The University of Reading
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Modelling Glacier Flow

by

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Declaration

I confirm that this is my own work and the use of all materials from other sources have been properly and fully acknowledged.

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Abstract

In this dissertation a moving mesh method is used to produce a numerical approximation of a simple glacier model, which is a second order nonlinear diffusion equation, for the purpose of investigating how a glacier moves over time. The same model is also solved on a fixed grid for qualitive comparison with the moving mesh method.

Contents

Chapter 1

Introduction

This dissertation will provide a moving mesh method of the simple glacier equation, which is also the highly nonlinear second order PDE

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial H}{\partial x}\right) + M \tag{1.1}$$

with the snow term M and D defined as

$$D = cH^5\left(\frac{\partial H}{\partial x}\right)^2 \tag{1.2}$$

with boundary conditions $H = 0$ at the boundaries, and initial condition, when $t = 0$, $H = 0$ at all points.

In chapter 2 a brief introduction to glaciers will be given, with aspects such how glaciers are formed and the environmental impact of glaciers. In chapter 3 the simple model defined above shall be derived from the continutity equation, and conservation of mass principle. Chapter 4 will see the self-similar solution calculated, and the model shown to be scale invariant under the mappings defined. The model will be approximated on a fixed grid in chapter 5, and instabilities found will be smoothed

Chapter 2

Glaciers and Ice Sheets

A glacier can form in any climate zone where the input of snow exceeds the rate at which it melts. The amount of time required for a glacier to form depends on the rate at which the snow acculumates and turns to ice. Once a glacier has formed, its survival depends on the balance between acculumation and ablation (melting), this balance is known as mass balance and is largely dependant on climate.

The study of glacier mass balance is concerned with inputs to and outputs from the glacial system. Snow, hail, frost and avalanched snow are all inputs, output is generally from melting of snow, and calving of ice into the sea. If these inputs survive the summer ablation period the transformation into glacier ice will begin, the snow that has begun this transformation is called firn. The transformation involves three steps: compaction of the snow layers; the expulsion of the trapped air; and the growth of ice crystals. Again, the rate at which this transformation takes place depends on climate. If the rate of acculumation is high and significant melting occurs then the process can be very rapid since the older snow is buried by fresh snow which then compacts the firn; while alternate freezing and melting encourages the growth of new ice crystals. In contrast, if precipitation is low and little melting occurs, the process can be very slow.

During the period between 1550 and 1850 known as the Little Ice Age, glaciers increased. After this period, up unitl about 1940, glaciers worldwide retreated in response to an increase in climate temperature. This recession slowed during the period 1950 to 1980, in some cases even reversing, due to small global cooling. However, widespread retreat of glaciers has increased rapidly since 1980, increasing even more since 1995. Excluding the ice caps and sheets of the arctic and antarctic, the total surface area of glaciers worldwide has decreased by 50% since the end of the 19th century.

There are many different reasons for studying glaciers, the environmental impacts being at the forefront in recent years. Climatologists argue that increased temperature will lead to increased melting of glacial ice which can contribute to rising sea levels. The total ice mass on the Earth covers almost 14,000,000 km^2 of area, that is around 30,000,000 km^3 of ice. If all this ice melted the sea level would quiescent phase; during this period the glacier retreats significantly due to much lower velocities. Glacial surges can cause problems in shipping lanes where large amounts of ice discharge into the sea.

Chapter 3

The Simple Glacier Model

A glacier flows in the direction of decreasing surface elevation due to driving stress, resistance of this force can come from the glacier bed and at lateral margins, or it may be associated with gradients in longitudinal stresses. Most numerical models of ice-flow are based on the lamellar flow where the driving stress is taken to be opposed entirely by basal drag, so the longitudinal stresses and lateral shear are neglected.

3.1 The Continuity Equation

Numerical models of glaciers require that no ice may be created or lost. Changes in ice thickness at any point must be due to the flow of the ice and local snowfall or loss

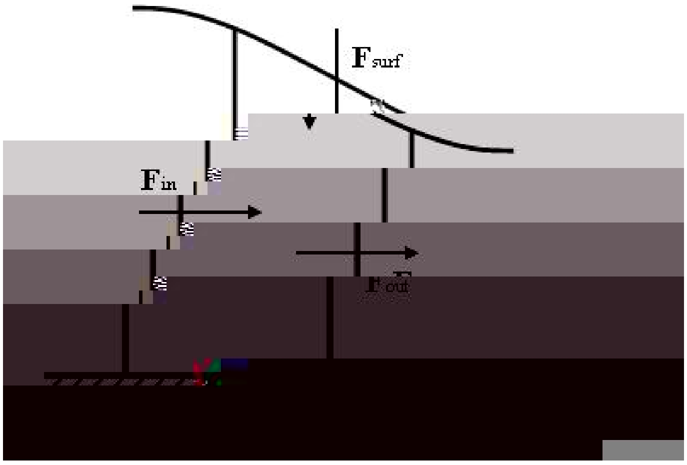

Figure 3.1: Diagram of ice fluxes into and out of a vertical column of ice extending from the bed to the surface

of ice flowing through the section per unit time per unit width in the cross-section direction. So the flux into the column is defined to be

$$F_{in} = HU(x) \tag{3.1}$$

and the ice flowing out of the column is defined as

$$F_{out} = HU(x + \Delta x) \tag{3.2}$$

If M is the accumulation rate of the ice, that is the snowfall and melting or calving, then the ice flux into the surface is defined to be

