principles for unsteady flow (3.82) and (3.85) — they are both functionals of one variable and are reciprocal in the sense defined earlier.

The particular relationship between the 'p' and 'r' principles (3.82) and (3.85) for unsteady flow has not survived the transition to principles for steady flow. The 'p' and 'r' functionals (3.90) and (3.91) cannot be constrained so that they each depend on just one function and have reciprocal constraints and natural conditions. The relationship of the 'p' and 'r' principles in unsteady motion is a result of the integrands being expressible in the form

$$\text{function of } (\mathbf{Q},d) \text{ or } (\mathbf{v},E) + \text{multiplier} \times \text{conservation law}$$

and once the variables are constrained to satisfy the relevant conservation law and, in the case of the 'p' principle, irrotationality, the functions of $(\mathbf{Q},d)$ or $(\mathbf{v},E)$ can each be written in terms of one variable. In the steady motion functional

(3.90) the pressure function p is still expressed as a function of $\mathbf{v}$ and E but, since E is a known function and no longer a variable of the problem, p is in fact a function of $\mathbf{v}$ alone. The flow stress P is also a function of one variable so that the constrained 'p' and 'P' principles for steady motion exhibit the same relationship as the constrained 'p' and 'r' principles for unsteady motion in terms of being reciprocal and depending on just one variable.

For the case of constant equilibrium depth h, where the energy E is a constant, the constrained 'p' and 'P' principles (3.94) and (3.95) are examples of Bateman's functions (Bateman (1929)), using the gas dynamics analogy.

Reciprocal 'r' and 'R' Principles

The function r depends on $\mathbf{Q}$ and d and cannot be written as a function of one variable by requiring the irrotationality condition or conservation of mass to be satisfied. The function R also depends on two variables and cannot be reduced to a function of one variable. However the 'r' and 'R' principles for steady motion can be constrained to give reciprocal principles.

Let $\Sigma_Q = \Sigma$. Then constraining the 'R' principle to satisfy the irrotationality condition gives

$$\delta\left\{\iint_D (-R(\boldsymbol{\nabla}\phi, d) + Ed)\, dx\, dy + \int_\Sigma C\phi\, d\Sigma\right\} = 0 \qquad (3.96)$$

which depends on ϕ and d. The variational principle (3.96) has natural conditions

$$\left.\begin{aligned} -R_d + E &= 0 \\ \boldsymbol{\nabla}.\left(R_{\boldsymbol{\nabla}\phi}\right) &= 0 \end{aligned}\right\} \quad \text{in} \quad ,$$

$$R_{\boldsymbol{\nabla}\phi}.\mathbf{n} - C = 0 \qquad \text{on } \Sigma,$$

the second of which is conservation of mass in the form

$$(\quad) = 0$$

Let $\Sigma \quad = \Sigma$ and $\quad = \quad$ on Σ. Then constraining the 'r' principle to satisfy conservation of mass gives

$$(\,(\quad) + \quad) \qquad = 0 \qquad (3\ 97)$$

where $\quad = 0$, which has as natural conditions

$$+ \quad = 0$$

and the irrotationality condition in .

The 'r' and 'R' principles are reciprocal since the constraint of one principle is a natural condition of the other. Unlike the constrained 'p' and 'P' principles

The domain of the problem is the channel

$$= \;(\quad) : \quad [\quad]; \qquad \frac{(\;)}{2}\;\frac{(\;)}{2}$$

as in Section 2.3.

Consider the functionals (3.68), (3.69), (3.78) and (3.79) for time-dependent shallow water flows. Corresponding functionals for quasi one-dimensional flows may be derived from these by a'9+J)a'9+˙HV):FiF[+6)2.3060gromF˜060˜0"+r th˜060˜0"+derivswvsr

–

(3.78) and (3.79) the boundary functions are = ( ) and = ( )

for = 1 2 which, in the natural conditions of the variational principles, yield boundary conditions for and respectively at and . Let = so that in the one-dimensional case only the boundary function for , given by = ( ) in for = 1 2, exists.

Making the above substitutions into (3.68), (3.69), (3.78) and (3.79) and integrating with respect to yields the one-dimensional functionals ( ), ( ), ( ) and ( ) given by

For the ‘P’ principle $\delta K \;\; = 0$,

$$P \;\; - \phi \;\; = 0$$

$$P \;\; - d = 0 \qquad \text{in } (x \;, x \;) \text{ for } t \in (t \;, t \;),$$

$$\phi \; + E - gh = 0$$

$$d \; + -(BQ) \;\; = 0$$

$$C \quad Q \quad = \quad 0 \quad \text{for } t \quad (t \;, t \;),$$

$$C \quad Q \quad = \quad 0 \quad \text{for } t \quad (t \;, t \;),$$

$$g \quad d \quad = \quad 0 \quad \text{in } (x \;, x \;) \text{ for } i = 1, 2.$$

For the ‘R’ principle $\delta K \;\; = 0$,

$$R \;\; + Q = 0$$

$$R \quad \phi \; + gh = 0 \qquad \text{in } (x \;, x \;) \text{ for } t \quad (t \;, t \;),$$

$$v \quad \phi \;\; = 0$$

$$d \; + -(BQ) \;\; = 0$$

$$C \quad Q \quad = \quad 0 \quad \text{for } t \quad (t \;, t \;),$$

$$C \quad Q \quad = \quad 0 \quad \text{for } t \quad (t \;, t \;),$$

$$g \quad d \quad = \quad 0 \quad \text{in } (x \;, x \;) \text{ for } i = 1, 2.$$

Thus each set of natural conditions includes the equations of motion for time-dependent quasi one-dimensional shallow water flow. The conservation of mass equation (2.33) is explicit in each set. The conservation of momentum equation, in its integrated form, is explicit in the natural conditions of the ‘p’ and ‘P’ principles but for the ‘r’ principle the relations $Q = dv$ and $E = gd + -v$ are needed and for the ‘R’ principle the relation $E = gd + -v$ is required. The

The natural conditions of $\delta K_1^c = 0$ are

$$\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\right)_t + \frac{1}{B}\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x B\right)_x = 0 \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned}
C_o + \left(\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x\right)\Bigg|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2),\\
C_e + \left(\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x\right)\Bigg|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2),\\
g_i + \frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\Bigg|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2,
\end{aligned}$$

which correspond to conservation of mass in the domain and boundary conditions on ϕ.

The 'r' principle, corresponding to the functional (3.99), may be constrained to satisfy conservation of mass by specifying

$$\left.\begin{aligned} d &= \tfrac{\psi_x}{B} \\ Q &= -\tfrac{\psi_t}{B} \end{aligned}\right\}, \tag{3.108}$$

for some function $\psi(x, t)$. The constrained 'r' principle depends on ψ and ϕ and is given by

$$\begin{aligned}
\delta K_2^c = \delta \Bigg\{ &\int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(\hat{r}(\psi) + g\frac{\psi_x}{B}h\right) B\,dx\,dt \\
&+ \int_{t_1}^{t_2} \left(\left(B\phi\left(C_o + \frac{\psi_t}{B}\right)\right)\Bigg|_{x_o} - \left(B\phi\left(C_e + \frac{\psi_t}{B}\right)\right)\Bigg|_{x_e}\right) dt \\
&- \int_{x_e}^{x_o} \left(\left(\phi\left(\frac{\psi_x}{B} - g_2\right)\right)\Bigg|_{t_2} - \left(\phi\left(\frac{\psi_x}{B} - g_1\right)\right)\Bigg|_{t_1}\right) B\,dx \Bigg\} = 0,
\end{aligned} \tag{3.109}$$

where

$$\hat{r}(\psi) = \frac{1}{2}\left(\frac{\psi_t^2}{\psi_x B} - g\frac{\psi_x^2}{B^2}\right).$$

The natural conditions of (3.109) are given by

$$-\Psi_t + \left(\frac{1}{2}\Psi^2 + g\frac{\psi_x}{B} - gh\right)_x = 0 \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$+ \text{—} \quad = 0 \quad \text{for} \quad (\quad)$$

$$+ \text{—} \quad = 0 \quad \text{for} \quad (\quad)$$

$$\frac{1}{2}\Psi \quad \text{—} + \quad = 0 \quad \text{for} \quad (\quad)$$

$$\frac{1}{2}\Psi \quad \text{—} + \quad = 0 \quad \text{for} \quad (\quad)$$

$$\text{—} \quad = 0 \quad \text{in} \ (\quad) \ \text{for} \quad = 1 \ 2$$

$$(\quad + \Psi) \quad = 0 \quad \text{in} \ (\quad) \ \text{for} \quad = 1 \ 2$$

where $\Psi = \text{—}$. The first of these conditions is the conservation of momentum equation in the domain. The equation $=$ is not derived but is essentially redundant anyway in quasi one-dimensional flow.

The variational principles (3.107) and (3.109) are reciprocal in that the natural condition in of (3.107) is the constraint of (3.109) and the natural condition in

$$C \quad Q \quad = 0 \quad \text{for } t \quad (t \ , t \ ),$$

$$g \quad P \quad = 0 \quad \text{in } (x \ , x \ ) \text{ for } i = 1, 2,$$

the second of which is the equation of conservation of mass.

The ‘R’ principle constrained to satisfy conservation of mass by specifying (3.108) is

$$\delta K \ = \delta \qquad R(v, \frac{\psi}{B}) \quad \frac{\psi}{\ } + \frac{\ }{\ }$$

$$+ \qquad + \frac{\ }{\ } \qquad + \frac{\ }{\ }$$

$$\frac{\ }{\ } \qquad \frac{\ }{\ } \qquad = 0 \quad (3.111)$$

which has the natural conditions

$$\frac{\ }{\ } = 0 \qquad \text{in } (\quad) \text{ for } \quad (\quad)$$

$$\dot{\ } \quad + \quad = 0$$

$$+ \frac{\ }{\ } \quad = 0 \quad \text{for} \quad (\quad)$$

$$+ \frac{\ }{\ } \quad = 0 \quad \text{for} \quad (\quad)$$

$$\dot{\ } + \quad = 0 \quad \text{for} \quad (\quad)$$

$$\dot{\ } + \quad = 0 \quad \text{for} \quad (\quad)$$

$$\frac{\ }{\ } \quad = 0 \quad \text{in } (\quad) \text{ for } \quad = 1 \ 2$$

$$(\quad) \quad = 0 \quad \text{in } (\quad) \text{ for } \quad = 1 \ 2$$

the second of which is the equation of conservation of momentum.

Thus the constraint on (3.110) is one of the natural conditions of (3.111) and the constraint on (3.111) is one of the natural conditions of (3.110), that is, these constrained ‘P’ and ‘R’ principles are reciprocal in the sense defined earlier.

in order to satisfy conservation of momentum.

The natural conditions of $\delta M_1 = 0$, $\delta M_2 = 0$, $\delta M_3 = 0$ and $\delta M_4 = 0$ may be deduced from the general formulae (3.36) and (3.37) and are as follows.

For the 'p' principle $\delta M_1 = 0$,

$$\left.\begin{array}{r} p_v + Q = 0 \\ v - \phi' = 0 \\ (BQ)' = 0 \end{array}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'r' principle $\delta M_2 = 0$,

$$\left.\begin{array}{r} r_Q - \phi' = 0 \\ r_d + E = 0 \\ (BQ)' = 0 \end{array}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'P' principle $\delta M_3 = 0$,

$$\left.\begin{array}{r} P_Q - \phi' = 0 \\ (BQ)' = 0 \end{array}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'R' principle $\delta M = 0$,

$$R + Q = 0$$
$$R + E = 0 \qquad \text{in } (\quad)$$
$$v \quad \phi = 0$$
$$(BQ) = 0$$

$$(\ ) = 0$$
$$(\ ) = 0$$

Thus each set of natural conditions contains the conservation of mass equation, the formula $=$, expressed in different variables in the cases of the 'p' and 'r' principles, and also the boundary conditions $(\ ) =$ and $(\ ) =$ —, that is, the equations of motion, as required. Conservation of momentum is implicit from using $= \tilde{} +$.

The variational principles for steady quasi one-dimensional flows can be constrained in the same way as the two-dimensional versions.

The 'p' principle based on the functional (3.112) can be constrained to depend on only one variable by substituting $=$ into the integrand. The constrained 'p' principle is given by

$$= \quad (\quad) \quad + \quad (\ (\ ) \quad (\ )) = 0 \qquad (3\ 116)$$

where = ( ), which has the natural conditions

$$\frac{1}{} \quad \frac{1}{2} \quad = 0 \quad \text{in } (\quad)$$

$$\frac{1}{} \quad \frac{1}{2} \quad = 0$$

$$\frac{1}{} \quad \frac{1}{2} \quad = 0$$

the first of which is the conservation of mass equation.

The 'P' principle may be constrained to satisfy conservation of mass by specifying = in [ ]. The constrained 'P' principle is given by which has the natural conditions

$$\left.\begin{array}{r} (BR_{\phi'})' = 0 \\ \\ -R_d + E = 0 \end{array}\right\} \quad \text{in } (x_e, x_o),$$

$$CB_e - (BR_{\phi'})\big|_{x_e} \;=\; 0,$$

$$CB_e - (BR_{\phi'})\big|_{x_o} \;=\; 0,$$

that is, conservation of mass and the definition of E as a function of ϕ and d in the domain and boundary conditions on the mass flow.

The reciprocity of the constrained variational principles that occurred in two-dimensional flows and time-dependent one-dimensional flows has not survived to the steady one-dimensional case since there is essentially only the one equation (the conservation of mass equation) which can be used as a constraint.

3.8 Discontinuous Flows

The variational principles considered so far are only valid for continuous shallow water flows since, in deriving the natural conditions using integration by parts and the divergence theorem, the variables have been assumed to be differentiable. In this section variational principles for time-independent discontinuous flows in one and two dimensions are derived.

In the variational principles of Sections 3.6 and 3.7 for steady state shallow water flows the conservation of momentum equation is satisfied implicitly by specifying $E = \tilde{E} + gh$ in , where $\tilde{E}$ is a constant and E is the energy defined by either (2.35) or (2.19), depending on whether the flow being considered is one-dimensional or two-dimensional. For discontinuous flows there is a jump in the value of E on crossing the discontinuity and this property is used in generating the variational principles for discontinuous flows.

Let the domain be of the form

$$= \left\{(x, y) : x \in [x_e, x_o]; y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2}\right]\right\},$$

that is, a channel, where there is flow into the channel at $x = x_e$ and out of the channel at $x = x_o$.

Consider a discontinuous flow in which has energy $E = E_e$ at inlet and $E = E_o$ at outlet, where E_e and E_o are constants such that $E_e > E_o$. Let the channel bed be horizontal so that the undisturbed fluid depth h is a constant for $x \in [x_e, x_o]$. Then, for a time-independent flow, $E = E_e$ everywhere on the inlet side of the discontinuity and $E = E_o$ everywhere on the outlet side. Substituting these values for the energy into the functionals (3.112)–(3.115) and (3.90)–(3.93) and allowing the variables to be discontinuous yields functionals whose corresponding variational principles have as natural conditions the equations of discontinuous motion in one and two dimensions.

3.8.1 Two-dimensional Flows

Let Σ_s be the line in across which the flow is discontinuous. Assume that it divides into the two regions $_e$ and $_o$, where $_e$ borders the inlet boundary and $_o$ the outlet boundary.

