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AN L^∞ REGULARISATION STRATEGY TO THE INVERSE SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC EQUATIONS

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Abstract. In this paper we utilise new methods of Calculus of Variations in L^∞ to provide a regularisation strategy to the ill-posed inverse problem of identifying the source of a non-homogeneous linear elliptic equation, satisfying Dirichlet data on a domain. One of the advantages over the classical Tykhonov regularisation in L^2 is that the approximated solution of the PDE is uniformly close to the noisy measurements taken on a compact subset of the domain.

1. Introduction

Let $n \in \mathbb{N}$ and $\Omega \Subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ regular boundary $\partial\Omega$. Let also L be the linear non-divergence differential operator

$$\mathrm{L}[u] := A : \mathrm{D}^2 u + b \cdot \mathrm{D}u + cu \tag{1.1}$$

which is assumed to be uniformly elliptic with bounded continuous coefficients:

$$\left\{ \begin{array}{l} A \in (C^0 \cap L^\infty)(\Omega; \mathbb{R}^{n\times n}_s),\ b \in (C^0 \cap L^\infty)(\Omega; \mathbb{R}^n),\ c \in (C^0 \cap L^\infty)(\Omega); \\ \qquad \text{and exists } a_0 > 0 : \ A : \xi \otimes \xi \geq a_0 |\xi|^2, \text{ for all } \xi \in \mathbb{R}^n. \end{array} \right. \tag{1.2}$$

In the above, the notations ":" and "$\cdot$" symbolise the Euclidean inner products in the space of symmetric matrices $\mathbb{R}^{n\times n}_s$ and in $\mathbb{R}^n$ respectively, whilst $\mathrm{D}u = (\mathrm{D}_i u)_{i=1\dots n}$, $\mathrm{D}^2 u = (\mathrm{D}^2_{ij} u)_{i,j=1\dots n}$ and $\mathrm{D}_i \equiv \partial/\partial x_i$. The direct (or forward) Dirichlet problem for the above operator has the form

$$\left\{ \begin{array}{rll} \mathrm{L}[u] = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{array} \right. \tag{1.3}$$

and asks to determine u, given a source f and boundary data g. This is a classical problem which is essentially textbook material, see e.g. [19, Ch. 9]. In particular, it is well-posed (in the sense of Hadamard) and, given $f \in L^\infty(\Omega)$ and $g \in W^{2,\infty}(\Omega)$, there exists a unique solution u in the locally convex (Frechet) space

$$W^{2,\infty}_g(\Omega) := \bigcap_{1<p<\infty} \Big\{ u \in W^{2,p} \cap W^{1,p}_g(\Omega) : \ \mathrm{L}[u] \in L^\infty(\Omega) \Big\}. \tag{1.4}$$

Note that due to the failure of the L^p elliptic estimates when $p = \infty$ (see e.g. [18]), in general $u \notin W^{2,\infty}(\Omega)$. Let us also note with the assumptions (1.2) on L, the case of divergence operators with C^1 matrix coefficient A is included as a special case:

$$\mathrm{L}'[u] = \mathrm{div}(A\mathrm{D}u) + b \cdot \mathrm{D}u + cu.$$

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The *inverse problem* associated to (1.3) consists of the question of nding f, given the boundary data g and some *partial information on the solution u, typically* can only determine a unique biharmonic function u in Ω with $\Delta^2 u = 0$. Another popular choice in the literature for the observation operator Q consist of one of the terms in the separation of variables formula (when L = Δ on rectangular domains), as e.g. in [36]. To the best of our knowledge, (1.10) has not been studied before in this generality.

Herein we follow an approach based on recent advances in Calculus of Variations in the space L^∞ (see [22, 23, 24, 25]) developed recently for functionals involving higher order derivatives. The field has been initiated in the 1960s by Gunnar Aronsson (see e.g. [3, 4, 5, 6, 7]) and is still a very active area of research; for a review of the by-now classical theory involving scalar first order functionals we refer to [21]. To this end, we provide a *regularisation strategy* inspired by the classical Tykhonov regularisation strategy in L^2 (see e.g. [27, 30]), but for the next L^∞ "error" functional:

$$(1.11)\qquad \mathrm{E}_\infty(u) := \big\| \mathrm{Q}[u] - q \big\|_{L^\infty(\Omega;H)} + \alpha \big\| \mathrm{L}[u] \big\|_{L^\infty(\Omega)}, \quad u \in W^{2,\infty}_g(\Omega),$$

where $\alpha > 0$ is a fixed regularisation parameter for the penalisation term $|\mathrm{L}[u]|$. In the variational language, it serves to make the functional coercive in the space. The benefit of finding a best fitting solution in L^∞ is apparent: we can keep the error term $|\mathrm{Q}[u] - q|$ due to the noise effects *uniformly small, not merely small on average*, which would happen if one chose to minimise the integral of the error instead of the supremum.