$$F_{surf} = M\Delta x \tag{3.3}$$

The ice becomes thicker or thinner when these three terms are not in balance. The density of the ice is taken to be zero so that the densification in the upper firn layers can be neglected, therefore the conservation of mass corresponds to conservation of volume. By definition, the rate of change of the ice thickness is $\frac{\partial H}{\partial x}$ and so the rate of change of volume of the column is defined as $\frac{\partial H}{\partial t}\Delta x$. Therefore conservation of mass (or volume) is

$$\begin{aligned}\frac{\partial H}{\partial t}\Delta x &= F_{in} - F_{out} + F_{surf} \\ &= HU(x) - HU(x+\Delta x) + M\Delta x \end{aligned} \tag{3.4}$$

dividing both sides by Δx and taking the limit as $\Delta x \to 0$ gives the continuity equation

$$\frac{\partial H}{\partial t} = -\frac{\partial (HU)}{\partial x} + M \tag{3.5}$$

3.2 Derivation of Simple Model

From [1] the vertical-mean velocity is given by

$$U = \frac{2AH}{n+2}\tau_{dx}^{n} + U_s \tag{3.6}$$

where τ_{dx} is the driving stress, U_s is the sliding velocity, and A and n are the flow parameters from Glen's Flow Law. Again, from [1] the driving stress is given by

$$\tau_{dx} = -\rho g H\frac{\partial h}{\partial x} \tag{3.7}$$

where ρ is the ice density, g is the acceleration due to gravity, and h is the surface elevation of the ice.

x_b is the right hand boundary. A is set to be a constant for this model, a common value used is 0.8×10^{-16}, and n shall be set as $n = 3$ from Glen's Flow Law. Glen's Flow Law gives a relationship between effective stress rate and strain rate, further information can be found in chapter 2 of [1]. The density of ice is set as 910kgm^{-3}, so the constant $c = 0.000022765$.

This makes the simple glacier model

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(\frac{2A}{5}(\rho g)^3 H^5 \left(\frac{\partial H}{\partial x}\right)^3\right) + b(x_b - ax) \tag{3.14}$$

with conditions $H = 0$ at the boundaries, and initial condition $H = 0$ for all x. This is the model that is used in this dissertation. In the next chapter the scale invariance of the simple model shall be looked at.

Chapter 4

Scale Invariance

An equation is said to be scale invariant if, for the partial differential equation

$$u_t = f(x, u, u_x, u_{xx} \dots) \tag{4.1}$$

a transformation can be defined mapping the system (u, x, t) to a new system $(\hat{u}, \hat{x}, \hat{t})$ such that the PDE remains unchanged. That is that the physical quantities of the PDE are not dependant on the the system in which they are observed. For any nonlinear partial differential equation that is satisfied by the system (u, x, t), the transformation mapping to the new system $(\hat{u}, \hat{x}, \hat{t})$ is defined as

$$u = \lambda^{\gamma}\hat{u}, \quad x = \lambda^{\beta}\hat{x}, \quad t = \lambda^{\alpha}\hat{t}. \tag{4.2}$$

for some arbitrary positive parameter, λ. The solution is self-similar if it is invariant under the mappings defined above.

Consider the second order simple glacer model, defined on the system (H, x, t)

$$\frac{\partial H}{\partial t} = c\frac{\partial}{\partial x}\left(H^5\left(\frac{\partial H}{\partial x}\right)^3\right) + b(x_b - ax) \tag{4.3}$$

with boundary conditions $H = 0$ at the boundaries and initial condition $H = 0$ everwhere; where a, b are constants, x_b is the right hand boundary and $c = \frac{2A}{5}(\rho g)^3$ is constant. Then, using the transformations (4.2) with H substituted for u gives

$$\frac{\partial H}{\partial x} = \frac{\partial(\lambda^\gamma \hat{H})}{\partial(\lambda^\alpha \hat{t})} = \lambda^{\gamma-\alpha}\frac{\partial \hat{H}}{\partial \hat{t}} \tag{4.4}$$

$$c\frac{\partial}{\partial x}\left[H^5\left(\frac{\partial H}{\partial x}\right)^3\right] = c\frac{\partial}{\partial(\lambda^\beta \hat{x})}\left[(\lambda^\gamma \hat{H})^5\left(\frac{\partial(\lambda^\gamma \hat{H})}{\partial(\lambda^\beta \hat{x})}\right)^3\right] \tag{4.5}$$

$$b(x_b - ax) = \lambda^\beta b(\hat{x}_b - a\hat{x}) \tag{4.6}$$

and so,

$$\lambda^{\gamma-\alpha}\frac{\partial \hat{H}}{\partial \hat{t}} = c\lambda^{8\gamma-4\beta}\frac{\partial}{\partial \hat{x}}\left[\hat{H}^5\left(\frac{\partial \hat{H}}{\partial \hat{x}}\right)^3\right] + \lambda^\beta(\hat{x}_b - a\hat{x}) \tag{4.7}$$

In order for (4.3) to be scale invariant $\lambda^{\gamma-\alpha}$, $\lambda^{8\gamma-4\beta}$, and λ^β must cancel, i.e.