The functionals for discontinuous flows in two dimensions, derived from (3.90)–(3.93), are

$$N_1 = \iint_{D_e} (p(\mathbf{v}, E_e) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi))\, dx\, dy$$
$$+ \iint_{D_o} (p(\mathbf{v}, E_o) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi))\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.118}$$
$$N_2 = \iint_{D_e} (r(\mathbf{Q}, d) + E_e d - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \iint_{D_o} (r(\mathbf{Q}, d) + E_o d - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.119}$$
$$N_3 = \iint_{D_e} (P(\mathbf{Q}, E_e) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \iint_{D_o} (P(\mathbf{Q}, E_o) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.120}$$
$$N_4 = \iint_{D_e} (-R(\mathbf{v}, d) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi) + E_e d)\, dx\, dy$$
$$+ \iint_{D_o} (-R(\mathbf{v}, d) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi) + E_o d)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.121}$$

where $N_1 = N_1(\mathbf{v}, \mathbf{Q}, \phi, \Sigma_s)$, $N_2 = N_2(d, \mathbf{Q}, \phi, \Sigma_s)$, $N_3 = N_3(\mathbf{Q}, \phi, \Sigma_s)$ and $N_4 = N_4(d, \mathbf{v}, \mathbf{Q}, \phi, \Sigma_s)$. The section of the boundary Σ_Q in the functionals (3.90)–(3.93) is taken to be Σ, the whole boundary of , and the boundary function C is set to zero on the parts of the boundary across which there is no flow; Σ_e is the inlet boundary and Σ_o is the outlet boundary.

The natural conditions of $\delta N_1 = 0$, $\delta N_2 = 0$, $\delta N_3 = 0$ and $\delta N_4 = 0$ may be deduced from the general formulae (3.60)–(3.67).

The natural conditions of the ‘p’ principle $\delta N_1 = 0$ are

$$\left.\begin{aligned} p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in} \quad {}_e \cup {}_o,$$

$$
\begin{aligned}
C - \quad . \quad &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\
\left[p + \quad . \quad - \quad . \ \frac{\partial \phi}{\partial \tau} \right] &= 0, \\
\left[\quad . \ \frac{\partial \phi}{\partial \tau} \right] &= 0, \\
[\quad . \quad] &= 0,
\end{aligned}
$$

where the brackets $[\cdot] \quad = \cdot| \quad - \cdot| \quad$, + denotes the downstream side of Σ_s and − the upstream side. The first four of these conditions are the same as for the continuous case but are valid separately in $\quad_e$ and $\quad_o$ and on Σ_e and Σ_o. Using the $\quad = \quad \phi$ natural condition the last three natural conditions can be rewritten as

$$
\begin{aligned}
[p + (\quad . \quad)(\quad . \quad)] &= 0, \\
[\quad . \quad] &= 0, \\
[\quad . \quad] &= 0,
\end{aligned}
$$

which are the required jump conditions (2.84)–(2.86) for discontinuous shallow water flows.

The natural conditions of the 'r' principle $\delta N \quad = 0$ are

$$
\left.
\begin{aligned}
r \quad - \quad \phi &= \\
r_d + E &= 0 \\
. \quad &= 0
\end{aligned}
\right\} \quad \text{in } \quad_e \cup \quad_o,
$$

$$
\begin{aligned}
C - \quad . \quad &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\
\left[r + Ed - \quad . \quad \phi + \quad . \ \frac{\partial \phi}{\partial n} \right] &= 0, \\
\left[\quad . \ \frac{\partial \phi}{\partial \tau} \right] &= 0, \\
[\quad . \quad] &= 0.
\end{aligned}
$$

The natural conditions of the 'P' principle $= 0$ are

$$= \quad \text{in}$$
$$= 0$$

$$= 0 \quad \text{on } \Sigma \quad \Sigma$$
$$+ \quad — \quad = 0$$
$$— \quad = 0$$
$$[\quad] = 0$$

The natural conditions of the 'R' principle $= 0$ are

$$+ \quad =$$
$$+ \quad = 0 \quad \text{in}$$
$$=$$
$$= 0$$

$$= 0 \quad \text{on } \Sigma \quad \Sigma$$
$$+ \quad (\quad) + \quad + \quad — \quad = 0$$
$$— \quad = 0$$
$$[\quad] = 0$$

Using the relationships between and , and , (3.74), (3.76) and (3.71), the last three natural conditions of $= 0$, $= 0$ and $= 0$ can be seen to be the same as the last three natural conditions of $= 0$. Thus

3.8.2 One-dimensional Flows

Let x_s be the position of the discontinuity in (x_e, x_o). Then the functionals for discontinuous flow in one dimension, derived from (3.112)–(3.115), are

$$\begin{aligned} S_1 = &\int_{x_e}^{x_s} \left(p(v, E_e) + Q\,(v - \phi')\right) B\, dx \\ &+ \int_{x_s}^{x_o} \left(p(v, E_o) + Q\,(v - \phi')\right) B\, dx \\ &+ CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.122}$$

$$\begin{aligned} S_2 = &\int_{x_e}^{x_s} \left(r(Q, d) + E_e d - Q\phi'\right) B\, dx \\ &+ \int_{x_s}^{x_o} \left(r(Q, d) + E_o d - Q\phi'\right) B\, dx \\ &+ CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.123}$$

$$\begin{aligned} S_3 = &\int_{x_e}^{x_s} \left(P(Q, E_e) - Q\phi'\right) B\, dx \\ &+ \int_{x_s}^{x_o} \left(P(Q, E_o) - Q\phi'\right) B\, dx \\ &+ CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.124}$$

$$\begin{aligned} S_4 = &\int_{x_e}^{x_s} \left(-R(v, d) + Q\,(v - \phi') + E_e d\right) B\, dx \\ &+ \int_{x_s}^{x_o} \left(-R(v, d) + Q\,(v - \phi') + E_o d\right) B\, dx \\ &+ CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.125}$$

where $S_1 = S_1(v, Q, \phi, x_s)$, $S_2 = S_2(d, Q, \phi, x_s)$, $S_3 = S_3(Q, \phi, x_s)$ and $S_4 = S_4(d, v, Q, \phi, x_s)$.

The natural conditions of $\delta S_1 = 0$, $\delta S_2 = 0$, $\delta S_3 = 0$ and $\delta S_4 = 0$ may be deduced using (3.54)–(3.57) and are as follows.

For the 'p' principle $\delta S_1 = 0$,

$$\left.\begin{aligned} p_v + Q &= 0 \\ v - \phi' &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0, \\ [BQ]_{x_s} &= 0, \\ [(p + Qv)\,B]_{x_s} &= 0, \end{aligned}$$

where $[\cdot]_{x_s} = \cdot|_{x_{s+}} - \cdot|_{x_{s}}$, the $+$ denotes the x_o side of x_s and the $-$ the x_e side; E is given the value E_e in $[x_e, x_s)$ and E_o in $(x_s, x_o]$.

For the 'r' principle $\delta S_2 = 0$,

$$\left.\begin{aligned} r_Q - \phi' = 0 \\ r_d + E = 0 \\ (BQ)' = 0 \end{aligned}\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0, \\ [BQ]_{x_s} &= 0, \\ [(r + Ed)\,B]_{x_s} &= 0. \end{aligned}$$

For the 'P' principle $\delta S_3 = 0$,

$$\left.\begin{aligned} P_Q - \phi' = 0 \\ (BQ)' = 0 \end{aligned}\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0, \\ [BQ]_{x} &= 0, \\ [BP]_{x} &= 0. \end{aligned}$$

For the 'R' principle $\delta S \ = 0$,

$$\left.\begin{aligned} -R_v + Q &= 0 \\ -R_d + E &= 0 \\ v - \phi \ &= 0 \\ (BQ) \ &= 0 \end{aligned}\right\} \quad \text{in } (x \ , x \ ) \cup (x \ , x \ ),$$

$$\begin{aligned} C - Q(x \ ) \quad &= \quad 0, \\ CB \ - Q(x \ )B \quad &= \quad 0, \\ [BQ] \quad &= \quad 0, \\ [(-R + Qv + Ed)\, B] \quad &= \quad 0. \end{aligned}$$

Thus the natural conditions include the equations of shallow water motion in $(x \ , x \ )$ and $(x \ , x \ )$ and boundary conditions on the mass flow at x and x . The first jump condition in each case is the condition of no jump in the mass flow (2.77). Using one-dimensional versions of the equations (3.74), (3.76) and (3.71), which relate p to r, P and R, the second jump condition in each case can be recognised as the condition of no jump in the value of the flow stress, defined by (2.72), on crossing the point of discontinuity x , that is condition (2.76).

Ch pter

Approxim tions to Qu si One-dimension l Sh llow W ter Flows

The variational principles of Section 3.7 have as natural conditions the equations of steady state quasi one-dimensional motion in shallow water. This chapter is concerned with using some of these variational principles to generate approximations to the variables of shallow water flows in channels.

The particular variational principles used here are the 'p' and 'r' principles based on the functionals (3.112) and (3.113). The constrained versions of these principles, (3.116) and (3.117), both depend on only one variable and are used to develop algorithms for generating approximations, defined on fixed grids, to the velocity and depth of flow in a channel. The constrained 'p' principle is also used to generate approximations on adaptive grids.

The final section of this chapter is concerned with deriving approximations to discontinuous flows in channels. The constrained 'r' principle is used to generate approximations to the depth on a grid with one moving node which is placed at the position of the discontinuity. This algorithm is extended to give a method for approximating discontinuous depth profiles on adaptive grids.

The domains of the problems to be considered are channels with breadth $B(x)$ for $x \in [x_e, x_o]$ and undisturbed fluid depth $h(x)$ for $x \in [x_e, x_o]$, where B and h are functions to be defined later.

4.1 The onstrained 'r' Principle

The 'r' principle based on the functional (3.113), with variations constrained to satisfy the conservation of mass equation, can be used to generate approximations to the depth of fluid for shallow water flow in a channel.

The functional of the constrained principle (3.117) is given by

$$M_2^c(d) = \int_{x_e}^{x_o} (r(Q,d) + Ed)\, B\, dx, \tag{4.1}$$

where Q and E are known functions of x, namely,

$$Q(x) = \frac{CB_e}{B(x)} \qquad \text{for } x \in [x_e, x_o], \tag{4.2}$$

from the conservation of mass constraint, and

$$E(x) = \tilde{E} + gh(x) \qquad \text{for } x \quad [x\ , x\ ], \tag{4.3}$$

corresponding to conservation of momentum. In practice the constants C and $\tilde{E}$ are calculated from given values of two of the three variables mass flow, depth and velocity at the inlet boundary. Given the values of two of these variables at $x = x$ the value of the third can be deduced from (2.34). Then, using (2.35),

$$C \;=\; Q(x\ )$$

$$\text{and} \qquad \tilde{E} \;=\; gd(x\ ) + \frac{1}{2}v(x\ ) \qquad gh(x\ ).$$

For a continuous flow to exist notice that C and $\tilde{E}$ must satisfy

$$CB \quad \frac{1}{g} \quad \frac{2\ \ \tilde{E} + gh(x)}{3} \qquad B(x) \qquad \text{for } x \quad [x\ , x\ ], \tag{4.4}$$

using (2.62) and (2.56).

The function d which satisfies $\delta M\ \ = 0$ is the depth of fluid in the channel. The nature of the stationary value of M can be determined by considering the second derivative

$$\frac{d\ M}{dd} = \qquad r \quad B\,dx.$$

From the definition of r, (3.103),

$$r \;\; = \frac{Q}{d} \qquad g, \tag{4.5}$$

which is positive if — $> gd$, that is (using (2.34)), if $v \;\; > gd$ and the flow is supercritical and negative if — , that is, if the flow is subcritical. Thus, if the flow is supercritical in the whole of [ ], the function satisfying $= 0$ minimises and, if the flow is subcritical in the whole of [ ], the stationary function d maximises M . If the flow is critical at isolated points in the channel then these statements still hold but, if both subcritical and supercritical flows exist in the channel, it is not possible to say whether the stationary function minimises or maximises M .

The method used here for generating approximations to d is to substitute into the functional (4.1) finite element expansions for d and to find the parameters of the expansions for which M is stationary with respect to variations in the parameters.

Let the interval [ ] be divided into 1 regular intervals by the points ... given by

$$= \frac{(\quad 1)}{(\quad 1)} (\qquad) + \qquad = 1 \; \ldots \qquad (4\ 6)$$

Let ( ) ... ( ) be finite element basis functions, defined on the grid given by (4.6), and let

$$(\ ) = \qquad (\ ) \qquad (4\ 7)$$

be the approximation to , where the (= 1 ...) are parameters of the solution, to be determined.

Consider the finite dimensional version of the functional (4.1),

$$(\ ) = \qquad (\qquad) +$$

where = (...) and and are given by (4.2) and (4.3).

The parameters for which (4.7) is an approximation to are such that is stationary with respect to variations in . They are found by solving the non-linear set of equations

$$(\ ) = \frac{}{} = \quad (\quad + \quad) \qquad = 0 \qquad = 1\ \ldots \qquad (4\ 8)$$

There is more than one solution of the set of equations (4.8). One possible solution involves negative values of and is not considered since it has no physical meaning. In the case of approximations to non-critical flows there are two other solutions — one which approximates subcritical flow and one which approximates supercritical flow. In the case of flows which become critical at a point in the domain there is a further possibility, that is, an approximation to transitional flow.

In the present work (4.8) is solved using Newton's method. The Jacobian is given by

$$(\ ) = \qquad = \frac{}{} = \frac{}{} = \qquad (4\ 9)$$

and is the Hessian of .

Given an approximation to the solution , Newton's method provides an updated approximation

$$= \quad + \qquad (4\ 10)$$

where

$$(\quad) \qquad = \quad (\quad) \qquad (4\ 11)$$

This yields a sequence of approximations to . The process is repeated until

$$\max \qquad \text{tolerance} \qquad (4\ 12)$$

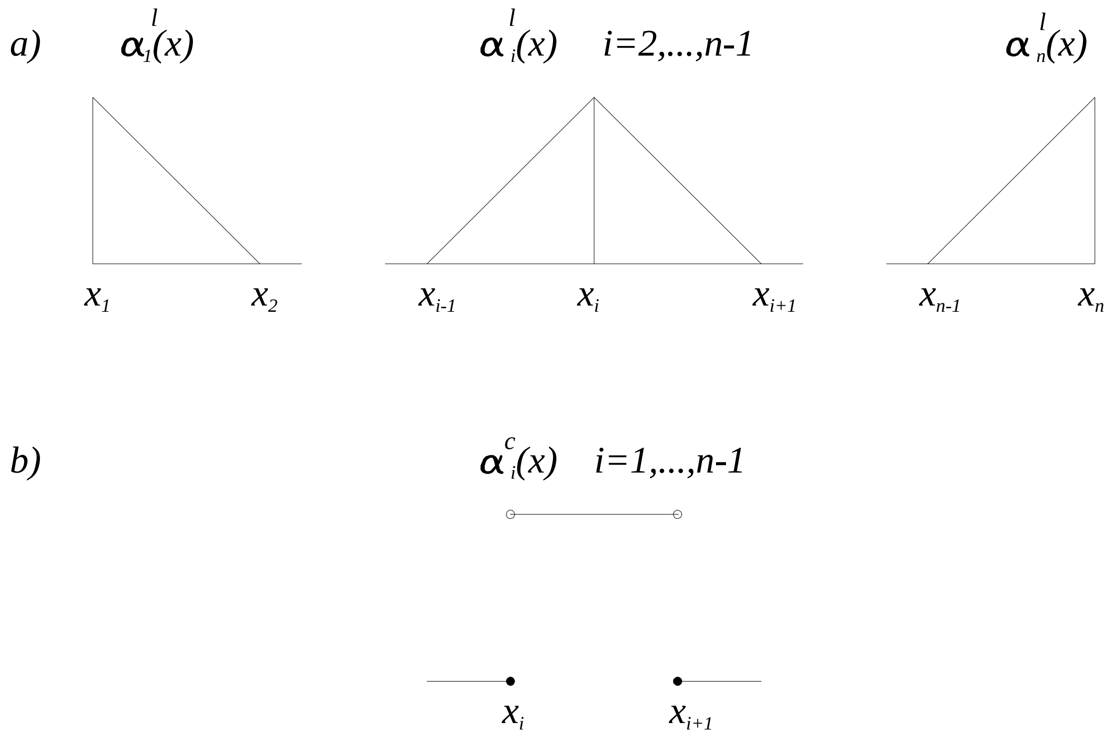
Then = for = 1 ... are Ox..of..at5Fs-40k05GnGIn6is[40F8leraf the0Lis bee0Limapk6

k05GL6ti8L5nGC50k05117kL58˜0GIGI55k06k8—L5CnRNL50k05dkL58588—05Ik8O8C50k0kGCL calculated using five point Gaussian quadrature, where it is assumed that the error introduced by the numerical integration is sufficiently small that the finite element solution, for a particular tolerance in (4.12), is unaffected.