As it is well known to the experts of Calculus of Variations in L^∞, mere (global) minimisers of supremal functionals, albeit typically easy to obtain with standard direct minimisation methods ([13, 16]), they are not truly optimal and they do not share the nice "local" minimality properties of minimisers of their integral counterparts ([10, 32]). A popular method is to use minimisers of L^p approximating functionals as $p \to \infty$ and prove appropriate convergence of such L^p minimisers to a limiting L^∞ minimiser. This method is fairly standard nowadays and provides a selection principle of L^∞ minimisers with additional favourable properties (see e.g. [9, 11, 12, 17, 22, 23]). This idea is inspired by the simple measure-theoretic fact that the L^p norm (of a fixed $L^1 \cap L^\infty$ function) converges to the L^∞ norm as $p \to \infty$. Accordingly, we will obtain *special* minimisers of (1.11) as limits of minimisers of

$$(1.12)\qquad \mathrm{E}_p(u) := \big\| |\mathrm{Q}[u] - q|_{(p)} \big\|_{L^p(\Omega;H)} + \alpha \big\| |\mathrm{L}[u]|_{(p)} \big\|_{L^p(\Omega)},$$

Theorem 1 (L^∞ and L^p regularisations of the inverse source identification problem). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and let also g be in $W^{2,\infty}(\Omega)$. Suppose also the operators (1.1) and (1.6) are given, satisfying the assumptions (1.2), (1.7), (1.8). Suppose further a function $q \in L^\infty(\Omega; H)$ is given which satisfies (1.9) for $\alpha > 0$. Let finally $\beta > 0$ be fixed. Then, we have the next results in relation to the problem (1.10):*

(i) [Existence] *There exist a global minimiser $u_\infty \equiv u_\infty^{\alpha,\beta} \in W^{2,\infty}_g(\Omega)$ of the functional E_∞ defined in (1.11). In particular, we have $E_\infty(u_\infty) \leq E_\infty(v)$ for all $v \in W^{2,\infty}_g(\Omega)$ and*

$$f_\infty \equiv f_\infty^{\alpha,\beta} := \mathrm{L}[u_\infty^{\alpha,\beta}] \in L^\infty(\Omega).$$

In addition, there exist signed Radon measures

$$\mu_\infty \equiv \mu_\infty^{\alpha,\beta} \in \mathcal{M}(\Omega); \quad \nu_\infty \equiv \nu_\infty^{\alpha,\beta} \in \mathcal{M}(\Omega)$$

such that the divergence PDE

$$K_r(\cdot, u_\infty, \mathrm{D}u_\infty)\,\mu_\infty - \mathrm{div}\big(K_p(\cdot, u_\infty, \mathrm{D}u_\infty)\,\mu_\infty\big) + \mathrm{L}^*[\nu_\infty] = 0, \tag{1.13}$$

is satisfied by the triplet $(u_\infty, \mu_\infty, \nu_\infty)$ in the distributional sense. In (1.13), the operator L^* is the formal adjoint of L, defined through duality, i.e.

$$\mathrm{L}^*[v] := \mathrm{div}(\mathrm{div}(Av)) - \mathrm{div}(bv) + cv$$

and K_r, K_p denote the partial derivatives of $K(x, r, p)$ with respect to $(r, p) \in \mathbb{R} \times \mathbb{R}^n$. Additionally, the error measure μ_∞ is supported in the closure of the subset of Ω of maximum noise, that is

$$\mathrm{supp}(\mu_\infty) \subseteq \Big\{ \big|Q[u_\infty] - q\big|^{\mathrm{F}} = \big\|Q[u_\infty] - q\big\|_{L^\infty(\Omega;H)} \Big\}, \tag{1.14}$$

where "$|\cdot|^{\mathrm{F}}$" symbolises the "essential limsup" with respect to the Radon measure $\mathcal{H}^n \llcorner \Omega$ on $\overline{\Omega}$, see Proposition 6 that follows. If additionally the measurement function q is continuous on $\overline{\Omega}$, (1.14) improves to

$$\mathrm{supp}(\mu_\infty) \subseteq \Big\{ \big|Q[u_\infty] - q\big| = \big\|Q[u_\infty] - q\big\|_{L^\infty(\Omega;H)} \Big\}. \tag{1.15}$$