$$\gamma - \alpha = 8\gamma - 4\beta = \beta$$

so,

$$8\gamma = 5\beta \Rightarrow \gamma = \frac{5}{8}\beta$$

Now, let $\alpha = 1$, then

$$\gamma - 1 = \beta \Rightarrow \beta = -\frac{8}{3}$$

and,

$$y = -\frac{5}{3}$$

So, the PDE is scale invariant under the transformations

$$H = \lambda^{-\frac{5}{3}}\hat{H}, \quad x = \lambda^{-\frac{8}{\ }}$$

Now consider the right hand side of (5.3), since

$$\frac{\partial \xi}{\partial x} = t^{8/3} \tag{4.11}$$

then

$$\begin{aligned} c\frac{\partial}{\partial x}\left(H^5\left(\frac{\partial H}{\partial x}\right)^3\right) &= c\frac{\partial \xi}{\partial x}\frac{d}{d\xi}\left((t^{-5/3}\eta)^5\left(\frac{\partial \xi}{\partial x}\frac{d}{d\xi}(t^{-5/3}\eta)\right)^3\right) \\ &= ct^{8/3}\frac{d}{d\xi}\left(t^{-25/3}\eta^5\left(t^{8/3}\frac{d}{d\xi}(t^{-5/3}\eta)\right)^3\right) \\ &= ct^{-8/3}\frac{d}{d\xi}\left(\eta^5\left(\frac{d\eta}{d\xi}\right)^3\right) \end{aligned} \tag{4.12}$$

The snow term becomes

$$b(x_b - ax) = t^{-8/3}b(\xi_b - a\xi) \tag{4.13}$$

and so,

$$\frac{8}{3}\xi t^{-8/3}\eta' - \frac{5}{3}t^{-8/3}\eta = ct^{-8/3}(\eta^5\eta'^3)' + t^{-8/3}b(\xi_b - a\xi) \tag{4.14}$$

cancelling the $t^{-8/3}$ terms gives

$$\frac{8}{3}\xi\eta' - \frac{5}{3}\eta = c\left(\eta^5\eta'^3\right)' + b(\xi_b - a\xi) \tag{4.15}$$

So the second order PDE in (H, x, t) becomes a second order ordinary differential equation in (ξ, η). This ODE can be solved numerically using finite differences to approximate the equation.

4.2 Numerical Approximation of the Self-Similar Solution

The self-similar solution is approximated over a normalised fixed grid with $\xi \in [0, 1]$, and $\eta = 0$ at both boundaries, corresponding to $H = 0$ at the boundaries; initially $\eta = 0$ at all points, again to correspond with $H = 0$ initially, i.e. there is no ice to start with. Using a central difference approximation, the following is obtained

$$\frac{8}{3}ih\left(\frac{\eta_{i+1}-\eta_{i-1}}{2h}\right) - \frac{5}{3}\eta_i = c\left(\frac{(\eta^5\eta'^3)_{i+\frac{1}{2}} - (\eta^5\eta'^3)_{i-\frac{1}{2}}}{h}\right) + b(1-aih) \tag{4.16}$$

expanding the right hand side of the equation

$$\begin{aligned} &= \frac{c}{h}\left(\left(\frac{\eta_{i+1}+\eta_i}{2}\right)^5\left(\frac{\eta_{i+1}-\eta_i}{h}\right)^3 - \left(\frac{\eta_i+\eta_{i-1}}{2}\right)^5\left(\frac{\eta_i-\eta_{i-1}}{h}\right)^3\right) \\ &+ b(1-aih) \end{aligned} \tag{4.17}$$

where ξ has been approximated by $\xi = ih$ for all i.

To approximate, (4.17) is made linear by splitting $\left(\frac{\eta_{i+1}-\eta_i}{h}\right)^3$ and $\left(\frac{\eta_i-\eta_{i-1}}{h}\right)^3$ into two components, one at level k and one at level $k+1$, such that

$$\left(\frac{8}{6}i + \frac{c}{h^2}M\right)\eta_{i+1}^{(k+1)} + \left(\frac{c}{h^2}L + \frac{c}{h^2}M - \frac{5}{3}\right)\eta_i^{(k+1)} + \left(-\frac{8}{6}i - \frac{c}{h^2}M\right)\eta_{i-1}^{(k+1)} = b(aih-1) \tag{4.18}$$

where

$$L = \left(\frac{\eta_{i+1}^{(k)}+\eta_i^{(k)}}{2}\right)^5\left(\frac{\eta_{i+1}^{(k)}-\eta_i^{(k)}}{h}\right)^2 \tag{4.19}$$

$$M = \left(\frac{\eta_i^{(k)}+\eta_{i-1}^{(k)}}{2}\right)^5\left(\frac{\eta_i^{(k)}-\eta_{i-1}^{(k)}}{h}\right)^2 \tag{4.20}$$

This gives a tridiagonal matrix that is inverted in the program using the Gauss-Seidel method.

4.3 Numerical Results of Self-Similar Solution

Figure 4.1 shows the self-similar solution of the simple glacier model with $h = 0.005$.