Let ˆ satisfy (ˆ) = and let 0 be such that the domain = : ˆ contains the point ˆ, where is an appropriate norm. Assume that the first derivatives of are continuous in and that is non-singular in . Then, there exists 0 such that Newton's method is quadratically convergent whenever ˆ (Johnson and Riess (1982)).

From (4.9) has the form of a weighted mass matrix, where is the weight function. Using (4.5) it can be seen that, if the approximate solution in [ ] is subcritical throughout the Newton iteration, is negative definite and the solution of (4.8) maximises . Alternatively, if the approximate solution is supercritical in [ ] for all iterations, is positive definite and the solution of (4.8) minimises .

and supercritical motion and if an approximation, to the approximate solution, at an iteration step has both subcritical and supercritical values the Jacobian is indefinite and Newton's method may fail to converge to the solution.



a)
$\alpha_1^l(x)$
$\alpha_i^l(x)$ $i=2,\ldots,n-1$
$\alpha_n^l(x)$
x_1 x_2
x_{i-1} x_i x_{i+1}
x_{n-1} x_n
b)
$\alpha_i^c(x)$ $i=1,\ldots,n-1$
x_i x_{i+1}

The energy $\tilde{E}$ is given the value 50. In order to guarantee that a continuous solution exists the value of mass flow at inlet C must satisfy

$$C \leq \frac{1}{g}\left(\frac{2\left(\tilde{E}+gh(x)\right)}{3}\right)^{\frac{3}{2}}\frac{B(x)}{B_e} \qquad \text{in } [x_e, x_o],$$

from (4.4). For the case $h = h_1$ this is just

$$C \leq \frac{1}{g}\left(\frac{2\tilde{E}}{3}\right)^{\frac{3}{2}}\frac{B_{\text{min}}}{B_e},$$

where

$$B_{\text{min}} = \min_{x\in[x_e,x_o]} B(x).$$

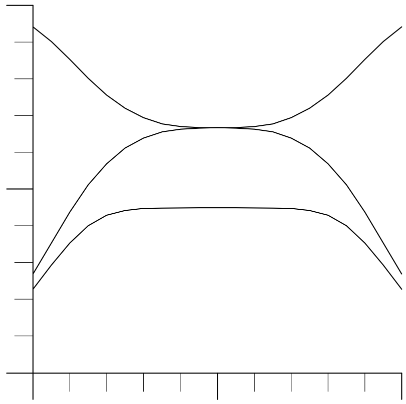
Thus, for the given breadth functions (4.15) and (4.16), C must have a value such that $C \leq C_*$, where

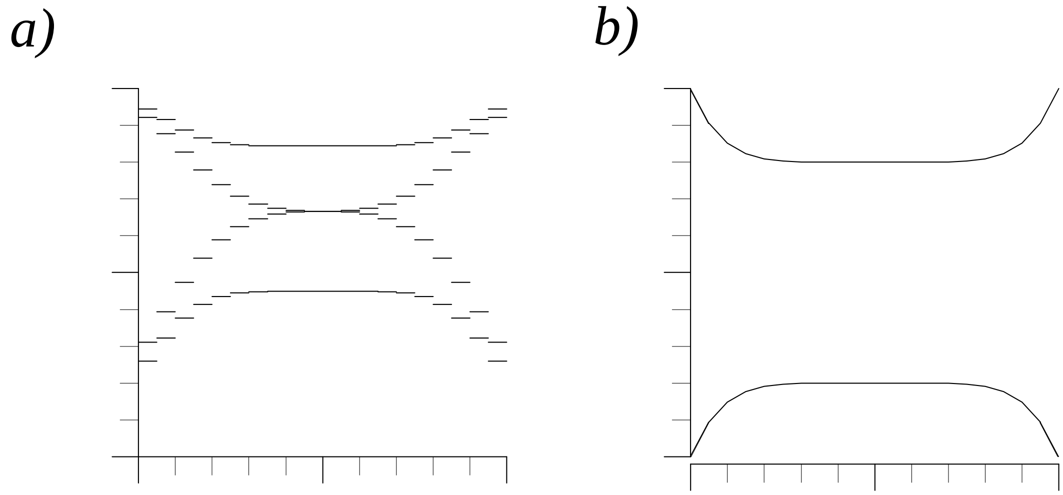
$$C_* = \frac{20}{\sqrt{3}}. \tag{4.20}$$

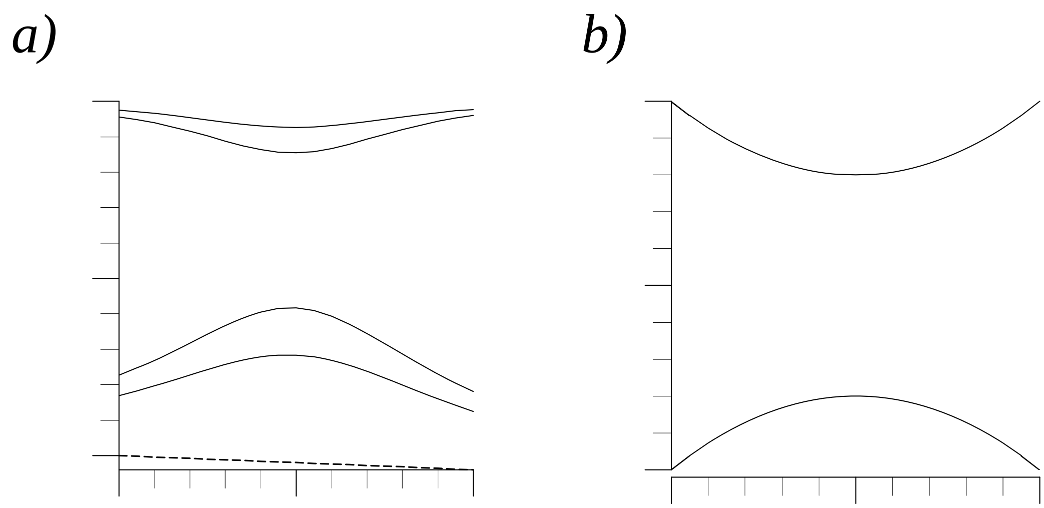
A value of $C = C_*$ yields flows which are critical at the point of minimum breadth. A value of $C = 10$ is used to give examples of non-critical flows.

The initial approximation $\mathbf{d}^0$ to the solution $\mathbf{d}$ determines whether the finite element solution is an approximation to subcritical or to supercritical flow. In practice subcritical approximations are obtained by specifying $d_i^0 > d_*$ for $i = 1, \ldots, n$, where d_* is the critical depth, given by (2.65). In this case, for $h = h_1$, $d_* = \frac{100}{30} \approx 3.33$. Supercritical approximations are obtained by specifying $d_i^0 < d_*$ for $i = 1, \ldots, n$. Transitional flows are not considered because of problems with the convergence of Newton's method with an indefinite Jacobian.

Let the tolerance on the Newton iteration be 10^{-3}. Consider the channel with breadth $B = B_{1,6}$. Using the piecewise linear basis functions (4.13) Newton's method converges to the supercritical approximation from the initial approximation $d_i^0 = 1$ for $i = 1, \ldots, n$ in 15 iterations for critical flow and 7 iterations




a)
b)


a)
b)

4.1.2 Error Bounds

An error bound for the piecewise constant approximations to d can be calculated.

Proposition The piecewise constant approximations, defined using (4.7) and (4.14) and generated from (4.8), converge linearly to the shallow water depth for wholly subcritical or wholly supercritical flows.

Proof The parameters of the approximation d^h are defined as those which satisfy (4.8), that is,

$$\int_{x_1}^{x_n} (r_{d^h} + E)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1. \tag{4.21}$$

The exact depth d satisfies the equation

$$r_d + E = 0,$$

from the definitions of r (3.103), mass flow (2.34) and energy (2.35). Thus

$$\int_{x_1}^{x_n} (r_d + E)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1. \tag{4.22}$$

Subtracting (4.22) from (4.21) gives

$$\int_{x_1}^{x_n} (r_{d^h} - r_d)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1,$$

and so

$$\int_{x_i}^{x_{i+1}} (r_{d^h} - r_d)\, B\, dx = 0 \qquad i = 1, \ldots, n-1, \tag{4.23}$$

using (4.14).

Both d and d^h are differentiable on each interval $[x_i, x_{i+1}]$ and thus, using the Mean Value Theorem,

$$r_{d^h} - r_d = \left(d^h - d\right) r_{\psi\psi}(Q, \psi)\big|_{\psi=\theta}, \tag{4.24}$$

for ( ) between ( ) and ( ), where = — , from (3.103). Thus if and are completely supercritical 0 and if and are completely subcritical 0 in [ ]. Therefore, substituting (4.24) into (4.23) to give

( ) = 0

implies that = 0 at at least one point (say = ˆ) in ( ) for completely subcritical or supercritical flows, since 0.

Now is constant on [ ] so, for [ ],

( ) = ( ) ( ) = ( ) ( ) = ( ) ( )

Thus

( ) ( ) = ( )

( ) max

= ( ) max

where max = max ( ) .

n	Δx	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	$\frac{10}{2}$	6.028×10^{-1}	8.608×10^{-1}	2.687×10^{-1}	5.097×10^{-1}
5	$\frac{10}{2^2}$	1.870×10^{-1}	3.521×10^{-1}	1.188×10^{-1}	2.654×10^{-1}
9	$\frac{10}{2^3}$	7.249×10^{-2}	1.550×10^{-1}	4.882×10^{-2}	1.218×10^{-1}
17	$\frac{10}{2^4}$	3.198×10^{-2}	7.249×10^{-2}	2.192×10^{-2}	5.772×10^{-2}
33	$\frac{10}{2^5}$	1.504×10^{-2}	3.504×10^{-2}	1.038×10^{-2}	2.805×10^{-2}
65	$\frac{10}{2^6}$	7.293×10^{-3}	1.722×10^{-2}	5.050×10^{-3}	1.382×10^{-2}
129	$\frac{10}{2^7}$	3.592×10^{-3}	8.539×10^{-3}	2.491×10^{-3}	6.860×10^{-3}
257	$\frac{10}{2^8}$	1.782×10^{-3}	4.252×10^{-3}	1.237×10^{-3}	3.417×10^{-3}
513	$\frac{10}{2^9}$	8.878×10^{-4}	2.121×10^{-3}	6.164×10^{-4}	1.705×10^{-3}
1025	$\frac{10}{2^{10}}$	4.431×10^{-4}	1.060×10^{-3}	3.077×10^{-4}	8.520×10^{-4}

Table 4.1: L_2 errors for piecewise constant depth approximations.

Thus the piecewise constant depth approximation converges linearly with n to the solution d.

The L_2 error is calculated for piecewise constant approximations on grids with different numbers of nodes for the example $B = B_{1,2}$, defined by (4.15), and $h = h_1$, defined by (4.17). The energy $\tilde{E} = 50$ and both $C = C_*$, defined by (4.20), and $C = 10$ are considered. The results are given in Table 4.1, from which it can be seen, more especially for larger n, that as the interval length Δx halves the L_2 error also halves.

The L_2 errors for the corresponding piecewise linear approximations are given in Table 4.2. It can be seen that the convergence is almost quadratic.

		critical flows		non-critical flows	
	Δ	subcritical	supercritical	subcritical	supercritical
3	—	1 178 10	9 217 10	1 087 10	3 975 10
5	—	2 087 10	2 084 10	6 606 10	9 668 10
9	—	4 395 10	5 122 10	1 155 10	1 769 10
17	—	9 825 10	1 235 10	2 842 10	4 714 10
33	—	2 304 10	3 135 10	6 858 10	1 249 10
65	—	5 595 10	7 657 10	1 651 10	3 976 10
129	—	1 280 10	2 234 10	4 401 10	1 453 10

Table 4.2: errors for piecewise linear depth approximations.

Finite element expansions for the mass flow and the velocity potential, as well as for the fluid depth, can be obtained using the unconstrained 'r' principle, based on the functional (3.113). The method used here is a simple extension of the algorithm in Section 4.1.