(ii) [Convergence] *For any $\alpha, \beta > 0$, the minimiser u_∞ can be approximated by a family of minimisers $(u_p)_{p>n} \equiv (u_p^{\alpha,\beta})_{p>n}$ of the respective L^p functionals (1.12) and the pair of measures $(\mu_\infty, \nu_\infty) \in \mathcal{M}(\Omega) \times \mathcal{M}(\Omega)$ can be approximated by respective absolutely continuous signed measures $(\mu_p, \nu_p)_{p>n} \equiv (\mu_p^{\alpha,\beta}, \nu_p^{\alpha,\beta})_{p>n}$, as follows:*

For any $p > n$, the functional (1.12) has a global minimiser $u_p \equiv u_p^{\alpha,\beta}$ in the space $(W^{2,p} \cap W^{1,p}_g)(\Omega)$ and there exists a sequence $p_j \to \infty$ as $j \to \infty$, such that

(1.16) $\Big\{$ $u_p \to u_\infty$, in $C^{1,\gamma}(\overline{\Omega})$; f4.60oxiTf 26 5.70315 202.547 Td [()]]TJ/F11 9.96211 9. p u

as $p \to \infty$ along the sequence. Further, for each $p > n$, the triplet $(u_p, \sigma_p, \lambda_p)$ solves the equation

$$\text{(1.18)} \qquad K_r(\cdot, u_p, \mathrm{D}u_p)\,\sigma_p - \mathrm{div}\big(K_p(\cdot, u_p, \mathrm{D}u_p)\,\sigma_p\big) + \mathrm{L}^*[\lambda_p] = 0,$$

in the distributional sense.

(iii) [L^∞ error estimates] *For any exact solution $u^0 \in W^{2,\infty}_g(\Omega)$ of (1.10) (with $f = \mathrm{L}[u^0]$ and $\mathrm{Q}[u^0] = q^0$) corresponding to measurements with zero noise, we have the estimate:*

$$\text{(1.19)} \qquad \big\| \mathrm{Q}[u_\infty^{\varepsilon,\lambda}] - \mathrm{Q}[u^0] \big\|_{L^\infty(\Omega;\mathbb{H})} \le 2\varepsilon + \lambda \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)},$$

for any $\varepsilon, \lambda > 0$.

(iv) [L^p error estimates] *For any exact solution $u^0 \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ of (1.10) (with $f = \mathrm{L}[u^0]$ and $\mathrm{Q}[u^0] = q^0$) corresponding to measurements with zero noise and for $p > n$, we have the estimate:*

$$\text{(1.20)} \qquad \big\| \mathrm{Q}[u_p^{\varepsilon,\lambda}] - \mathrm{Q}[u^0] \big\|_{L^p(\Omega;\mathbb{H})} \le 2\varepsilon + \lambda \|\mathrm{L}[u^0]\|_{L^p(\Omega)},$$

for any $\varepsilon, \lambda > 0$.

The estimate in part **(iv)** above is useful if we have merely that $\mathrm{L}[u^0] \in L^p(\Omega)$ for $p < \infty$ (namely when perhaps $\mathrm{L}[u^0] \notin$

- $\mathrm{Q}[u] := u(x,c)$, for $n = 2$ and $\Omega = (a,b) \times (c,d)$ being a rectangular domain (i.e., one of the products in the separation of variables when $\mathrm{L} = \Lambda$). This implies that (1.19) simplifies to

$$\left\| u_\infty^{\varepsilon,\delta}(\cdot,c) - u^0(\cdot,c) \right\|_{L^\infty((a,b);H^1)} \le \ldots 2 + \|\mathrm{L}[u^0]\|_{L^\infty((a,b)\times(c,d))} \quad \text{as } \varepsilon,\delta \to 0,$$

and similarly for its L^p-counterpart.