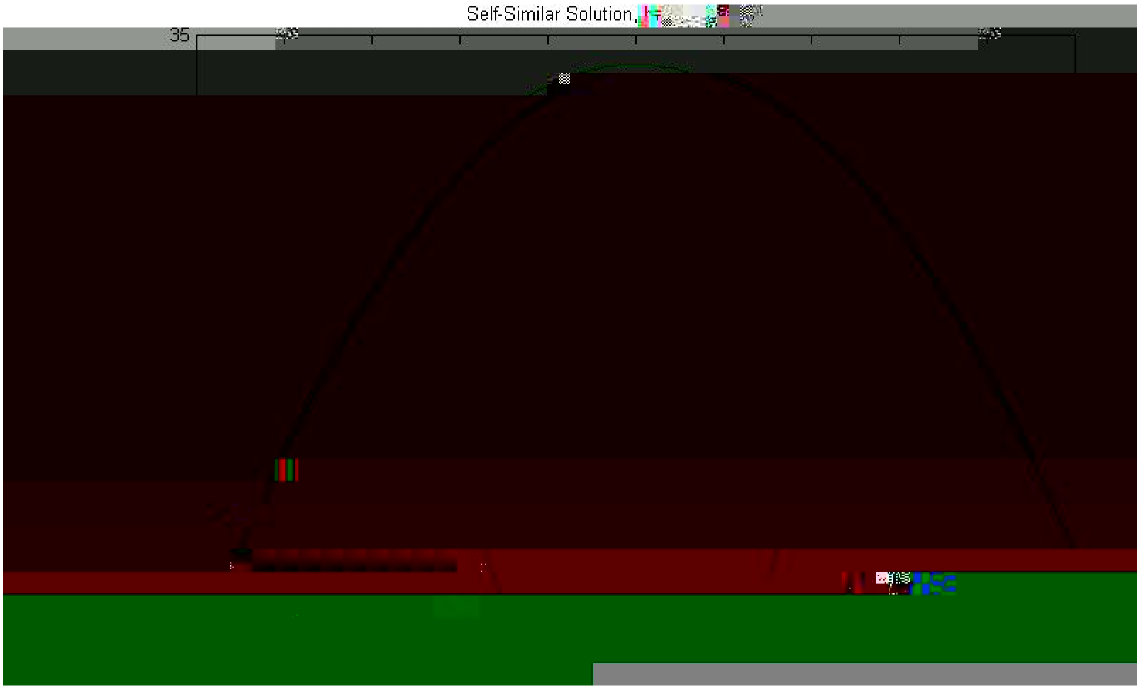

Figure 4.1: Self-Similar Solution of Model, $h = 0.005$

The self-similar solution is a particular solution of the simple glacier model that can be used to test the numerical approximations in chapters 5 and 6. The graph shows the solution after convergence of the solution has been reached.

The graph is quite symmetrical, it is not expected that the numerical approximations in the next chapters will reflect this property as the snow term in this approximation is of a slightly different form. In the following the chapters the snow term is unable to be negative, instead at these points it is set to zero; this is not reflected in the ODE solved here. It would be expected that there would be a bias of mass to the left if this was the case.

Chapter 5

A Fixed Grid Approximation

Basic numerical approximations are done on grids with a fixed number of nodes, and a fixed grid spacing, where a node is calculated using the values of the nodes that surround it, or the boundary conditions if the node is close to or on a boundary. There are many different methods for solving such approximations, most of which involve inverting matrices. These methods can be explicit, where the solution is calculated from previous time steps, or implicit; where the solution is calculated using information from both previous time steps and the current time step.

5.1 A Fixed Grid Approximation

$$D = cH^5 \left(\frac{\partial H}{\partial x} \right)^2 \tag{5.2}$$

as usual, a, b, c, x_b are constants. Using a finite difference approximation to approximate the spacial derivative of H at the time level $t = n$ gives the following approximation of the diffusivity

$$D_i^{(n)} = cH_i^{5(n)} \left(\frac{\left(H_{i+1}^{(n)} - H_{i-1}^{(n)} \right)}{2\Delta x} \right)^2 \tag{5.3}$$

The ice flux $F_{i\pm1/2}$ is approximated using an average of the diffusivity and a forward or backward difference approximation of $\frac{\partial H}{\partial x}$

$$F_{i+1/2} = \frac{1}{2} \left(D_{i+1}^{(n)} + D_i^{(n)} \right) \left(\frac{H_{i+1}^{(n+1)} - H_i^{(n+1)}}{\Delta x} \right)^2 \tag{5.4}$$

The flux term in (5.1) is approximated by $\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x}$, expanding gives

$$\frac{c}{2\Delta x} \left(H_i^{5(n)} \left(\frac{H_{i+1}^{(n)} - H_{i-1}^{(n)}}{2\Delta x} \right)^2 + H_{i+1}^{5(n)} \left(\frac{H_{i+1}^{(n)} - H_i^{(n)}}{\Delta x} \right)^2 \right) \left(\frac{H_{i+1}^{(n+1)} - H_i^{(n+1)}}{\Delta x} \right)$$
$$- \frac{c}{2\Delta x} \left(H_i^{5(n)} \left(\frac{H_{i+1}^{(n)} - H_{i-1}^{(n)}}{2\Delta x} \right)^2 + H_{i-1}^{5(n)} \left(\frac{H_i^{(n)} - H_{i-1}^{(n)}}{\Delta x} \right)^2 \right) \left(\frac{H_i^{(n+1)} - H_{i-1}^{(n+1)}}{\Delta x} \right)$$

The time derivative can be approximated using Euler's Explicit Scheme, i.e.

$$\frac{\partial H}{\partial t} = \frac{H_i^{(n+1)} - H_i^{(n)}}{\Delta t} \tag{5.5}$$