Consider the grid defined by the points (4.6), with $= 0$, $= 10$ and $= 21$. Let

$$( ) = \quad ( ) \quad ( ) = \quad ( ) \quad ( ) = \quad ( ) \qquad (4\ 26)$$

be approximations ta—

into the functional (3.113) yields the finite dimensional version

$$L(\mathbf{Q}, \mathbf{d}, \boldsymbol{\phi}) = \int_{x_1}^{x_n} \left(r(Q^h, d^h) + Ed^h - \phi^{h\prime} Q^h \right) B \, dx + CB_e \left(\phi^h(x_n) - \phi^h(x_1) \right), \tag{4.27}$$

where $\mathbf{Q} = (Q_1, \ldots, Q_n)^T$, $\mathbf{d} = (d_1, \ldots, d_n)^T$, $\boldsymbol{\phi}$=$(\phi_1, \ldots, \phi_n)^T$ and $E(x) = \tilde{E} + gh(x)$. The parameters $\mathbf{Q}$, $\mathbf{d}$ and $\boldsymbol{\phi}$ are calculated by solving

$$\frac{\partial L}{\partial Q_i} = 0 \,, \; \frac{\partial L}{\partial d_i} = 0 \,, \; \frac{\partial L}{\partial \phi_i} = 0 \quad \text{for } i = 1, \ldots, n. \tag{4.28}$$

Let the α_i be the piecewise linear basis functions defined by (4.13). Then equations $(4.28)_3$ yield

$$-\int_{x_1}^{x_n} \alpha_i' Q^h B \, dx + CB_e \left(\alpha_i(x_n) - \alpha_i(x_1) \right) = 0 \quad i = 1, \ldots, n,$$

which may be rewritten as

$$\begin{aligned} \sum_{j=1}^{2} Q_j \int_{x_1}^{x_2} \alpha_1' \alpha_j B \, dx &= -CB_e, \\ \sum_{j=i-1}^{i+1} Q_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i' \alpha_j B \, dx &= 0 \qquad i = 2, \ldots, n-1, \\ \sum_{j=n-1}^{n} Q_j \int_{x_{n-1}}^{x_n} \alpha_n' \alpha_j B \, dx &= CB_e, \end{aligned}$$

or as,

$$A_Q \mathbf{Q} = C_Q, \tag{4.29}$$

where A_Q is a constant $n \times n$ matrix and C_Q is a constant $n \times 1$ vector with only first and last entries non-zero. The matrix A_Q is of rank $n-1$ and is singular but, using the boundary condition $Q_1 = C$, the solution of (4.29) is unique. A_Q is tridiagonal and $\mathbf{Q}$ is calculated using Gaussian elimination and back substitution.

quations $(4.28)_2$ yield

$$\int_{x_1}^{x_n} \left(r_{d^h} + E \right) \alpha_i B \, dx = 0 \quad i = 1, \ldots, n,$$

which, once Q^h is known, can be solved for d^h by the method of Section 4.1.

quations $(4.28)_1$ give

$$\int_{x_1}^{x_n} \left(r_{Q^h} - \phi^{h\prime}\right) \alpha_i B\, dx = 0 \quad i = 1, \ldots, n,$$

which may be written as

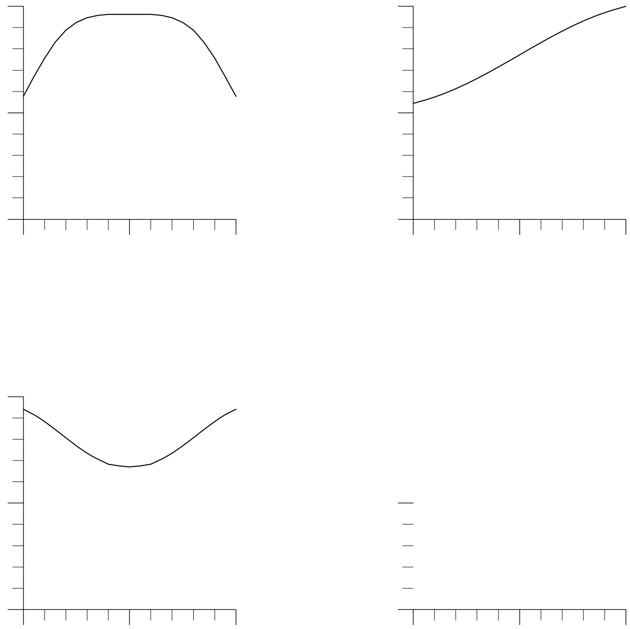
$$\begin{aligned}
\sum_{j=1}^{2} \phi_j \int_{x_1}^{x_2} \alpha_1 \alpha_j' B\, dx &= \int_{x_1}^{x_2} r_{Q^h} \alpha_1 B\, dx, \\
\sum_{j=i-1}^{i+1} \phi_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i \alpha_j' B\, dx &= \int_{x_{i-1}}^{x_{i+1}} r_{Q^h} \alpha_i B\, dx \quad i = 2, \ldots, n-1, \\
\sum_{j=n-1}^{n} \phi_j \int_{x_{n-1}}^{x_n} \alpha_n \alpha_j' B\, dx &= \int_{x_{n-1}}^{x_n} r_{Q^h} \alpha_n B\, dx,
\end{aligned}$$

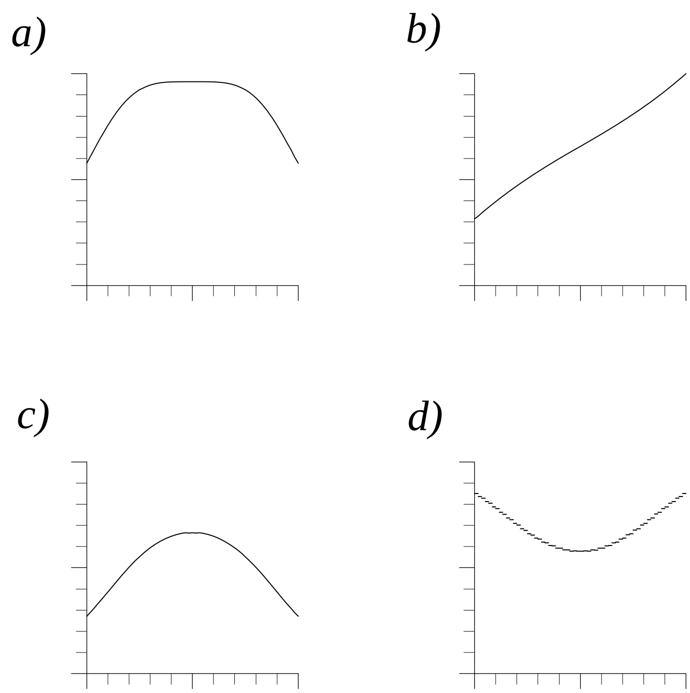
or as

$$A_\phi \boldsymbol{\phi} = C_\phi, \tag{4.30}$$

where A_ϕ is an $n \times n$ matrix and C_ϕ is an $n \times 1$ vector. Once Q^h and d^h are known $\boldsymbol{\phi}$ can be calculated directly. The matrix A_ϕ is of rank $n-1$ and singular but ϕ is a potential function and the important quantity is its gradient so one of the values, say ϕ_1, is specified arbitrarily. This procedure is equivalent to setting the arbitrary constant in ϕ by assigning its value at the boundary.

Results for critical flow in a channel with $B = B_{1,4}$, defined by (4.15), and $h = h_1$, defined by (4.17), are shown in Figure 4.6. The energy $\tilde{E}$ is taken to be 50. The piecewise linear approximation to the mass flow is shown in Figure 4.6*a*. The piecewise linear approximations to the velocity potential and depth for a supercritical flow are given in Figures 4.6*b* and 4.6*c*, respectively. Figure 4.6*d* shows the piecewise constant approximation to the supercritical velocity derived by taking the gradient of the piecewise linear velocity potential approximation in each interval $[x_i, x_{i+1}]$ for $i = 1, \ldots, n-1$. The Newton iteration to find d^h




a)
b)
c)
d)

the velocity) it is not ideal. However a variational principle exists which depends on the velocity potential alone, that is, the 'p' principle, based on the functional (3.112), constrained by $v = \phi'$. In using this constrained principle to seek an approximation to ϕ (and therefore v) no other approximations are made and more accurate results might be expected.

4.3 The onstrained 'p' Principle

The functional of the constrained 'p' principle (3.116) is given by

$$M_1^c(\phi) = \int_{x_e}^{x_o} p(\phi', E)B\,dx + CB_e\left(\phi(x_o) - \phi(x_e)\right), \tag{4.31}$$

where $E(x) = \tilde{E} + gh(x)$ and the constants $\tilde{E}$ and C are prescribed.

The velocity potential of a shallow water flow is the function ϕ which satisfies $\delta M_1^c = 0$. The nature of the stationary value of M_1^c can be deduced by considering

$$\frac{d^2 M_1^c}{d{\phi'}^2} = \int_{x_e}^{x_o} p_{\phi'\phi'} B\,dx.$$

From the definition of p, (3.102),

$$p_{\phi'\phi'} = \frac{1}{g}\left(\frac{3}{2}{\phi'}^2 - E\right). \tag{4.32}$$

Thus, from (4.32), if the flow is supercritical in the whole of $[x_e, x_o]$ then the solution ϕ of $\delta M_1^c = 0$ minimises M_1^c and if the flow is subcritical in the whole of $[x_e, x_o]$ the solution maximises M_1^c.

4.3.1 The Algorithm

The algorithm for generating an approximation to the velocity potential using (4.31) is similar to that of Section 4.1.

Let the x_i $(i = 1, \ldots, n)$, given by (4.6), define the grid. Let the finite element approximation to the velocity potential be given by

$$\phi^h(x) = \sum_{i=1}^{n} \phi_i \alpha_i(x),$$

where the α_i are the piecewise linear basis functions (4.13) and the ϕ_i are parameters of the solution. Thus the finite dimensional version of the functional of the constrained 'p' principle is given by

$$L(\boldsymbol{\phi}) = \int_{x_1}^{x_n} p(\phi^{h\prime}, E) B \, dx + C B_e \left(\phi^h(x_n) - \phi^h(x_1) \right),$$

where $E(x) = \tilde{E} + gh(x)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$. The approximation to the velocity potential is determined by the $\boldsymbol{\phi}$ which causes L to be stationary, that is, the $\boldsymbol{\phi}$ which satisfies

$$F_i(\boldsymbol{\phi}) = \frac{\partial L}{\partial \phi_i} = \int_{x_1}^{x_n} p_{\phi^{h\prime}} \alpha_i' B \, dx + C B_e \left(\alpha_i(x_n) - \alpha_i(x_1) \right) = 0 \quad i = 1, \ldots, n. \tag{4.33}$$

The solution of the non-linear set of equations (4.33) is found using Newton's method. The Jacobian is given by

$$J(\boldsymbol{\phi}) = \{J_{ij}\} = \left\{ \frac{\partial F_i}{\partial \phi_j} \right\} = \left\{ \frac{\partial^2 L}{\partial \phi_j \partial \phi_i} \right\} = \left\{ \int_{x_1}^{x_n} p_{\phi^{h\prime} \phi^{h\prime}} \alpha_i' \alpha_j' B \, dx \right\},$$

which is the Hessian of L and has the form of a weighted mass matrix, with weight $p_{\phi^{h\prime} \phi^{h\prime}} B$. From (4.32) J is negative definite for wholly subcritical flows and positive definite for wholly supercritical flows.

Given an initial approximation $\boldsymbol{\phi}^0$ to the solution $\boldsymbol{\phi}$ Newton's method produces a sequence of approximations $\boldsymbol{\phi}^k$ from

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta \boldsymbol{\phi}^k, \tag{4.34}$$

where

$$(\quad) \quad = \quad (\quad) \tag{4 35}$$

The sequence ends when

$$\max \qquad \text{tolerance} \tag{4 36}$$

The Jacobian and the vector are integrated exactly. The Jacobian is tridiagonal and (4.35) is solved by Gaussian elimination and back substitution. The initial approximation is given by

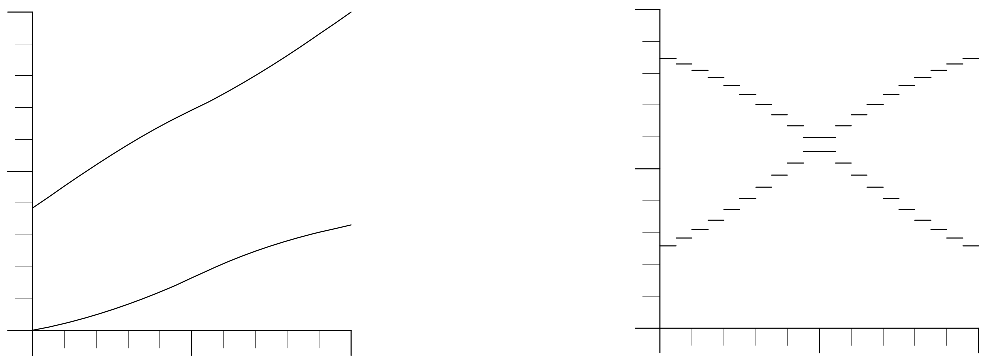
$$= (\quad 1) \qquad = 1 \ \ldots$$

where is assigned a value which determines whether the approximation being calculated is an approximation to subcritical or to supercritical flow. Let $=$ min , where is defined by (2.63). Then, if —— , the approximation will be subcritical. Let $=$ max . Then, if —— , the approximation will be supercritical.

The algorithm is implemented on the grid (4.6), with $= 0$, $= 10$ and $= 21$. The energy $\tilde{}$ is again taken to be 50. Approximations to flows in channels with breadths given by (4.15) and (4.16) and fluid depths below the level $= 0$ given by (4.17) and (4.18) are considered.

For $=$ the value of mass flow at inlet $=$, where is given by (4.20), is used to give examples of critical flows and $= 10$ is used to give examples of non-critical flows.

Consider the channel with breadth $=$ and let the tolerance in (4.36) be 10 . The method converges to the subcritical approximation in 4 iterations, using $= 1$, and to the supercritical upercriticalThe5imatCkO5GNNk5Fk06d[i4088k8658OI05LO *a)* *b)*



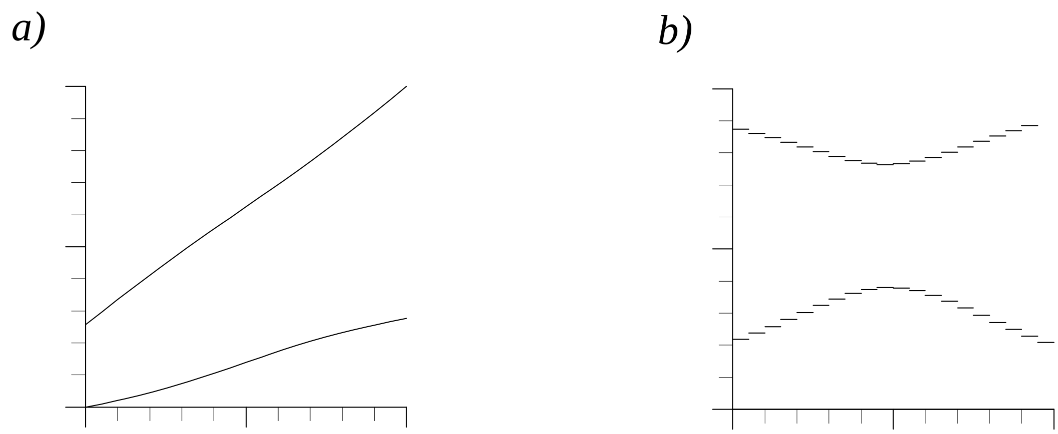


Figure 4.10:) Velocity potential and) velocity approximations for =

-

	Δ	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	—	3 266 10	2 744 10	1 933 10	1 444 10
5	—	1 582 10	1 352 10	8 914 10	6 549 10
9	—	7 782 10	6 653 10	4 449 10	3 261 10
17	—	3 861 10	3 296 10	2 227 10	1 632 10
33	—	1 923 10	1 640 10	1 114 10	8 164 10
65	—	9 597 10	8 179 10	5 569 10	4 082 10
129	—	4 794 10	4 084 10	2 784 10	2 041 10
257	—	2 396 10	2 041 10	1 392 10	1 021 10
513	—	1 198 10	1 020 10	6 961 10	5 103 10
1025	—	5 987 10	5 100 10	3 481 10	2 552 10

Table 4.3: errors for piecewise constant velocity approximations.

The approximations derived so far in this chapter have all been defined on the fixed regular grid given by the points (4.6). In this section a method of generating irregular grids using the constrained 'p' principle (3.116) is investigated.

The method of generating irregular grids and the corresponding approximations to the velocity potential using (3.116) is similar to the method of Section 4.3 in that a finite element expansion for the velocity potential is substituted into

solution for the velocity potential is found by solving

$$F_i(\boldsymbol{\phi}, \mathbf{x}) = \frac{\partial L}{\partial \phi_i} = \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) p_{\phi^{h\prime}} \alpha_i' B \, dx + C B_e \left[\alpha_i \right]_{x_1}^{x_n} = 0 \quad i = 1, \ldots, n \tag{4.37}$$

for $\boldsymbol{\phi}$ with the x_i fixed and given by (4.6). This is done using Newton's method, as in Section 4.3.