- $\mathrm{Q}[u] := \mathrm{D}u \cdot n$, where n is the outer normal vector on $\partial\Omega$. In this case, (1.21) simplifies to

$$\left\| n \cdot \left(\mathrm{D}u_\infty^{\varepsilon,\delta} - \mathrm{D}u^0\right) \right\|_{L^\infty(\partial\Omega;\mathcal{H}^{n-1})} \le \ldots 2 + \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)} \quad \text{as } \varepsilon,\delta \to 0,$$

and similarly for its L^p-counterpart.

We would like to note again that, due to the ill-posed nature of the problem, in general it is not possible to obtain an estimate on $n \cdot \Lambda$.

We now provide some clarifications regarding Theorem 1.

Remark 5. (i) We note that in (1.13) the distributional meaning of this PDE is

$$\int K_r(\cdot, u_\infty, \mathrm{D}u_\infty)\, \mathrm{d}\ldots_\infty + \int K_p(\cdot, u_\infty, \mathrm{D}u_\infty) \cdot \mathrm{D}\ldots\, \mathrm{d}\ldots_\infty + \int \mathrm{L}[\ldots]\, \mathrm{d}\ldots$$

measure respectively, the above is in fact equivalent to

$$K_r(\ ; u_p; \mathrm{D}u_p) \quad + K_p(\ ; u_p; \mathrm{D}u_p)\ \ \mathrm{D} \quad \frac{\mathrm{Q}[u_p] \quad q \quad {}^{p\ \ 2}_{(p)}\ \mathrm{Q}[u_p] \quad q}{j\mathrm{Q}[u_p] \quad q\ j_{(p)}\ {}^{p\ \ 1}_{L^p(\ \ ;H\ \ )}}\, \mathrm{d}H$$

$$+ \qquad \mathrm{L}[\ \ ]\frac{j\mathrm{L}[u_p]j^{p\ \ 2}_{(p)}\ \mathrm{L}[u_p]}{j\mathrm{L}[u_p]j_{(p)}\ {}^{p\ \ 1}_{L^p(\ \ )}}\, \mathrm{d}L^n = 0;$$

for all $\ 2\ C^2_c(\ \ )$.

(iii) Since we only prescribe boundary conditions $u = g$ on @ but impose no condition on the gradient (as opposed to e.g. [22], wherein an L^1 minimisation problem was considered by imposing Du = Dg on @ additionally to $u = g$ on @), we therefore have \natural boundary conditions" for the gradient on @ . We will make no particular further use of this observation.

The following two results are of independent interest and are utilised in the proof of Theorem 1 that follows. We state and prove them in considerably greater generality than that needed herein, as they have their own merits in the Calculus of Variations in L^1.

Proposition 6 (The essential limsup). *Let X R^n be a Borel set, endowed with the induced Euclidean topology and let also 2 $M(X)$ be a positive nite Radon measure on X. For any f 2 $L^1(X;\ )$, we de ne the function f^{F} 2 $L^1(X;\ )$ by setting*

$$f^{\mathrm{F}}(x) := \lim_{"\,!\,0} \quad \mathop{\text{ess sup}}_{y 2 \mathrm{B}_"(x)} f(y$$ 8 Td [(y).9626 Tf 9.9254 -14.048 Td [(3051)]TJ/F55 9.9

(i) *There exists a subsequence* $(k_i)_1^{\infty}$ *and a limit measure* $\ _{\infty} \in M(X)$ *such that*

$$\ _{k} \overset{*}{\rightharpoonup}\ _{\infty} \quad \text{in } M(X),$$

as $k_i \to \infty$.