The ice thickness, H, is approximated using

$$\frac{H_i^{(n+1)} - H_i^{(n)}}{\Delta t} = \frac{F_{i+1/2} - F_{1-1/2}}{\Delta x} + M \tag{5.6}$$

The matrix A will be inverted using the Thomas Algorithm, this algorithm only acts on the interior points and so first the matrix needs to be rearranged into

$$\begin{pmatrix} b_1 & c_1 & 0 & 0 & \dots \\ a_2 & b_2 & c_2 & 0 & \dots \\ \ddots & \ddots & \ddots & \dots & c_{N-2} \\ 0 & 0 & \dots & a_{N-1} & b_{N-1} \end{pmatrix} \begin{pmatrix} H_1^{(n+1)} \\ H_2^{(n+1)} \\ \vdots \\ H_{N-1}^{(n+1)} \end{pmatrix} = \begin{pmatrix} f_1 - a_1 H_0^{(n+1)} \\ f_2 \\ \vdots \\ f_{N-1} - c_{n-1} H_N^{(n+1)} \end{pmatrix}$$

The Thomas Algorithm then follows two distinct steps. Firstly, a forward sweep of the matrix "removes" the diagonal of a_i's to obtain a matrix with two diagonals. The new system is defined by

$$\begin{pmatrix} b'_1 & c'_1 & 0 & 0 & \dots \\ 0 & b'_2 & c'_2 & 0 & \dots \\ \ddots & \ddots & \ddots & \dots & c'_{N-2} \\ 0 & 0 & \dots & 0 & b'_{N-1} \end{pmatrix} \begin{pmatrix} H_1^{(n+1)} \\ H_2^{(n+1)} \\ \vdots \\ H_{N-1}^{(n+1)} \end{pmatrix} = \begin{pmatrix} f'_1 - a_1 H_0^{(n+1)} \\ f'_2 \\ \vdots \\ f'_{N-1} - c_{n-1} H_N^{(n+1)} \end{pmatrix}$$

where the new coeﬃcients are defined as

$$\begin{aligned} b'_i &= b_i - c'_{i-1} \frac{a_i}{b'_{i-1}} \\ c'_i &= c_i \\ f'_i &= f_i - f'_{i-1} \frac{a_i}{b'_{i-1}} \end{aligned}$$

for all $i = 2, 3 \dots, N-1$, and

$$\begin{aligned} b'_1 &= b_1 \\ f'_1 &= f_1 \end{aligned}$$

The second step involves a backward sweep of the new matrix in order the calculate the solution. Initially

$$H_{N-1}^{(n+1)} = \frac{f'_{N-1}}{b'_{N-1}} \tag{5.10}$$

then $H_{N-1}^{(n+1)}$ is used to calculate the next solution, and the loop continues with

$$H_i^{(n+1)} = \frac{f'_i - c'_i H_{i+1}^{(n+1)}}{b'_i} \tag{5.11}$$

for all $i = N-2, N-3\ldots, 1$.

Figure 5.1 shows the approximation of the simple glacier model on a fixed grid with $\Delta t = 0.0001$, $\Delta x = 0.1$, and $n = 200000$.

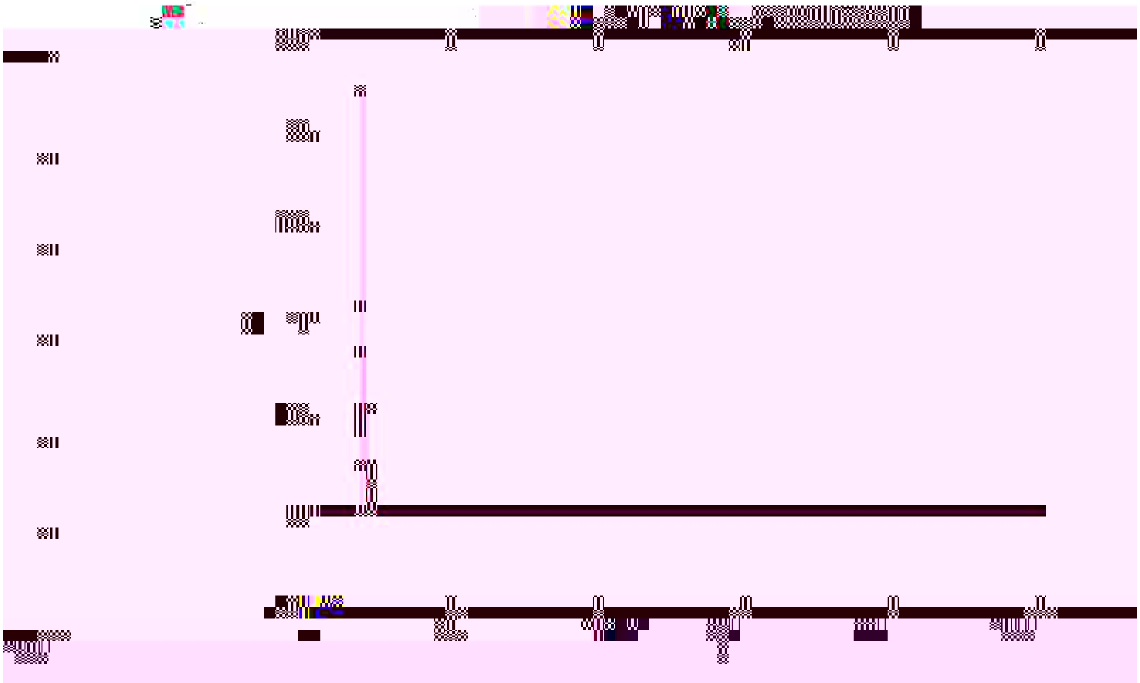

Figure 5.1: Unstable Fixed Grid Method, $\Delta t = 0.001$, $\Delta x = 0.1$, $n = 200000$

It can clearly be seen this is not an accurate representation of a glacier. The model becomes unstable and shoots up at one point. To make the approximation stable an extra diffusion term is added to the model. So the model is

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D \left(\frac{\partial H}{\partial x} \right) \right) + \frac{\partial^2 H}{\partial x^2} + M \tag{5.12}$$

where is a constant and D is as before.