New positions for the internal grid points are then found by solving

$$\begin{aligned} G_i(\boldsymbol{\phi}, \mathbf{x}) &= \frac{\partial L}{\partial x_i} \\ &= -\left[pB \right]_{x_i} + \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_i} B \, dx + C B_e \left[\frac{\partial \phi^h}{\partial x_i} \right]_{x_1}^{x_n} = 0 \\ & \qquad i = 2, \ldots, n-1, \end{aligned} \tag{4.38}$$

for x_i ($i = 2, \ldots, n-1$), by Newton's method. The Jacobian is the $(n-2) \times (n-2)$ matrix given by

$$\begin{aligned} J(\boldsymbol{\phi}, \mathbf{x}) &= \{J_{ij}\} = \left\{ \frac{\partial L}{\partial x_j \partial x_i} \right\} \\ &= \left\{ - \left[p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_j} B \right]_{x_i} - \left[p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_i} B \right]_{x_j} \right. \\ & \quad + \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) \left(p_{\phi^{h\prime}\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_j} \frac{\partial \phi^{h\prime}}{\partial x_i} + p_{\phi^{h\prime}} \frac{\partial^2 \phi^{h\prime}}{\partial x_j \partial x_i} \right) B \, dx \\ & \quad \left. + C B_e \left[\frac{\partial^2 \phi^h}{\partial x_j \partial x_i} \right]_{x_1}^{x_n} \right\}, \end{aligned}$$

which is tridiagonal so that the equation

$$J(\boldsymbol{\phi}, \mathbf{x}^k) \, \delta \mathbf{x}^k = -\mathbf{G}(\boldsymbol{\phi}, \mathbf{x}^k)$$

is solved for $\delta \mathbf{x}^k$ by Gaussian elimination and back substitution. A sequence of approximations to $\mathbf{x}$ is generated using

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \delta \mathbf{x}^k.$$

The process is repeated until

$$\max_i \left|\delta x_i^k\right| < \text{tolerance}. \tag{4.39}$$

The procedure is to return to (4.37) and find $\boldsymbol{\phi}$ on the new grid, using the solution for $\boldsymbol{\phi}$ on the previous grid as the initial approximation $\boldsymbol{\phi}^0$. quations (4.38) are then solved again to modify the grid further, the initial approximation $\mathbf{x}^0$ being given the values of the solution $\mathbf{x}$ at the previous iteration.

This process is repeated until

$$\max_i \left(|F_i|, |G_i|\right) \tag{4.40}$$

changes by less than some percentage between successive iterations on the positions of the grid nodes.

The energy $\tilde{E}$ is assigned the value 50. The two values of mass flow at inlet $C = C_*$, given by (4.20), which generates critical flows, and $C = 10$, which generates examples of non-critical flows, are considered. The criterion for convergence using (4.40) is that (4.40) changes by less than 5% between two successive iterations.

Results are given for the channel with $x_e = 0$, $x_o = 10$ and breadth $B = B_3$, where

$$B_3(x) = 8 + 2\cos\left(\frac{\pi x}{5}\right),$$

which is shown in Figure 4.11. The fluid depth below the reference level $z = 0$ is $h = h_1$ (equation (4.17)).

The tolerance on the Newton iteration for $\boldsymbol{\phi}$ is taken to be 10^{-7} and on (4.39) to be $\frac{x_o - x_e}{n-1}10^{-4}$, where n is the number of grid points. The approximations to subcritical and supercritical velocities for $C = C_*$, derived as the gradients of the piecewise linear approximations to the velocity potential, for $n = 5$, 7 and 11 are

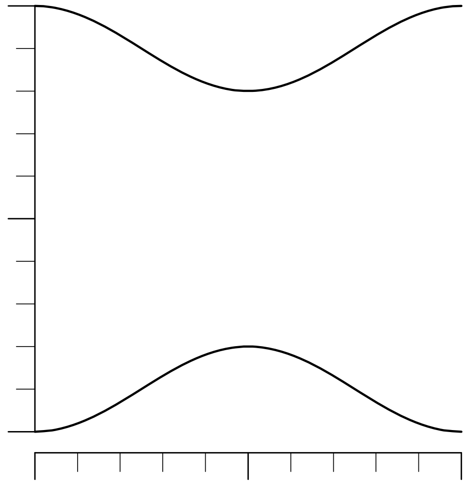


Figure 4.11: The breadth function ( ).

shown in Figure 4.12. The dots represent the final positions of the grid points. The subcritical approximation with = 5, Figure 4.12 , requires 28 sets of iterations to converge and the grid points have moved from their initial equi-spaced positions. The subcritical approximations with = 7 and 11, Figures 4.12 and 4.12 respectively, both converge after one set of iterations, the grid points have moved slightly towards the line = 5, although this is not obvious from the figure. The supercritical approximations with = 5, 7 and 11, Figures 4.12 , 4.12 and 4.12 respectively, all converge after one set of iterations; there is no discernible motion of the grid points in these cases.

Table 4.4 gives the errors of the approximate solutions for various , in the same channel and with the same conditions as above. For comparison, the corresponding errors for approximations generated on fixed, equi-spaced grids

to satisfy conservation of mass, is

$$(\quad) = \quad ((\quad) + \quad) \quad + \quad ((\quad) + \quad) \tag{4 41}$$

where $(\ ) =$ ——. The equilibrium fluid depth is assumed constant so that the energy , defined by (2.35), has the constant value in [ ) and the constant value in ( ]. The values of and are deduced from boundary conditions and, from (2.78), are such that . The natural conditions of the first variation of are

$$\begin{aligned} + \quad &= 0 \quad \text{in } (\quad) \\ + \quad &= 0 \quad \text{in } (\quad) \\ [+ \quad] &= 0 \end{aligned} \tag{4.42}$$

where the coefficients of the total variation of on either side of have been equated, that is, the equation

$$+ \quad = \quad + \tag{4 43}$$

is assumed satisfied. It is not obvious how, in practice, it might be possible to construct variations that satisfy (4.43). It is the assumption that (4.43) is true which gives rise to the natural jump condition (4 42) . So, if variations satisfying (4.43) cannot be found then, in order to generate approximations to the depth in discontinuous shallow water flows, (4 42) must be enforced in some way.

The method of finding approximations is based on that of Section 4.1 in that finite element expansions for in the regions of the domain before and after the discontinuity are substituted into a finite dimensional version of (4.41). Then the node which separates the pre- and post-discontinuity approximations must be repositioned in order to satisfy (4 42) . An algorithm based on this is given in Section 4.5.1. The method is then extended in Section 4.5.2 to give an algorithm generating approximations on grids where all of the internal grid nodes are positioned using (4 42) .

Let the domain of the problem [ ] be divided into 1 adjacent regular intervals [ ] by the points (= 1 ...) defined by (4.6). One of these nodes must be chosen as being the initial approximation to the position of the discontinuity and the number of the node nearest to the actual position of the hydraulic jump needs to be deduced. Let be the initial guess for the jump position.

$$\alpha_N^e(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}} & x \in [x_{N-1}, x_N] \\ 0 & x \notin [x_{N-1}, x_N] \end{cases}.$$

Let the approximation to the depth in the post-jump region $[x_N, x_n]$ be

$$d^o(x) = \sum_{i=N}^{n} d_i^o \alpha_i^o(x),$$

where

$$\alpha_N^o(x) = \begin{cases} \dfrac{x_{N+1} - x}{x_{N+1} - x_N} & x \in [x_N, x_{N+1}] \\ 0 & x \notin [x_N, x_{N+1}] \end{cases},$$

$$\alpha_i^o(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = N+1, \ldots, n-1,$$

$$\alpha_n^o(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\ 0 & x \notin [x_{n-1}, x_n] \end{cases}.$$

The algorithm is in two parts. Firstly the two finite element approximations d^e and d^o are derived by finding the values of $\mathbf{d}^e = (d_1^e, \ldots, d_N^e)^T$ and $\mathbf{d}^o = (d_N^o, \ldots, d_n^o)^T$ such that

$$L(\mathbf{d}^e, \mathbf{d}^o) = \int_{x_1}^{x_N} \left(r(Q, d^e) + E_e d^e\right) B\, dx + \int_{x_N}^{x_n} \left(r(Q, d^o) + E_o d^o\right) B\, dx$$

is stationary with respect to variations in $\mathbf{d}^e$ and $\mathbf{d}^o$. This requires solving the two sets of equations

$$\frac{\partial L}{\partial d_i^e} = 0 \;\; i = 1, \ldots, N \quad \text{and} \quad \frac{\partial L}{\partial d_i^o} = 0 \;\; i = N, \ldots, n,$$

using Newton's method, as described in Section 4.1. The initial approximation to $\mathbf{d}^e$ must be supercritical in order that the supercritical flow in the region before the jump is approximated and the initial approximation to $\mathbf{d}^o$ must be subcritical.

The second stage of the algorithm is to alter the position of by employing the jump condition (4 42) . If is the exact position of the jump and is the exact solution then, from (4 42) ,

$$((\quad) + \quad) \qquad ((\quad) + \quad) \qquad = 0$$

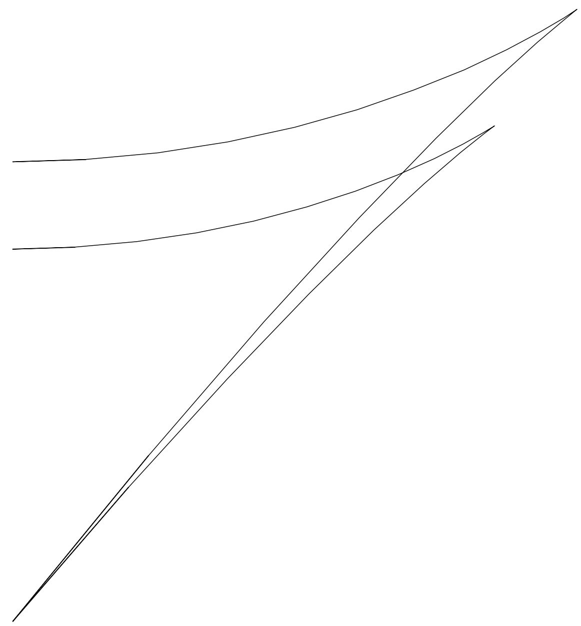
If the approximation satisfies

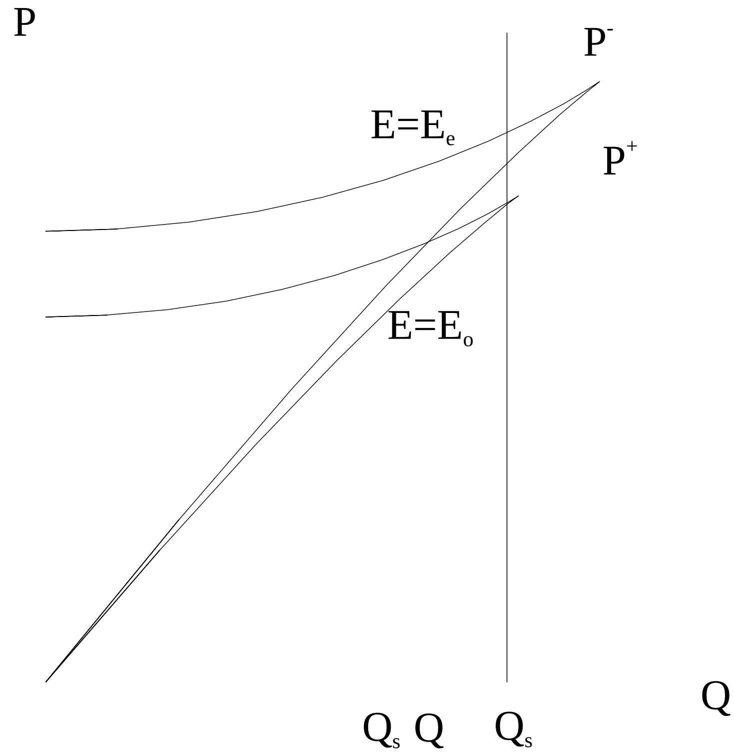
$$((\quad) + \quad) \qquad ((\quad) + \quad) \qquad \text{tolerance} \tag{4 44}$$

for some specified tolerance, then the approximate solution has been found and

issatisfies

The process which occurs on solving (4.45) is explained more fully in the





P
P-
E=Ee
P+
E=Eo
Q
Qs Q
Qs

gives an expression for as a function of , that is,

$$= \overline{2(\qquad)}$$

Let

$$= \overline{2(\qquad)} \quad \text{and} \quad = \overline{2(\qquad)}$$

where is the approximation to on the side of , the approximation to the jump position, and is the approximation on the side of . Let

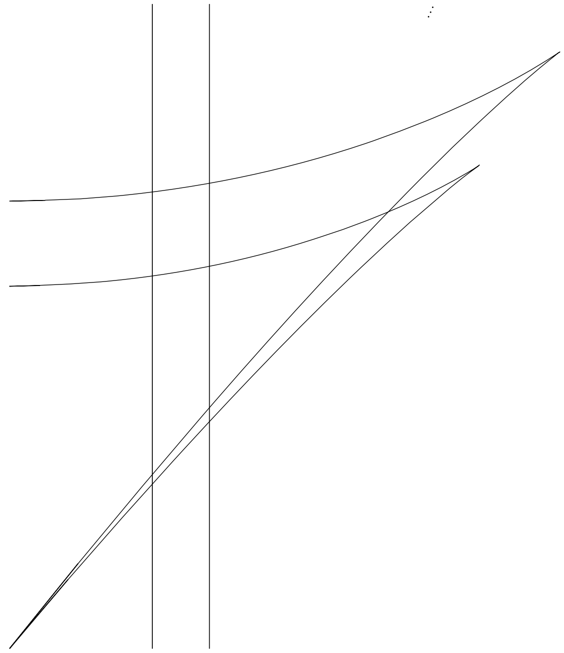
$$(\ ) = (\qquad) +$$

and

$$(\ ) = (\qquad) + \qquad .$$

Figure 4.15 shows a sketch of the curves and on a graph of as a function of for two different values of , . Notice that touches the curve at = and touches the curve at = . Note also that neither nor is necessarily equal to the mass flow ( ) = ——. The point of intersection of the and curves gives the value of equivalent to solving (4.45). There are four possible situations arising.

1.kF6that[40F8kL6N8OI—k5FLkVn5F—U.[4LnC=FC=25F—vF—U.max)OLc)GCkk0F8LI41.kF



the value of is found. If ( )( ) 0 then lies between and . Otherwise the process is repeated until the node is found, where ( )( ) 0. Then, if , the number of the node to be moved to the jump position is ; otherwise = 1.

Once the number of the node to be moved to the jump position has been established in this way, is moved to . The finite element approximations and are recalculated on the modified grid and, if (4.44) is still not satisfied, (4.45) and (4.46) are used to reposition and the process is repeated until (4.44) is satisfied. The approximate solution has then been found and is an approximation to the jump position.