(ii) *If there exists* $f_{\infty} \in L^{\infty}(X;\ ) \setminus \{0\}$ *such that*

$$\sup_X |f_k - f_{\infty}| \to 0 \quad \text{as } k \to \infty,$$

then the limit measure is supported in the set where (the by our assumptions on L and the Hölder inequality we have

$$\begin{aligned}\mathrm{E}_p(v) &\geq \|\mathrm{L}[v]\|_{L^p(\Omega)}\\ &\geq \frac{1}{C(p;A,b,c)}\|v\|_{W^{2,p}(\Omega)} - \|g\|_{W^{2,p}(\Omega)}\\ &\geq \frac{1}{C(p;A,b,c)}\|v\|_{W^{2,p}(\Omega)} - \|g\|_{W^{2,\infty}(\Omega)}\end{aligned}$$

for some $C = C(p;A,b,c) > 0$ and any $v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$. Let $(u_p^m)_1^\infty$ be a minimising sequence of E_p:

$$\mathrm{E}_p(u_p^m) \longrightarrow \inf\Big\{\mathrm{E}_p(v) \,:\, v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)\Big\},$$

as $m \to \infty$. Then, by the above estimates, we have the uniform bound

$$\|u_p^m\|_{W^{2,p}(\Omega)} \leq C$$

for some $C > 0$ depending on p but independent of $m \in \mathbb{N}$. By standard weak and strong compactness arguments in Sobolev spaces, there exists a subsequence $(u_p^{m_k})_1^\infty$ and a function $u_p \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ such that, along this subsequence we have

$$\begin{cases} u_p^m \longrightarrow u_p, & \text{in } L^p(\Omega),\\ \mathrm{D}u_p^m \longrightarrow \mathrm{D}u_p, & \text{in } L^p(\Omega;\mathbb{R}^n),\\ \mathrm{D}^2u_p^m \rightharpoonup \mathrm{D}^2u_p, & \text{in } L^p(\Omega;\mathbb{R}^{n\times n}_s),\end{cases}$$

as $m_k \to \infty$. Additionally, since $p > n$, by the regularity of the boundary we have the compact embedding $W^{2,p}(\Omega) \Subset C^{1,\kappa}(\overline{\Omega})$ as a consequence of the Morrey estimate. Hence,

$$u_p^m \longrightarrow u_p \quad \text{in } C^{1,\gamma}(\overline{\Omega}), \text{ for } \gamma \in \left(0, 1-\frac{n}{p}\right),$$

as $m_k \to \infty$. The above modes of convergence and the continuity of the function K defining the operator Q imply that $\mathrm{Q}[u_p^m] \longrightarrow \mathrm{Q}[u_p]$ uniformly on Ω as $m_k \to \infty$. Therefore,

$$\Big\| |\mathrm{Q}[u_p^m] - q|_{(p)}\Big\|_{L^p(\Omega;H)} \longrightarrow \Big\| |\mathrm{Q}[u_p] - q|_{(p)}\Big\|_{L^p(\Omega;H)}$$

as $m_k \to \infty$. Additionally, by the linearity of the operator L and because its coefficients are L^∞, we have that

$$\mathrm{L}[u_p^m] \rightharpoonup \mathrm{L}[u_p] \quad \text{in } L^p(\Omega),$$

as $m_k \to \infty$. Since the functional

$$\big\| |\cdot|_{(p)}\big\|_{L^p(\Omega)} \,:\, L^p(\Omega) \longrightarrow \mathbb{R}$$

is convex on this reflexive space and also it is strongly continuous, it is weakly lower semi-continuous and therefore

$$\big\| |\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \big\| |\mathrm{L}[u_p^{m_k}]|_{(p)}\big\|_{L^p(\Omega)}.$$

By putting all the above together, we see that

$$\mathrm{E}_p(u_p) \leq \liminf_{k\to\infty} \mathrm{E}_p(u_p^{m_k}) \leq \inf\Big\{\mathrm{E}_p(v) \,:\, v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)\Big\},$$

which concludes the proof.

Lemma 9. *For any ; > 0, there exists a (global) minimiser u_1 2 $W_g^{2;1}(\ )$ and a sequence of minimisers $(u_{p_i})_1^1$ of the respective E_p-functionals constructed in Lemma 8, such that (1.16) holds true.*