So the new coefficients of the matrix A are defined to be

$$\begin{aligned}
a_i &= \frac{k}{4} H_i^{5(n)} \left(H_{i+1}^{(n)} - H_{i-1}^{(n)} \right)^2 + k H_{i+1}^{5(n)} \left(H_{i+1}^{(n)} - H_i^{(n)} \right)^2 + \frac{}{(\Delta x)^2} \\
b_i &= -\frac{k}{2} H_i^{5(n)} \left(H_{i+1}^{(n)} - H_{i-1}^{(n)} \right)^2 - k H_{i+1}^{5(n)} \left(H_{i+1}^{(n)} - H_i^{(n)} \right)^2 - k H_{i-1}^{5(n)} \left(H_i^{(n)} - H_{i-1}^{(n)} \right)^2 \\
&\quad -1 - \frac{2}{(\Delta x)^2} \\
c_i &= \frac{k}{4} H_i^{5(n)} \left(H_{i+1}^{(n)} - H_{i-1}^{(n)} \right)^2 + k H_{i-1}^{(n)} \left(H_i^{(n)} - H_{i-1}^{(n)} \right)^2 + \frac{}{(\Delta x)^2}
\end{aligned}$$

This has the effect of smoothing the unstable point and is therefore a much better approximation.

5.2 Numerical Results of Fixed Grid Approximation

Figure 5.2 shows the approximation with the smoothing term added.

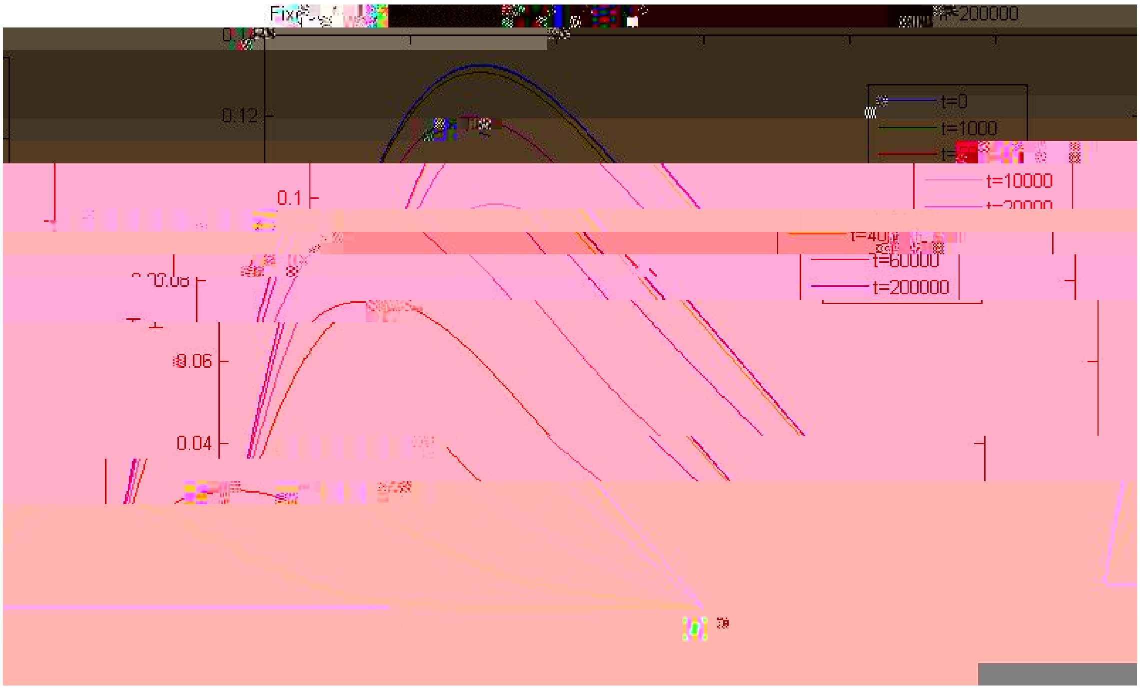

Figure 5.2: Fixed Grid Approximation with Extra Di□usion, Δt = 0.0001, Δx = 0.1, n = 200000

The model gains mass from the snow term before $x_i = 50$ then once a su□cient amount of ice mas has built up the glacier is forced to move to the right. The model cannot move to the left as the boundary condition $H_0 = 0$ prevents this, this conditions represents a deep ocean into which the glacier cannot extend.

Since this is a fixed grid approximation the right hand boundary is also fixed, so the model cannot extend beyond this boundary. In the next chapter a moving mesh method is used to approximate the glacier. Using this method, the right hand boundary is able to move.