The algorithm is applied to a grid with = 0, = 10 and = 21. The energy at inlet is given the value 50 and the mass flow at inlet = , where is defined by (4.20), to give a critical flow in a channel with breadth = , defined by (4.15). The depth at outlet is given for each case and is used to deduce the value of , using the definitions of mass flow (2.34) and energy (2.35). From the conservation of mass equation ( ) = ——, which yields

$$= \quad + \frac{1}{2} \quad \frac{\quad}{(\quad)}$$

The piecewise linear approximation to the discontinuous depth profile with = 4 69 and breadth = is given in Figure 4.16 . For a tolerance on the Newton iteration of 10 and on the jump condition (4.44) of 10 , the method converges in 3 iterations on the position of the discontinuity, once the

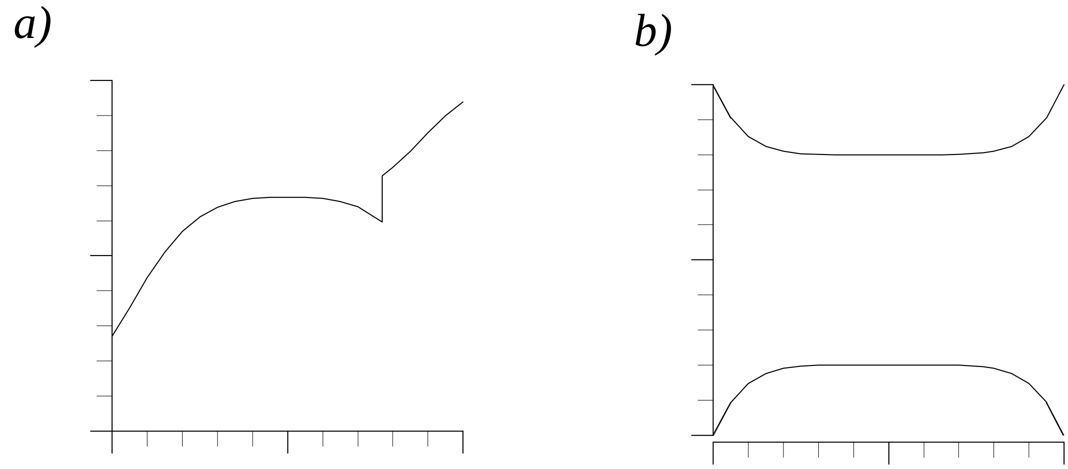


Figure 4.16: a) Piecewise linear depth approximation for $d_o = 4.69$ and b) $B_{1,6}(x)$.

and $d_i^o = 4.69$ $(i = N, \ldots, n)$. Once the number of the node to approximate the jump position is found subsequent approximations to the finite element solutions use the approximation on the previous grid as the first guess in Newton's method to find the approximation on the new grid. Figure 4.16b shows the breadth $B_{1,6}$.

The piecewise linear approximation for $d_o = 3.86$ is shown in Figure 4.17. This converges in 3 iterations on the position of node 20, which is selected by the algorithm to be moved to approximate the jump position, requiring 15, 4 and 4 Newton iterations.

The algorithm in this section generates approximations to the depth for discontinuous flows in channels, where the approximations are defined on grids in which all of the grid points except one are fixed. The one movable grid point is positioned, using the jump condition $(4.42)_3$, in such a way that $(4.42)_3$ is approximately satisfied. In Section 4.5.2 this method is extended, by allowing all of

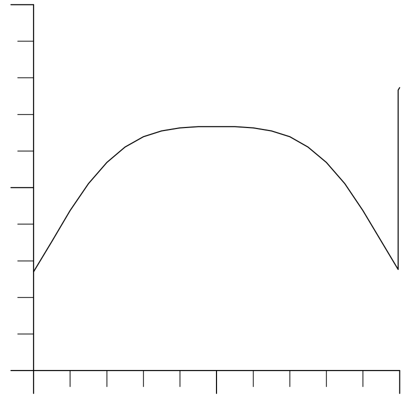


Figure 4.17: Piecewise linear depth approximation for $d \; = 3.86$.

the internal grid points to move, in order to generate irregular grids.

The domain of the problem $[x \; , x \;]$ is divided into $n - 1$ regular intervals by the points $x \;\; (i = 1, \ldots, n)$ defined by (4.6). Finite element approximations to the depth are generated separately on each interval $[x \; , x \quad]$ and the jump condition (4.42) is used at each internal node to reposition the node. Instead of having just two finite element approximations coupled at a point, as in Section 4.5.1, there will be $n - 1$ approximations coupled at the $n - 2$ internal grid points.

Let

$$d \;(x) = d \;\alpha \;(x) + d \;\alpha \;(x) \tag{4.47}$$

be the finite element approximation to d in the i th element $[x \; , x \quad]$, where

$$\alpha \;(x) \;=\; \begin{cases} \dfrac{x \qquad x}{x \qquad x} & x \quad [x \; , x \quad] \\ 0 & x \quad [x \; , x \quad] \end{cases} \qquad i = 1, \ldots, n \quad 1,$$

$$\alpha_i^R(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_i, x_{i+1}] \end{cases} \qquad i = 1, \ldots, n-1.$$

Let N be the number of the node chosen to be the initial approximation to the position of the hydraulic jump. Then, in the element $[x_i, x_{i+1}]$,

$$E = E_e \qquad \text{if } i+1 \leq N$$

and

$$E = E_o \qquad \text{if } i \geq N,$$

where E_e is the value of the energy E at inlet and E_o is the value at outlet.

The finite element solution on each element is given by the values of $\mathbf{d}_i = (d_i^L, d_i^R)$ such that

$$\begin{aligned} L(\mathbf{d}_1, \ldots, \mathbf{d}_{n-1}) &= \sum_{i=1}^{N-1} \left(\int_{x_i}^{x_{i+1}} \left(r(Q, d_i^h) + E_e d_i^h \right) B\, dx \right) \\ &+ \sum_{i=N}^{n-1} \left(\int_{x_i}^{x_{i+1}} \left(r(Q, d_i^h) + E_o d_i^h \right) B\, dx \right), \end{aligned}$$

where $Q(x) = \frac{CB_e}{B(x)}$, is stationary with respect to variations in $\mathbf{d}_i$ $(i = 1, \ldots, n-1)$. The solutions of the $n-1$ sets of non-linear equations

$$\frac{\partial L}{\partial d_i^L} = 0 \;\;,\;\; \frac{\partial L}{\partial d_i^R} = 0 \qquad i = 1, \ldots, n-1,$$

each with two unknowns, are found using Newton's method.

Once the $\mathbf{d}_i$ have been calculated on the initial grid the jump condition is applied at each internal node. If

$$\left| \left(r(Q, d_i^L) + E_1 d_i^L \right)\Big|_{x_i} - \left(r(Q, d_{i-1}^R) + E_2 d_{i-1}^R \right)\Big|_{x_i} \right| < \text{tolerance}, \tag{4.48}$$

where

$$E_1 = E_2 = E_e \qquad \text{if } i < N,$$

$$E_1 = E_o,\; E_2 = E_e \quad \text{if } i = N,$$

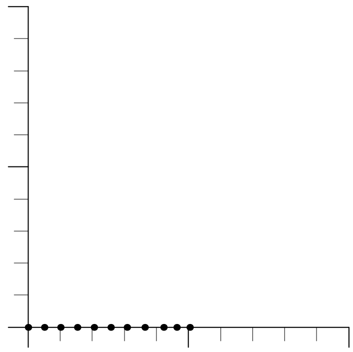
$$E_1 = E_2 = E_o \qquad \text{if } i > N,$$

for all $= 2 \ldots \quad 1$ and a specified tolerance, the required approximate solution has been found. If (4.48) is not satisfied for a particular value of then

$$(\quad) + \quad (\quad) \quad = 0 \qquad (4\ 49)$$

is solved for and the new position of the grid point is found from using the conservation of mass law and bisection.

The grid point closest to the jump position in the regular grid defined by (4.6) is found in the same way as in Section 4.5.1. The approximation to the jump position is initially taken to be ; equation (4.49) then yields the new approximation . The process is repeated using as the approximation to the jump position and then stepping backwards along the channel to each grid point in turn until $(\quad)(\quad) \quad 0$ for some . Then, if



Ch pter 5

Approxim tions to Continuous wo-dimension l Sh llow W ter Flows

In this chapter the constrained variational principles derived in Section 3.6.2 are used to generate approximations to two-dimensional shallow water flows. The method is an extension of the method used in Chapter 4 to approximate one-dimensional flows.

The functionals of the constrained variational principles for steady state flows, (3.94), (3.97), (3.95) and (3.96), are

$$
\begin{aligned}
L_1^c(\phi) &= \iint_D p(\boldsymbol{\nabla}\phi, E)\,dx\,dy + \int_\Sigma C\phi\,d\Sigma, && (5.1)\\
L_2^c(\mathbf{Q}, d) &= \iint_D (r(\mathbf{Q}, d) + Ed)\,dx\,dy, \\
L_3^c(\mathbf{Q}) &= \iint_D P(\mathbf{Q}, E)\,dx\,dy, \\
L_4^c(\phi, d) &= \iint_D (-R(\boldsymbol{\nabla}\phi, d) + Ed)\,dx\,dy + \int_\Sigma C\phi\,d\Sigma, && (5.2)
\end{aligned}
$$

the grid.

Substituting (5.4) into (5.1) gives the finite dimensional version of the functional, that is,

$$(\quad) = \qquad (\qquad) \qquad + \qquad \Sigma + \qquad \Sigma$$

where $= (\quad \ldots \quad) \,,\ (\quad) = \tilde{\ } + \ (\quad)$ and $\tilde{\ }$ is approximated by $\ ,$ the region covered by the triangular grid. The finite element solution is given by the which satisfies

$$(\quad) = \text{——} = \qquad + \qquad \Sigma + \qquad \Sigma = 0$$

for $= 1 \ \ldots$ and is found using Newton's method in the same way as before. The Jacobian is given by

$$(\quad) = \qquad = \text{——} = \text{————} =$$

where

$$= \quad + - \qquad + \qquad + - \qquad +$$

and is negative definite for wholly subcritical flow and indefinite for wholly supercritical flow.

Given an initial approximation to the solution a sequence of approximations is generated, using Newton's method, from

$$= \quad + \qquad (5\ 5)$$

where

The process is continued until

$$\frac{\max\limits_i \left|\delta\phi_i^k\right|}{\max\limits_i \ \phi_i^k} < \text{tolerance.} \tag{5.7}$$

Using the piecewise linear basis functions, defined on a triangular grid, the integrands of the Jacobian J and the vector $\quad = (F_1, \ldots, F_l)^T$ are constants over each element so J and $\quad$ are integrated exactly.

The Jacobian is no longer tridiagonal, as it was in the one-dimensional examples, although it is symmetric and banded. quation (5.6) may still be solved efficiently for $\delta\ \ ^k$ using a pre-conditioned conjugate gradient method (Golub and Van Loan (1989)), provided that J is not indefinite. The matrix J is preconditioned by its diagonal entries, that is, by the matrix $P = \text{diag}(J_{11}, \ldots, J_{ll})$. The system

$$P\ \ ^{1}(\ \ ^{k})\,J(\ \ ^{k})\,P\ \ ^{1}(\ \ ^{k})\,\delta\ \ ^{k} = \quad P\ \ ^{1}(\ \ ^{k})\ \ (\ \ ^{k})$$

is solved for $\delta\ \ ^k$ by the conjugate gradient method. Then the solution $\delta\ \ ^k$ of (5.6) is given by

$$\delta\ \ ^{k} = P\ \ ^{1}(\ \ ^{k})\,\delta\ \ ^{k}.$$

The effect of this pre-conditioning should be to improve the convergence rate of the conjugate gradient iteration. If $p\ \ _{\phi}\ \ _{\phi}$ is a constant in $\ ^h$ then preconditioning the system using the matrix P will improve the convergence rate of the conjugate gradient iteration (Wathen (1987)).

The initial approximation to $\quad$ is given by

$$\phi_i^0 = \frac{x_i \quad x_1}{x_2 \quad x_1}\ v^0 \qquad\qquad i = 1, \ldots, l, \tag{5.8}$$

where v^0 is assigned a value which determines whether the solution being calculated is an approximation to subcritical or to supercritical flow. The energy $\tilde{E}$ is taken to be 50.

The boundary function is given the value ( ) = , where is a constant, on the inlet boundary Σ and ( ) = —— on the outlet boundary

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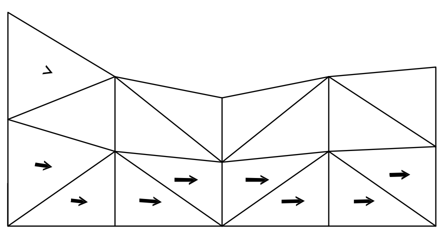
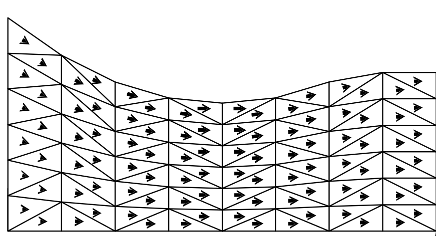
–

– –

Let the grid of points be defined by (5.3) with = 5 and = 3. Consider the channel with breadth and let = 0. Newton's method, with a tolerance in (5.7) of 10 , converges to the subcritical approximation in 4 iterations, requiring 7, 7, 7 and 4 conjugate gradient iterations, with a tolerance of 10 . On a refined





taking in the range —— —— min $\overline{2}$ in (5.8), where is the maximum critical speed in a particular channel for a flow with given values of and ˜. However, the Jacobian is indefinite for supercritical approximations and (5.6) must be solved for by an alternative to the conjugate gradient method, such as can be found in Golub and Van Loan (1989).

The functional of the constrained 'R' principle, given by (5.2), is used to generate approximations to the velocity potential and to the depth of flow. By considering the matrix of second derivatives of it can be shown that is maximised by subcritical solutions of the shallow water equations and has a saddle point for supercritical solutions.

Let

$$(\quad) = \quad (\quad) \quad \text{and} \quad (\quad) = \quad (\quad) \qquad (5\ 12)$$

be approximations to the velocity potential and the depth, where the are the two-dimensional piecewise linear basis functions defined earlier and the and the are parameters of the solutions whose values are to be determined.

Substituting (5.12) into (5.2) gives the finite dimensional version of the functional, that is,

$$(\quad) = \quad (\quad) + \quad + \quad \Sigma + \quad \Sigma$$

where $= (\ \ldots\ )$, $= (\ \ldots\ )$ ogzFhA]b[;LGa+(Σb:;][˜F˜[+b:;zz˜:+G[+b:;:fiepbzz[z˜

stationary with respect to variations, that is, and satisfy

$$F_i(\;,\;) = \frac{\partial L}{\partial \phi_i} = \int_D R \;\; _\phi \cdot \;\; \beta_i \, dx\, dy + \quad C\beta_i \, d\Sigma + \quad C\beta_i \, d\Sigma = 0,$$

$$F_{i \;\; l}(\;,\;) = \frac{\partial L}{\partial d_i} = \int_D (R_d \quad E)\, \beta_i \, dx\, dy = 0,$$

for $i = 1, \ldots, l$.