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Proof. For eacu

Then, the triplet $(u_p; \ _p; \ _p)$ *satisfies the PDE* (1.18) *in the distributional sense. In fact, the following stronger assertion holds: we have*

$$K_r(\ ; u_p; \mathrm{D}u_p) \quad + \ K_p(\ ; u_p; \mathrm{D}u_p) \ \ \mathrm{D} \quad \frac{\mathrm{Q}[u_p] \quad q \ \ {}^{p\ \ 2}_{(p)} \ \mathrm{Q}[u_p] \quad q}{|\mathrm{Q}[u_p] \quad q\,|_{(p)} \ {}^{p\ \ 1}_{L^p(\ ;H\ )}} \, \mathrm{d}H$$

$$+ \qquad \mathrm{L}[\ ] \frac{|\mathrm{L}[u_p]|^{p\ \ 2}_{(p)} \, \mathrm{L}[u_p]}{|\mathrm{L}[u_p]|_{(p)} \ {}^{p\ \ 1}_{L^p(\ )}} \, \mathrm{d}L^n \ = \ 0;$$

for all $\ \ 2 \ W^{2;p}_0(\ )$.

for any $\in W_0^{2;p}(\ ) \quad C^1(\overline{\ })$, because of the continuity of $K(x;r;p)$ in x and the C^1 regularity in $(r;p)$.

Lemma 11. *For any $\ ;\ > 0$, consider the minimiser u_∞ of E_∞ constructed in Lemma 9 as sequential limit of minimisers $(u_p)_{p>n}$ of the functionals $(E_p)_{p>n}$ as $p_i \to \infty$. Then, there exist signed Radon measures $\ _\infty \in M(\ )$ and $\ _\infty \in M(\ )$ such that the triplet $(u_\infty;\ _\infty;\ _\infty)$ satisfies the PDE (1.13) in the distributional sense, that is*

$$K_r(\ ;u_\infty;Du_\infty) \quad + K_p(\ ;u_\infty;Du_\infty)\ D \quad d\ _\infty + \quad L[\ ]\,d\ _\infty = 0;$$

for all $\ \in C_c^2(\ )$. Additionally, there exists a further subsequence along which the weak modes of convergence of (1.17) hold true as $p \to \infty$.*

Proof. As noted in the beginning of the proof of Lemma 10, we have the p-uniform total variation bounds $\|\ _p\|(\ ) \quad 1$ and $\|\ _p\|(\ ) \quad 1$. Hence, by the sequential weak* compactness of the spaces of Radon measures

$$M(\ ) = C_0^0(\ )\ ; \quad M(\ ) = C^0(\ )\ ;$$

there exists a further subsequence denoted again by $(p_i)_1^\infty$ such that $\ _p \ \overset{*}{\ }\ \ _\infty$ in $M(\ )$ and $\ _p \ \overset{*}{\ }\ \ _\infty$ in $M(\ )$, as $p_i \to \infty$. Fix now $\ \in C_c^2(\ )$. By Lemma 10, we have that the triplet $(u_p;\ _p;\ _p)$ satisfies (1.18), that is

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$$K_r(\ ;u_p;Du_p) \quad + K_p(\ ;u_p;Du_p)\ D$$

the (($\alpha;\varepsilon$)-dependent) minimiser u_p of E_p (constructed in Lemmas 8-11), satisfies the error bounds (1.20)*, that is:*

$$\big\|\mathrm{Q}[u_p]-\mathrm{Q}[u^0]\big\|_{L^\infty(\Omega;\mathbb{H})} \le 2\varepsilon + \|\mathrm{L}[u^0]\|_{L^p(\Omega)}.$$

If additionally $u^0 \in W^{2,\infty}_g(\Omega)$, then the (($\alpha;\varepsilon$)-dependent) minimiser u_∞ of E_∞ (constructed in Lemmas 8-11), satisfies the error bounds (1.19)*, that is:*

$$\big\|\mathrm{Q}[u_\infty]-\mathrm{Q}[u^0]\big\|_{L^\infty(\Omega;\mathbb{H})} \le 2\varepsilon + \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)}.$$

Proof. Let us use the symbolisation $q^0 := \mathrm{Q}[u^0]$, noting also that $q^0 \in C^0(\overline{\Omega})$ and that we have the estimate

$$\|q - q^0\|_{L^\infty(\Omega;\mathbb{H})} \le \varepsilon.$$

For any $p \in (n,\infty)$, the function u_p is a global minimiser of E_p in $(W^{2,p} \cap W^{1,p}_g)(\Omega)$. Therefore,

$$\mathrm{E}_p(u_p) \le \mathrm{E}_p(u^0).$$

This implies the estimate

$$\begin{aligned}\big\|\mathrm{Q}[u_p]-q\big\|_{L^p(\Omega;\mathbb{H})} + \big\|\mathrm{L}[u_p]\big\|_{L^p(\Omega)} \\ \le \big\|\mathrm{Q}[u^0]-q\big\|_{L^p(\Omega;\mathbb{H})} + \big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)}.\end{aligned}$$