Chapter 6

A Moving Mesh Method

There are a number of techniques used to improve the accuracy of the basic numerical meshes such as the fixed grid used in chapter 5, the main three being h-refinement, p-refinement and, the method to be adopted here, r-refinement. In h-refinement extra nodes are added to small areas of the grid to improve local resolution; p-refinement involves the use of higher order numerical approximations to improve local accuracy. For r-refinement, the number of nodes in the grid remains fixed but they are strategically relocated at each time step to where they can provide the most information.

Most techniques for generating adaptive moving meshes can be categorised into one of two groups: location based methods and velocity based methods. Location based methods, as the name suggests, are concerned with directly controlling the location of the mesh points. Velocity methods, such as the the method applied in this dissertation, compute the velocity of the mesh, $\frac{dx}{dt} = \dot{x} = v$. The new mesh points can then be found from this equation using time integration.

6.1 A Moving Mesh Method

Consider the second order simple glacier model derived in chapter 3

$$\frac{\partial H}{\partial t} = c\frac{\partial}{\partial x}\left(cH^5 \frac{\partial H}{} \right)$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} H dx = \int_{a(t)}^{b(t)} b(x_b - ax) \tag{6.6}$$

6.2 Numerical Approximations of The Moving Mesh Method

Equation (6.6) will be used to calculate the ice thickness in each interval so setting $b(t) = x_{i+1}$ and $a(t) = x_{i-1}$ gives

$$\frac{d}{dt}\int_{x_{i-1}}^{x_{i+1}} H dx = \int_{x_{i-1}}^{x_{i+1}} b(x_b - ax)dx \tag{6.7}$$

Using Euler's Explicit Scheme to approximate the time derivative gives

$$\int_{x_{i-1}}^{x_{i+1}} H dx|_{t=n+1} = \int_{x_{i-1}}^{x_{i+1}} H dx|_{t=n} + \Delta t \int_{x_{i-1}}^{x_{i+1}} b(x_b - ax)dx \tag{6.8}$$

then, applying the Midpoint Rule to approximate the H integrals gives

$$\frac{H_i^{(n+1)}}{2}(x_{i+1}^{(n+1)} - x_{i-1}^{(n+1)}) = \frac{H_i^{(n)}}{2}(x_{i+1}^{(n)} - x_{i-1}^{(n)}) + b\Delta t \left[x_b x_i^{(n)} - a\frac{x_i^{2(n)}}{2}\right]_{x_{i-1}}^{x_{i+1}} \tag{6.9}$$

Rearranging,

$$H_i^{(n+1)} = \frac{H_i^{(n)}(x_{i+1}^{(n)} - x_{i-1}^{(n)}) + 2b\Delta t \left[x_b(x_{i+1}^{(n)} - x_{i-1}^{(n)}) - \frac{a}{2}(x_{i+1}^{2(n)} - x_{i-1}^{2(n)})\right]}{(x_{i+1}^{(n+1)} - x_{i-1}^{(n+1)})} \tag{6.10}$$

This is the equation used to calculate the ice thickness, H, with boundary conditions $H(x_0, t) = H(x$
x

For the velocity, recalling (6.6) and integrating from x_0 to x_i gives

$$-H_i\left(\frac{\partial x}{\partial t}\right)_i + H_0\left(\frac{\partial x}{\partial t}\right)_0 = cH_i^5\left(\frac{\partial H}{\partial x}\right)_i^3 - cH_0^5\left(\frac{\partial H}{\partial x}\right)_0^3 \tag{6.11}$$

since $H_0 = 0$ and with $v = \frac{dx}{dt}$, then

$$-H_i v_i = cH_i^5\left(\frac{\partial H}{\partial x}\right)_i^3 \tag{6.12}$$

and so

$$v_i = -cH_i^4\left(\frac{\partial H}{\partial x}\right)_i^3 \tag{6.13}$$

This is the equation that is used to determine the velocity of each node.

To approximate the velocity (6.13) is first rearranged as

$$\begin{aligned} v_i &= -\left(H^{4/3}\frac{\partial H}{\partial x}\right)^3 \\ &= -\left(\frac{3}{7}\frac{\partial}{\partial x}H^{7/3}\right)^3 \end{aligned} \tag{6.14}$$

Then a forward difference approximates the derivative, giving

$$v_i = -\left(\frac{3}{7}\right)^3\left(\frac{\left(H^{7/3}\right)_{i+1} - \left(H^{7/3}\right)_i}{x_{i+1} - x_i}\right)^3 \tag{6.15}$$

This is the equation used to approximate the velocity. From the velocity, the new grid point can be obtained using time integration. By definition $v = \frac{dx}{dt}$, so applying Euler's Explicit Scheme again gives

6.3 Numerical Results of The Moving Mesh Method

Figure 6.1 shows the moving mesh approximation without the smoothing term included.