The solution is found using Newton's method. The Jacobian is given by

$$J(\;,\;) = \quad J_{ij} \quad ,$$

where

$$J_{ij} = \int_D R \quad _\phi \quad _\phi \quad \beta_i . \quad \beta_j \, dx\, dy,$$

$$J_{i\,j \;\; l} = \int_D \beta_i . R \quad _{\phi \;\; d} \, \beta_j \, dx\, dy,$$

$$J_{i \;\; l\, j} = \quad _m \; \beta \, R \quad . \quad \beta \;\; dx\, dy,$$

$$J \quad = \quad \beta \, R \quad \beta \;\; dx\, dy,$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, l$.

Given initial approximations and to and Newton's method yields a sequence of approximations,

$$= \quad + \delta \quad ,$$

where

$$J(\;,\;)\, \delta \quad = \quad (\;,\;). \tag{5.13}$$

The sequence ends when

$$\frac{\max \; \delta\phi}{\max \; \phi} < \text{tolerance} \quad \text{and} \quad \frac{\max \; \delta d}{\max \; d} < \text{tolerance}, \tag{5.14}$$

for some specified tolerance. The Jacobian and the vector are evaluated using 7 point Gaussian quadrature for integrating over triangles.

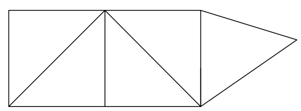
The Jacobian is symmetric and banded and (5.13) is solved, when is not indefinite, using a pre-conditioned conjugate gradient method, with the pre-conditioning matrix = diag(. . .).

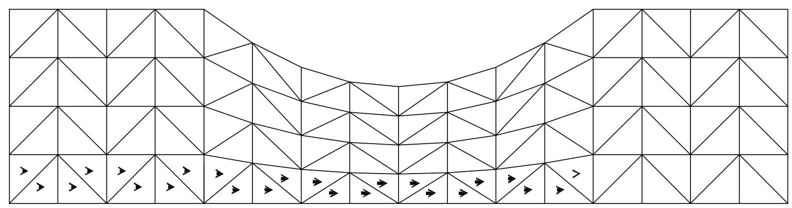
The initial approximation to is given by (5.8) and the initial approximation to is given by = ˆ, for = 1 . . . , where ˆ is a constant. The values of ˆ and , in (5.8), must be consistent with one another, that is, if ˆ is assigned a value corresponding to a subcritical depth then must be given a value in the range 0 ——— , where = min and is defined by (2.63). If ˆ has a value corresponding to a supercritical depth then must lie in the range

——— ——— min $\sqrt{2}$, where = max .

The constant ˜ is given the value 50. The boundary function is defined in the same way as in Section 5.1, that is, ( ) = on Σ and ( ) = ——— on Σ ; is taken to be 10. The channel breadths considered here are those given by (5.9), (5.10) and (5.11). The depth of fluid below the reference level = 0 is taken to be identically zero.

Consider the channel with breadth and let = 5. Then, with = 9 and = 3, Newton's method, with a tolerance of 10 in (5.14), converges in 26 iterations, with a tolerance on the conjugate gradient iterations of 5 10 . On a refined grid with = 17 and = 5 Newton's method converges in 32 iterations, with the same tolerances as before. In both cases the initial data is





generated, as above, by using the 'R' principle based on (5.2). Then

$$e = \frac{v \quad v}{(v \ + v \ )},$$

where $I = 2(n \quad 1)(m \quad 1)$ is the number of elements in the domain, is a measure of the difference between the two velocity approximations.

Consider the channel with breadth $B\ (x)$, given by (5.9), where $L = 5$, and let the fluid depth below the reference level $z = 0$ be identically zero. The values of e for grids with different values of n and m are given in Table 5.1. The results suggest that, as the number of elements increases, the differences between the approximations derived from the two principles decrease.

n	m	e
9	3	4.9 10
13	5	3.6 10
17	5	3.2 10

Table 5.1: Values of for various and .

Ch pter 6

Further Applic tions

In Chapters 4 and 5 the variational principles for steady state shallow water flows in one and two dimensions, derived in Chapter 3, are used to generate approximations to the corresponding flows. There are, however, other variational principles in Chapter 3 which can be used to generate approximate solutions to other problems. Two such problems are considered here.

The approximations generated so far have all been for solutions of the shallow water equations, in which it is assumed that the component of velocity perpendicular to the xy plane, that is the vertical component, is negligible, see the statement (2.14). The variational principle (3.7), based on Luke's principle (Luke (1967)) is satisfied for solutions of the equations of time-dependent free surface flows of an inviscid, homogeneous fluid in three dimensions. If an approximation to three-dimensional flow can be generated then it can be used to investigate the accuracy of the assumption that, under the conditions of shallow water theory, the magnitude of the vertical component of the velocity is negligible.

In Section 6.1 the functional of (3.7) is reduced to a functional whose corresponding variational principle has as its natural conditions the equations of time-independent free surface flow in two dimensions, that is, the solutions of these equations are functions of the vertical coordinate z and the one horizontal coordinate x. An attempt is made to extend the algorithms of Chapter 4 to generate approximations to free surface flows using the new functional.

In Section 6.2 the 'p' functional for time-dependent quasi one-dimensional shallow water flow (3.98) is used in an attempt to seek approximations to time-dependent flows in a channel of slowly varying breadth. The problems caused by using the functional (3.98) are also mentioned.

6.1 Two-dimensional Free Surface Flows

The functional of the modified version of Luke's principle (3.7) is given by

$$\int_{t_1}^{t_2} \iint_D \int_{-h}^{\eta} \rho \left\{ - \left(\chi_t + gz + \frac{1}{2}\mathbf{u}.\mathbf{u} \right) + \mathbf{u}. \left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\chi \right) \right\} dz\, dx\, dy\, dt, \qquad (6.1)$$

where $\eta = \eta(x, y, t)$, $h = h(x, y)$, $\chi = \chi(x, y, z, t)$, $\mathbf{u} = \mathbf{u}(x, y, z, t)$ and $\tilde{\boldsymbol{\nabla}}$ is defined by (2.2). The functional (6.1) is used in Section 6.1.1 to derive a functional which has as its natural conditions of the first variation the equations of time-independent motion in the x and z directions.

6.1.1 The Functional

The required functional is generated from (6.1) by first making the assumption that the flow variables are independent of time and evaluating the time integral and then assuming that the domain is a channel of slowly varying breadth so that the variables are functions of the coordinates x and z only and the integral with respect to y can be evaluated. It is also necessary to add in boundary terms so that variations can be allowed which do not necessarily vanish on the inlet and outlet boundaries of the channel.

Time-independent flows

First make the assumption that the free surface flow does not vary with time. Then the flow variable $\mathbf{u}$ and the height of the free surface above the reference level $z = 0$ are independent of time, that is, $\mathbf{u} = \mathbf{u}(x, y, z)$ and $\eta = \eta(x, y)$. The variation of the velocity potential χ with respect to time needs to be deduced. The flow is assumed to be irrotational so, from (2.5),

$$\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi.$$

By assumption $\mathbf{u}_t \equiv \mathbf{0}$ and so

$$\tilde{\boldsymbol{\nabla}}\chi_t \equiv \mathbf{0},$$

which implies that χ is of the form

$$\chi(x, y, z, t) = \hat{\chi}(x, y, z) + f(t),$$

for arbitrary functions $\hat{\chi}$ and f, where

$$\tilde{\boldsymbol{\nabla}}\chi = \tilde{\boldsymbol{\nabla}}\hat{\chi} \quad \text{and} \quad \chi_t = f'.$$

Let

$$\hat{E} = -\frac{1}{T}\int_{t_1}^{t_2} \chi_t \, dt = -\frac{1}{T}\left(f(t_2) - f(t_1)\right),$$

where $T = t_2 - t_1$. Then, making these substitutions into the functional (6.1) and integrating with respect to time gives

$$\iint_D \int_{-h}^{\eta} \rho T \left(\hat{E} - gz - \frac{1}{2}\mathbf{u}.\mathbf{u} + \mathbf{u}.\left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\hat{\chi}\right)\right) dz\, dx\, dy. \qquad (6.2)$$

Two-dimensional flows

Let   be the domain

$$= \left\{(x, y) : x \in [x_e, x_o], y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2}\right]\right\},$$

where $B(x)$, for $x \in [x_e, x_o]$, is the breadth of the channel and is a slowly varying function of x. Let h, the depth of fluid below the reference level $z = 0$, depend on the x coordinate alone.

Now make the assumption that, in this domain, all of the variables are independent of the y coordinate and redefine the variables as follows. The velocity (x, y, z) becomes $=$ (x, z), where $= (u, w)$, the velocity potential $\hat{\chi}(x, y, z)$ becomes $\hat{\chi}(x, z)$ and $\eta(x, y)$ becomes $\eta(x)$. The operator $\tilde{}$ is replaced by its two-dimensional counterpart $\hat{} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$.

Making these substitutions in (6.2) and integrating with respect to y gives

$$\int_{x}^{x} \int_{-h}^{\eta} \rho T \left(\hat{E} - gz - \frac{1}{2} \quad . \quad + \quad . \left(\quad - \quad \hat{} \; \hat{\chi}\right)\right) B \, dz \, dx.$$

As the final stage in the construction of the required functional, boundary terms must be added so that variations of the functional do not necessarily have to vanish at the ends of the channel, that is, at $x = x_e$ and $x = x_o$.

The required functional is

$$J(\eta, \quad , \chi) = \int_{x}^{x} \int_{-h}^{\eta} \left(\hat{E} - gz - \frac{1}{2} \quad . \quad + \quad . \left(\quad - \quad \hat{} \; \chi\right)\right) B \, dz \, dx$$
$$+ \int_{-h \; x}^{\eta \; x} C_o \, \chi|_x \, dz - \int_{-h \; x}^{\eta \; x} C_e \, \chi|_x \, dz, \qquad (6.3)$$

where the constant ρT has been set equal to unity and the $\hat{}$ notation on the velocity potential has been dropped for simplicity.

The natural conditions of $= 0$ are given by

$$\hat{\ } \quad = \qquad \text{for} \quad (\quad); \quad (\quad (\ ) \quad (\ ))$$
$$\hat{\ } \ (\quad) = 0$$

$$= 0 \quad \text{on} \quad = (\ ) \text{ for} \quad (\quad)$$

$$+ \quad = 0 \quad \text{on} \quad = \quad (\ ) \text{ for} \quad (\quad)$$

$$(\ ) \quad = 0 \quad \text{for} \quad (\quad (\ ) \quad (\ ))$$

$$(\ ) \quad = 0 \quad \text{for} \quad (\quad (\ ) \quad (\ ))$$

$$\hat{\ } \quad + \frac{1}{2} \quad \hat{\ } \quad = 0 \quad \text{on} \quad = (\ ) \text{ for} \quad (\quad)$$

which are, respectively, the irrotationality condition and the conservation of mass equation for ( ) and (( ) ( )), the condition of no flow across the free surface, the condition of no flow through the channel bed, boundary conditions on the horizontal component of velocity at the inlet and outlet boundaries and the dynamic free surface condition, as required. Notice that the first six natural conditions are due to variations in the variables and , while the last natural condition is due to the variation in .

The basic finite element technique, as used in Chapters 4 and 5, can only be applied to functionals in which the integration is over a fixed region. Various algorithms (for example, Aitchison (1979), Ikegawa and Washizu (1973)) have is based on the method used to approximate the position of a hydraulic jump in Section 4.5.

The functional (6.3) depends on the three functions $\eta(x)$, $\;(x,z)$ and $\chi(x,z)$. A functional depending on only two functions can be derived by making the substitution $\;=\;\hat{}\,\chi$ in (6.3), giving

$$\hat{J}(\eta,\chi)=\int_{x}^{x}\int_{h}^{\eta}\left(\hat{E}\quad gz\quad \frac{1}{2}\,\hat{}\,\chi.\,\hat{}\,\chi\right)B\,dz\,dx$$
$$+\int_{h\;x}^{\eta\;x} C_o\,\chi_{\;x}\;dz\quad \int_{h\;x}^{\eta\;x} C_e\,\chi_{\;x}\;dz. \qquad (6.4)$$

The variational principle corresponding to (6.4) is equivalent to the variational principle for (6.3), constrained to satisfy the irrotationality condition. The natural conditions of $\delta\hat{J}=0$ are

$$\hat{}\,.\left(B\,\hat{}\,\chi\right) = 0 \quad \text{for } x \quad (x\;,x\;);\; z \quad (\quad h(x),\eta(x)), \qquad (6.5)$$
$$\chi\;\eta\quad\;\chi = 0 \quad \text{on } z=\eta(x) \text{ for } x \quad (x\;,x\;), \qquad (6.6)$$
$$\chi\;h\;+\chi = 0 \quad \text{on } z=\quad h(x) \text{ for } x \quad (x\;,x\;), \qquad (6.7)$$
$$C\quad B(x\;)\,\chi = 0 \quad \text{for } z \quad (\quad h(x\;),\eta(x\;)), \qquad (6.8)$$
$$C\quad B(x\;)\,\chi = 0 \quad \text{for } z \quad (\quad h(x\;),\eta(x\;)), \qquad (6.9)$$
$$\hat{E}\quad gz\quad \frac{1}{2}\,\hat{}\,\chi.\,\hat{}\,\chi = 0 \quad \text{on } z=\eta(x) \text{ for } x \quad (x\;,x\;). \qquad (6.10)$$

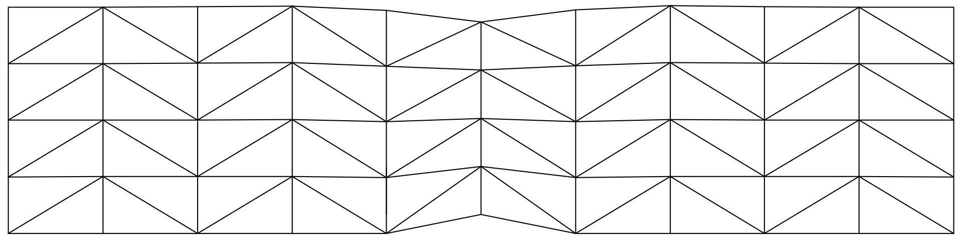
Let

$$\eta\;(x)=\sum\eta\;\alpha\;(x) \qquad (6.11)$$

be an approximation to $\eta(x)$ and let

$$\chi\;(x,z)=\sum\chi\;\beta\;(x,z) \qquad (6.12)$$

be an approximation to $\chi(x,z)$, where the α are the one-dimensional piecewise linear basis functions (4.13), the β are two-dimensional piecewise linear basis



Let

$$G_i(LkLhnnheEnnLnninEeniDhDnxhLkDnLkDnzDnDnnzcDhxnLkDnLkDn) = \frac{\partial L}{\partial \chi} \int \int \hat{\chi} \hat{\beta} B\, dz\, dx$$

$$+ \int \quad C \ \beta \quad dz \quad \int \quad C \ \beta \quad dz \quad i = 1, \ldots, l, \tag{6.13}$$

then the vector $= (G \ , \ldots , G \ )$ may be written as

$$(\ , \ ) = \quad A(\ ) \ + \ (\ ),$$

where

$$A(\ ) = \left\{ \int \quad \int \quad \hat{} \ \beta \ . \ \hat{} \ \beta \ B \, dz \, dx \right\} \tag{6.14}$$

is symmetric, positive definite and banded and $= (b \ , \ldots , b \ )$,where

$$b \ (\ ) = \int \quad C \ \beta \quad dz \quad \int \quad C \ \beta \quad dz. \tag{6.15}$$

The functional L is stationary with respect to variations in if $=$, that is, if

$$A(\ ) \quad = \quad (\ ). \tag{6.16}$$

Therefore, for a fixed , can be calculated from (6.16). Let $\eta = \eta$ in (6.4). Then the solution of (6.16) gives an approximation to the function χ satisfying (6.5)–(6.9), for the given domain, since these natural conditions are due solely to the variations of χ in $\delta \hat{J} = 0$.