The latter estimate together with the Minkowski and Hölder inequalities, in turn yield

$$\begin{aligned}\big\|\mathrm{Q}[u_p]-\mathrm{Q}[u^0]\big\|_{L^p(\Omega;\mathbb{H})} &\le \big\|\mathrm{Q}[u^0]-q\big\|_{L^p(\Omega;\mathbb{H})} \\ &\quad + \big\|\mathrm{Q}[u^0]-q\big\|_{L^p(\Omega;\mathbb{H})} + \big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)} \\ &= 2\|q-q^0\|_{L^\infty(\Omega;\mathbb{H})} + \big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)} \\ &\le 2\varepsilon + \|\mathrm{L}[u^0]\|_{L^p(\Omega)},\end{aligned}$$

as claimed. To obtain the corresponding estimate for u_∞ in the case that additionally $u^0 \in W^{2,\infty}_g(\Omega)$, we may pass to the limit as $p \to \infty$ in the last estimate above: indeed, consider the subsequence $p_i \to \infty$ along which we have the strong convergence $u_p \longrightarrow u_\infty$ in $C^1(\overline{\Omega})$ and therefore $\mathrm{Q}[u_p] \longrightarrow \mathrm{Q}[u_\infty]$ uniformly on $\overline{\Omega}$. Since by assumption $\mathrm{L}[u^0] \in L^\infty(\Omega)$, the conclusion follows by letting $i \to \infty$ in the last estimate.

We now establish Proposition 6.

Proof of Proposition 6. (i) Let $\mathbb{B}^n_\rho(x)$ be the open ρ-ball of $\mathbb{R}^n$ centred at x. By the Lebesgue differentiation theorem (see e.g. [16]) applied to the measure $\mu \llcorner X$ (namely to μ extended to $\mathbb{R}^n$ by zero on $\mathbb{R}^n \setminus X$) and by recalling that $\mathbb{B}_\rho(x)$ symbolises the open ball in X, we have

$$\begin{aligned}f(x) &= \lim_{\rho\to 0} ⨍_{\mathbb{B}^n_\rho(x)} f \, d(\mu \llcorner X) \\ &= \lim_{\rho\to 0} \frac{1}{\mu(\mathbb{B}_\rho(x))}\int_{\mathbb{B}_\rho(x)} f \, d\mu\end{aligned}$$

and therefore

$$f(x) \quad \lim_{!\,0} \frac{1}{(\mathrm{B}\ (x))} {}_{\mathrm{B}\ (x)} f\,\mathrm{d}$$

$$\lim_{!\,0} \quad \operatorname*{ess\,sup}_{\mathrm{B}\ (x)} f$$

By the Lebesgue-Besicovitch di erentiation theorem (see e.g. [16]), -a.e. point $x \in X$ has density 1, namely

$$\lim_{\varepsilon \to 0} \frac{(X(\ ) \cap B^n_\varepsilon(x))}{(B^n_\varepsilon(x))} = 1;$$

where $B^n_\varepsilon(x)$ is the open ε-ball centred at x with respect to $\mathbb{R}^n$. Hence, since

$$B_\varepsilon(x) = X \cap B^n_\varepsilon(x);$$

for any > 0, there exists $x \in X(\ )$ such that

$$X(\ ) \cap B_\varepsilon(x) = X(\ ) \cap B^n_\varepsilon(x) > 0.$$

Therefore, since

$$\operatorname*{ess\,sup}_{y \in X} f(y) + f(x); \qquad \text{a.e. } x \in X(\ );$$

we deduce

$$\begin{aligned} \operatorname*{ess\,sup}_{y \in X} f(y) & + \operatorname*{ess\,sup}_{y \in B_\varepsilon(x) \cap X(\ )} f(y) \\ & + \operatorname*{ess\,sup}_{y \in B_\varepsilon(x)} f(y). \end{aligned}$$

By letting $\varepsilon \to 0$ in the above inequality, we infer that

$$\begin{aligned} \operatorname*{ess\,sup}_{x \in X} f(x) & + \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{y \in B_\varepsilon(x)} f(y) \\ & = + f^F(x) \\ & + \sup_{x \in X} f^F(x); \end{aligned}$$

for any > 0. By letting $\to 0$, we obtain

$$\operatorname*{ess\,sup}_{x \in X} f(x) \quad \sup_{x \in X} f^F(x);$$

as desired. This inequality completes the proof.