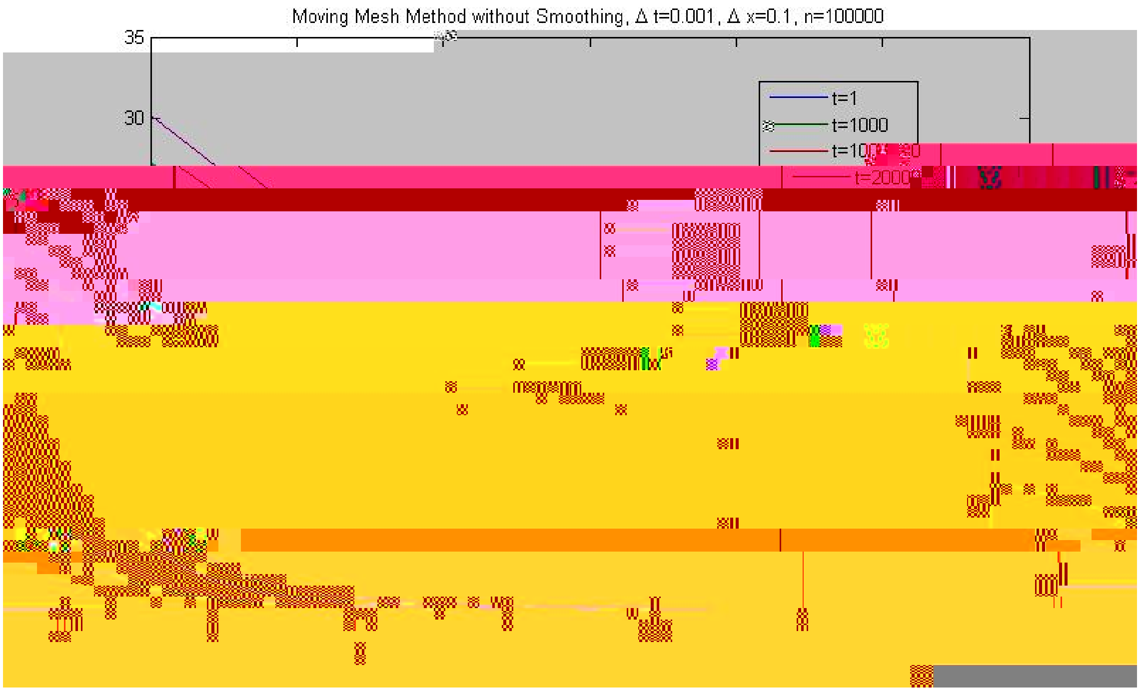

Figure 6.1: Moving Mesh Method with No Smoothing, $\Delta t = 0.001$, $\Delta x = 0.1$, $n = 100000$

Figure 6.2 shows the evolution of the glacier from $t = 1$ to $t = 100000$, with a time step of $\Delta t = 0.001$ after the smoothing term has been included.

At $t = 1$ the glacier has had input from the snow term up to $x = 0.5$ and so begins to build. At $t = 1000$ a distinct build up of mass has occured, though not enough to force the glacier to start to move yet. It can be seen that at later times a sufficient amount of ice mass has built up to force the boundary to start to move, allowing the glacier to flatten and spread out. The left hand boundary condition has been set at $H(x_0) = 0$ to represent a deep ocean at this point, this means that the

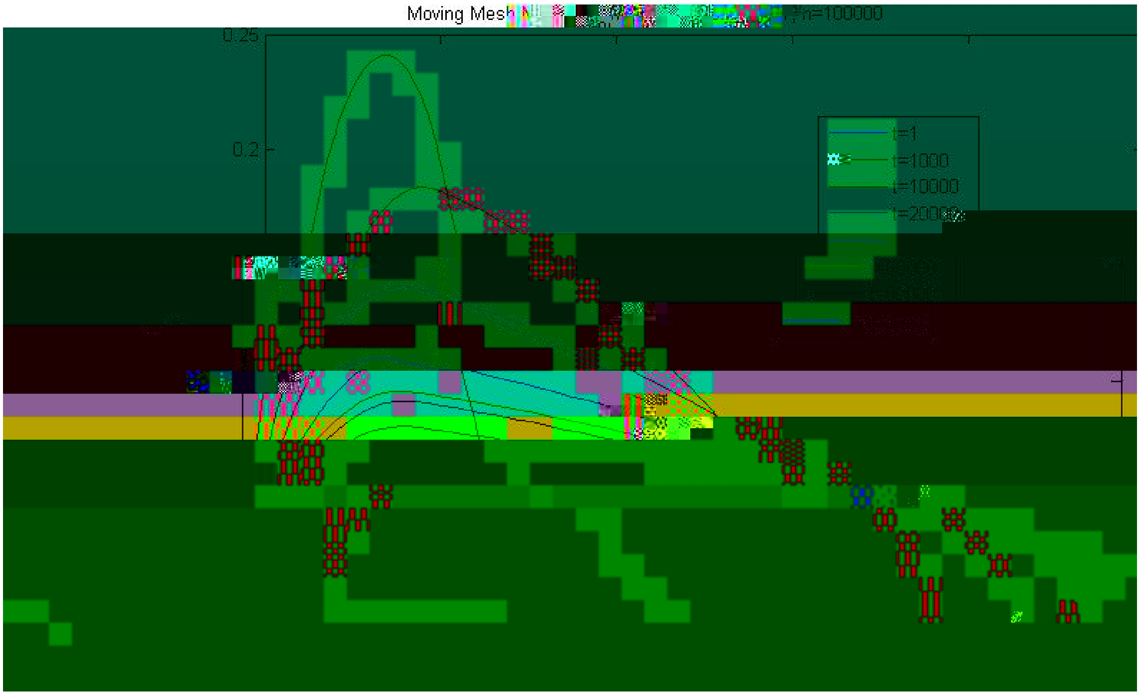

Figure 6.2: Moving Mesh Method, $\Delta t = 0.001$, $n = 100000$

the moving mesh, Δt could not be taken any larger than $\Delta t = 0.001$ as the solution blew up extremely quickly.

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