The problem remains to find η , that is, to find such that L is stationary with respect to variations in . This could be done by adapting the method in Aitchison (1979), which is for a functional written in terms of a stream function. The finite dimensional version of the functional (6.4) can be written as

$$L(\ , \ ) = \ \frac{1}{2} \quad A(\ ) \ + \quad (\ ) \ + c(\ ),$$

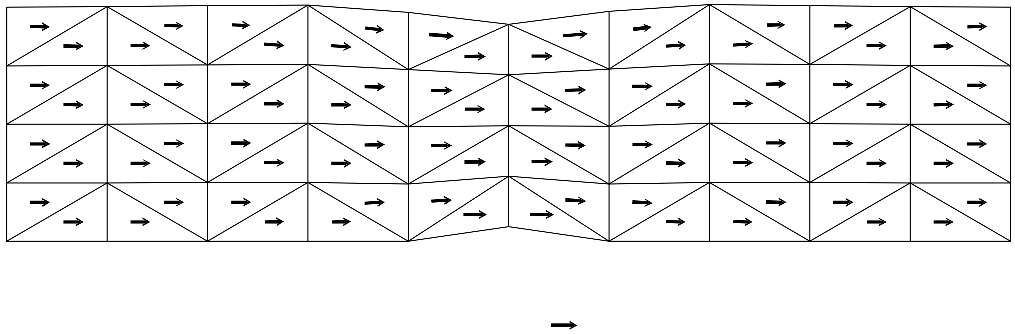


Figure 6.2: Piecewise constant velocity approximation for a subcritical free surface flow.

below approximately 10 , for the value = 0 05 of the relaxation parameter.

In this section a constrained version of the 'p' principle based on (3.98) is used to develop an algorithm for generating approximations to time-dependent quasi

Let

$$\phi \ (x,t) = \quad \phi \, \beta \, (x,t), \tag{6.19}$$

be an approximation to ϕ, where the β are two-dimensional piecewise linear basis functions, as in Figure 5.2, the ϕ are parameters of the solution, to be determined, and $l = nm$. The finite dimensional version of (6.18) is generated by substituting (6.19) for ϕ in (6.18) to give

$$L(\ ) = \quad \hat{p}(\phi \ )B \, dx \, dt + \quad C \quad B\phi \quad C \quad B\phi \quad dt$$

$$+ \quad \phi \quad g \quad \phi \quad g \quad B \, dx,$$

where $= (\phi \ , \ldots , \phi \ ) \ .$

The approximation for ϕ is given by (6.19), where satisfies

$$F(\ ) = \frac{\partial L}{\partial \phi} = \quad \frac{1}{g} \ \phi \quad gh + \frac{1}{2}\phi \quad \frac{\partial \beta}{\partial t} + \phi \ \frac{\partial \beta}{}$$

$$+ \quad + \quad = 0 \tag{6.20}$$

for $= 1 \ \ldots \ .$

One way of solving (6.20) is by using Newton's method. Given an approximation to an updated approximation is obtained from

$$= \quad +$$

where

$$(\ ) \quad = \quad (\ ) \tag{6 21}$$

and

$$(\ ) = \quad \frac{1}{} \quad \frac{}{} + \quad \frac{}{} \quad \frac{}{} + \quad \frac{}{}$$

$$+ \quad + \frac{1}{2} \quad \frac{}{}\frac{}{}$$

The Jacobian J is symmetric and banded and (6.21) is solved, when J is not indefinite, using the pre-conditioned conjugate gradient method, with the pre-conditioning matrix $P = \text{diag}(J_{11}, \ldots, J_{ll})$. Both J and $\mathbf{F} = (F_1, \ldots, F_l)^T$ are evaluated exactly. The process is continued until

$$\frac{\max_i \left|\delta\phi_i^k\right|}{\max_i \left|\phi_i^k\right|} < \text{tolerance}, \tag{6.22}$$

for some specified tolerance.

The initial approximation $\boldsymbol{\phi}^0$ is given by

$$\phi_i^0 = \frac{x_i - x_1}{x_2 - x_1} v^0 + (T_i - T_1)\,\bar{E} \qquad i = 1, \ldots, l, \tag{6.23}$$

for some constants v^0 and $\bar{E}$.

Let $x_e = 0$, $x_o = 10$, $t_1 = 0$ and $t_2 = 10$. The algorithm is implemented in a channel with $B(x) = 10$ and $h(x) = 0$ for $x \in [0, 10]$. The boundary functions $C_e(t)$, $C_o(t)$, $g_1(x)$ and $g_2(x)$ also need to be prescribed. There is an obvious difficulty with defining $g_2(x)$, the depth for $x \in [0, 10]$ at the time t_2. Here $g_1(x)$ and $g_2(x)$ are defined by $g_1(x) = g_2(x) = \hat{d}$, where $\hat{d} > 0$ is either the subcritical or the supercritical root of

$$g\hat{d}^3 - \tilde{E}\hat{d}^2 + \frac{1}{2}C^2 = 0,$$

where $\tilde{E}$ is given the value 50 and $C = 10$, that is, g_1 and g_2 are the depths in the channel for a steady state flow with energy $\tilde{E} = 50$ and mass flow at inlet $C = 10$. The functions $C_e(t)$ and $C_o(t)$ are defined by

$$C_e(t) = \begin{cases} 10 - \hat{C}\,t & t \in [0, 2] \\ 10 - \hat{C}(4 - t) & t \in [2, 4] \\ 10 & t \in [4, 10] \end{cases},$$

$$
(\;) \; = \; \begin{array}{lll} 10 \quad \hat{} & & [0 \;\; 2] \\ 10 \quad \hat{}\,(4 \qquad) & & [2 \;\; 4] \\ 10 & & [4 \;\; 10] \end{array}
$$

where $\hat{}$ is a given constant in the range 0 $\hat{}$ 10.

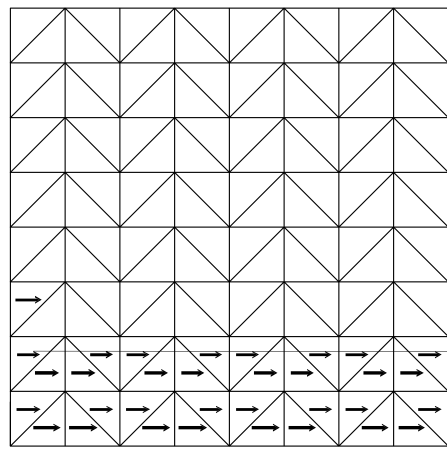
Thus, in this example, any time-dependence of the resulting flow is due to the changes in the mass flow at inlet and outlet with time. The conditions given above could be generated in practice by taking an initial steady flow with energy 50, and values of mass flow at inlet and outlet of 10, where these values are controlled by using a weir or a sluice gate. The values of the mass flow at inlet and outlet could be altered for [0 4] and then returned to their original values. The flow might then be expected eventually to return to a steady state.

Let $= 9$ and $= 9$. Then the algorithm converges to a subcritical approximation in 4 Newton iterations for $\hat{} = 0$, in 5 Newton iterations for $\hat{} = 1$ and in 5 Newton iterations for $\hat{} = 3$, for a tolerance in (6.22) of 10 . The initial approximation is, in each case, given by (6.23), with $= 2\;5$ and $\bar{} = 50$.

The value of in each element may be thought of as being an average of the velocity taken over the time period covered by the element. The piecewise constant velocity approximations for $= 9$ and $= 9$, in the three cases $\hat{}$

effect is more pronounced in Figure 6.3 , where the reduction in the mass flow at inlet and outlet is larger.

The success of the algorithm is heavily dependent on choosing boundary conditions which are consistent with one another. In particular, for certain choices, there may be no solution at all or the solution mt8cjay be dilazˆ+s8z0kqm8re'S˜ˆ8zo8+,



Ch pter 7

Concluding Rem rks

The central part of the work described in this thesis can be thought of as having two distinct components. The first component deals with the derivation of variational principles for free surface flows while in the second part a selection of these variational principles is used to generate numerical approximations to free surface flows. Variational principles for three-dimensional free surface flows are stated in Chapter 3 and used to derive principles for shallow water flows. Approximations to shallow water flows are generated in Chapters 4, 5 and 6, and Chapter 6 also contains an algorithm for approximating three-dimensional steady flows in a channel of constant breadth.

Two variational principles for general three-dimensional flows are used — one based on Hamilton's principle, (3.12), and the other based on Luke's principle (Luke (1967)). By approximating the variables by their shallow water counterparts, performing the integration with respect to the vertical coordinate z and adding on appropriate boundary terms, these two principles are reduced to give variational principles for shallow water. The process of changing variables in the functionals of the two shallow water principles derived in this way then allows the integrands to be expressed in terms of the p and r functions, defined by (3.27) and (3.29) respectively, and multiples of the conservation laws, (2.20) and (2.24), and the irrotationality condition, (2.15).

The function p has the values of vertically averaged pressure while r may be thought of as a Lagrangian density since its value at a point is the difference between kinetic and potential energy of a particle at that point. By recognising that p and r are related by means of a Legendre transform, two further functions — denoted by P and R — are constructed, so that p, r, P and R constitute a quartet of functions related to one another by a closed set of Legendre transforms, as shown in Figure 3.3. The function P has the values of flow stress and the value of R at a point is the total energy of a particle at that point.

Benjamin and Bowman (1987) consider conservation laws and symmetry properties of Hamiltonian systems, including shallow water, for which they derive four functions, two of which — identified by them as a Hamiltonian density and a flow force — have the values of the functions R and P respectively, apart from constant multipliers. The approach described here is more direct.

A set of four functionals — based on the p, r, P and R functions — is comprised of the two functionals derived from the variational principles for three-dimensional free surface flows and the two functionals generated by substituting P and R for p and r, respectively, in these functionals using the Legendre transforms. By making the assumption that the flow is independent of time, functionals for steady state flows are derived. Then, constraining the variations in the 'p' principle for steady flow to satisfy irrotationality, giving (3.94), and constraining the variations in the 'P' principle for steady flow to satisfy the conservation of mass equation and a boundary condition on the mass flow, giving (3.95), the gas dynamics analogy may be invoked to identify (3.94) and (3.95) as examples of Bateman's functions (Bateman (1929)). Sewell (1963) re-examined the relationships between these principles, in the context of Legendre transforms, for three-dimensional steady flows in perfect fluids. Use has been made of these variational principles, by, for example, Lush and Cherry (1956) and Wixcey (1990), to generate approximate solutions to the equations of motion for compressible gas flows.

For the case of shallow water there exist the extra variational principles — the 'r' and 'R' principles and all of their constrained versions — which may be used to approximate solutions of the shallow water equations. These principles are of particular value since they contain functionals of the depth of flow and can thus be used directly for generating approximations to the depth, unlike the constrained 'p' and 'P' principles.

The implementation of variational principles for finding approximations to time dependent flows reveals several inherent problems, which are discussed below. Therefore, with one exception, the numerical methods are applied to variational principles for steady state flows.

The constrained 'r' principle (3.117) for steady quasi one-dimensional flow depends on only one variable — the depth of flow — which makes it a natural candidate for developing an algorithm to generate approximations to the depth function. The constrained 'p' principle for steady flow (3.94) and the version of the variational principle for steady quasi one-dimensional flow are useful too since they also depend on only one variable each — the velocity potential. Other variational principles are used as well, namely, the unconstrained 'r' principle for steady quasi one-dimensional flow, which depends on the depth, mass flow and velocity potential functions, the 'R' principle for steady flow constrained to satisfy irrotationality, which depends on the depth and velocity potential functions, the constrained 'p' principle for unsteady quasi one-dimensional flow and a version of Luke's free surface principle (Luke (1967)), which depends on the velocity potential and the height of the free surface.

The same basic algorithm is applied to all of the variational principles and is, on the whole, successful. The variables in the variational principles are expressed as expansions in terms of finite element basis functions — piecewise linear and piecewise constant basis functions in one dimension and piecewise linear basis functions in two dimensions. The parameters of the solutions are determined as the values which cause the functionals of the variational principles to be stationary with respect to variations in the finite dimensional space spanned by the finite element basis functions. In each case this leads to one or more sets of equations, at least one of which is non-linear.

The method chosen to solve these non-linear sets of equations is Newton's method, which has quadratic convergence to the approximate solution, given an initial guess sufficiently close to the solution. The Jacobian in each case is symmetric and banded and, in fact, tridiagonal for the equations generated from the functionals for steady quasi one-dimensional flow. For tridiagonal Jacobians update is found using a pre-conditioned conjugate gradient method.

In this way it is possible to find approximations to the shallow water variables in cases where the flow is continuous.

A slightly different approach is taken in order to approximate discontinuous flows and in using a version of Luke's principle for free surface flows to approximate flows which do not necessarily satisfy the assumptions of shallow water theory. The constrained 'r' principle for steady quasi one-dimensional flow is used to generate approximations to discontinuous depth functions. In both cases the flow variables — depth in the 'r' principle case and velocity potential in the version of Luke's principle — are expanded in terms of the finite element basis the depth function. The piecewise constant approximation is found to converge linearly, in the norm, to the depth of flow. Numerical experiments show that the piecewise linear approximation to the depth, generated using the constrained 'r' principle, is quadratically convergent, in the norm, to the exact solution. The error in the piecewise constant approximation to the velocity, derived from the piecewise linear approximation to the velocity potential generated using the constrained 'p' principle for steady quasi one-dimensional flow, is considered in functional of the constrained 'R' principle, both of which are used in Chapter 5 outlet points of the channel, over the whole time interval being considered, and values of the depth at every point in the channel at the initial and final times. In order to prescribe this last condition the solution at the final time must be tion for shallow water flows in the domain of integration, assuming that variations vanish at the initial and final times so that the time boundary terms also vanish (the solutions are assumed known on the time boundaries and so these terms are constants).

In order to use the variational principles for time-dependent shallow water to generate time-dependent approximations, boundary terms are added in Chapter 3 so that non-zero variations are allowed on the time boundaries and no assumption need be made about knowing the solution at the final time. However, it can be seen that this only rephrases the basic problem since the solutions at the ends of the time interval are precisely the functions which are required for the boundary terms.

The numerical methods employed in this thesis have been successful in generating approximations to continuous steady flows which are wholly subcritical, for both quasi one-dimensional and two-dimensional flows, or wholly supercritical, for quasi one-dimensional flows. Success is also achieved in approximating discontinuous steady quasi one-dimensional flows. However, as described above, the application of the methods to the time-dependent case as F9 cto iffi0S7U(c0ffq19uS(o1q‘7U(y so tUfimjffSdo‘(o170S(he79)o170Sdede mations to two-dimensional flows and approximations to discontinuous flows in two-dimensions. In the first case, if an attempt is made to solve the non-linear set of equations obtained from the finite dimensional versions of the functionals by Newton's method, the Jacobian is indefinite so that a more sophisticated technique for solving a system with an indefinite matrix must be investigated; there

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