By invoking Proposition 7 whose proof follows, we readily obtain (1.14)-(1.15) by choosing

$$X = \ ; \quad = H \ \mathsf{X} \ ; \quad f_k = Q[u_{p_k}] \quad q \ ; \quad f_1 = Q[u_1] \quad q \ .$$

Proof of Proposition 7. (i) By the de nition of $_k$, we have for any continuous function $\in C^0(X)$ with $|\ | \ 1$ that

$$\begin{aligned} \int_X \, d_k & \frac{1}{\| |f_k|^{(k)} \|^{k-1}_{L^k(X;\ )}} \int_X |f_k|^{(k)\,k-2} f_k \ d \\ & \frac{1}{\| |f_k|^{(k)} \|^{k-1}_{L^k(X;\ )}} \int_X |f_k|^{(k)\,k-1} d \ . \end{aligned}$$

Hence, by Hölder inequality, we have the total variation bound

$$\begin{aligned} \| \ _k \|(X) & \ \| |f_k|^{(k)} \|^{1-k}_{L^k(X;\ )} \left(\int_X |f_k|^{(k)\,k} d \right)^{\frac{k-1}{k}} \\ & = 1. \end{aligned}$$

By the sequential weak* compactness of the space $M(X) = C^0(X)$

By the above, for any $\varepsilon > 0$ small enough (recall that $f_\infty \neq 0$) and for any $k \geq k(\varepsilon)$, we have the estimate

$$\frac{d\mu_k}{d\nu} \leq \frac{1}{\mu(X)} \left(\frac{\frac{1}{k} + |f_\infty| + \frac{\varepsilon}{4}}{\|f_\infty\|_{L^\infty(X;\nu)} - \frac{\varepsilon}{2}} \right)^{k-1}, \qquad \text{a.e. on } X.$$

By choosing $k(\varepsilon)$ even larger if needed, we can arrange

$$\frac{d\mu_k}{d\nu} \leq \frac{1}{\mu(X)} \left(\frac{2|f_\infty| + \varepsilon}{2\|f_\infty\|_{L^\infty(X;\nu)} - \varepsilon} \right)^{k-1}, \qquad \text{a.e. on } X.$$

Since by Proposition 6 we have $|f_\infty| \leq |f_\infty|^{F}$ ν-a.e. on X, we obtain

$$\frac{d\mu_k}{d\nu} \leq \frac{1}{\mu(X)} \left(\frac{2|f_\infty|^{F} + \varepsilon}{2\|f_\infty\|_{L^\infty(X;\nu)} - \varepsilon} \right)^{k-1}, \qquad \text{a.e. on } X.$$

Consider now for any $\varepsilon > 0$ the ν-measurable set

$$X_\varepsilon := \left\{ |f_\infty|^{F} < \|f_\infty\|_{L^\infty(X;\nu)} - 2\varepsilon \right\}.$$

Notice also that X_ε is in fact open in X because $|f_\infty|^{F}$ is upper semicontinuous (Proposition 6). Additionally, we have the estimate

$$\frac{d\mu_k}{d\nu} \leq \frac{1}{\mu(X)} \left(\frac{2\|f_\infty\|_{L^\infty(X;\nu)} - 3\varepsilon}{2\|f_\infty\|_{L^\infty(X;\nu)} - \varepsilon} \right)^{k-1}, \qquad \text{a.e. on } X_\varepsilon.$$

The above estimate together with the Lebesgue Dominated Convergence theorem imply that for any $\varepsilon > 0$ small enough we have

$$\frac{d\mu_k}{d\nu} \longrightarrow 0 \quad \text{in } L^1(X_\varepsilon;\nu), \quad \text{as } k \to \infty.$$

Consider now the sequence of nonnegativedaria1 rg 11 /[m11.9easures 11 /[9.9626 Tf 3.267 0 Td [(303000)]TJ

Therefore, since $X_{''}$ is open in X

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