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by

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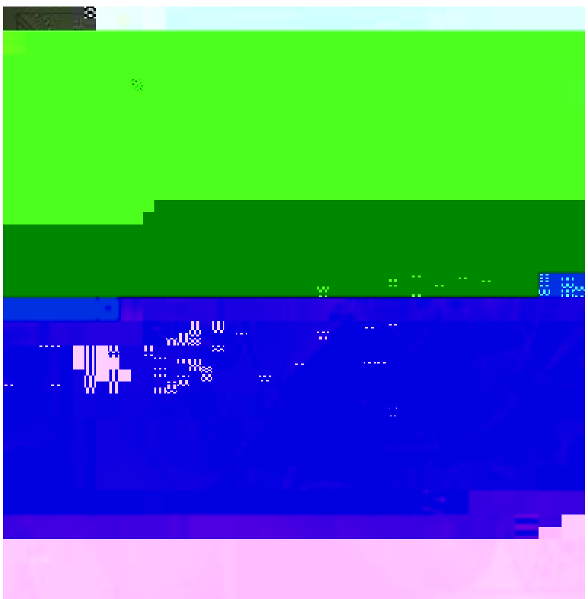

Counterexamples in Calculus of Variations in L^1 through the vectorial Eikonal equation

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Abstract

We show that for any regular bounded domain s in L^1

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1 -Laplacien apparaissant

pour p ! 1 des cartes

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1. Introduction

Calculus of Variations in L^1 is concerned with the variational study of supremal functionals, as well as with the necessary conditions governing their extrema. The archetypal model of interest is the functional

$$E_1(u;O) := \operatorname{ess\,sup}_O |Du|; \quad \text{for } u \ 2 \ W^{1;1}(\;;\ R^N);\ O \quad \text{measurable}; \tag{1}$$

where $n;N \ 2 \ N$, R^n is a xed open set and $Du(x) = (D_i u(x))_{i=1\,:::n}^{=1\,:::N} \ 2 \ R^{N \ n}$ is the gradient matrix. We note that our general notation is either self-explanatory or standard. In (1) and throughout

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We note that the results above improve and supersede one of the main results in [17] which required Ω to be a punctured ball. Since the unique solution to the Dirichlet problem for $\Delta_p u = 0$ in Ω with $u = \mathrm{id}$ on $\partial\Omega$ is $u(x) \equiv x$ when $p < \infty$, it follows that none of our diffeomorphisms is a limit of p-harmonic maps as $p \to \infty$. Thus, we confirm that (2) by itself cannot suffice to identify limits of p-harmonic maps and that additional selection criteria are needed to have a situation analogous to the scalar case.

The proof of Theorem 1.1 is based on the next result of independent interest.

Proposition 1.1 Let n, m, Ω be as in Theorem 1.1. Then, the nonlinear problem

$$|Du|^2 + 2\,\mathrm{div}\, u \equiv C \ \text{ in } \Omega \quad \text{and} \quad u = 0 \ \text{ on } \partial\Omega,$$

has infinitely many non-trivial solutions $(u, C) \in (C^m \cap C^0_0)\big(\overline{\Omega}; \mathbb{R}^n\big) \times (0, \infty)$. Additionally, the set of all solutions has the trivial solution $(0, 0)$ as an accumulation point with respect to the topology of $C^m\big(\overline{\Omega}; \mathbb{R}^n\big)$.

Since the proofs of the above results are non constructive, we include in Section 3 explicit examples of smooth ∞-Harmonic maps defined on annular domains which coincide with affine maps on the boundary.

2. Proofs

We begin with the proof of Proposition 1.1, which is an immediate consequence of the next lemma and of the Morrey estimate, in the form of inclusion of spaces $H^{m+2}(\Omega; \mathbb{R}^n) \subseteq C^m\big(\overline{\Omega}; \mathbb{R}^n\big)$ (since $n \in \{2, 3\}$).

Lemma 2.1 Let n, m, Ω be as in Theorem 1.1 and let us define the nonlinear mapping

$$\mathcal{M} : (H^{m+2} \cap H^1_0)(\Omega; \mathbb{R}^n) \longrightarrow H^{m+1}_\sharp(\Omega) := \Big\{ w \in H^{m+1}(\Omega) : \int_\Omega w(x)\, dx = 0 \Big\}$$

by setting (here the slashed integral denotes the average)

$$\mathcal{M}[u] := \frac{1}{2}|Du|^2 + \mathrm{div}\, u - \frac{1}{2}\int_\Omega |Du(x)|^2\, dx.$$

Then, the inverse image $\mathcal{M}^{-1}[\{0\}]$ contains infinitely-many elements accumulating at zero. In addition, for any $\varepsilon > 0$, there exists $\varphi_\varepsilon \in (H^{m+2} \cap H^1_0)(\Omega; \mathbb{R}^n) \setminus \{0\}$ such that $\mathcal{M}[\varphi_\varepsilon] = 0$ and $\|\varphi_\varepsilon\|_{H^{m+2}(\Omega)} < \varepsilon$.

Proof of Lemma 2.1. First note that $\mathcal{M}$ is well defined, namely its image lies in the subspace $H^{m+1}_\sharp(\Omega)$ of zero average. Indeed, for any $u \in (H^{m+2} \cap H^1_0)(\Omega; \mathbb{R}^n)$, the divergence theorem gives

By noting that $M\,^0[0]|_V : V\ !\ \ H^{m+1}_{]}(\ )$ is a linear isomorphism, the canonical isomorphism between $\ker(M\,^0[0])\ \ V$ and $\ker(M\,^0[0])\ \ V$ allows us to view M as a map on $\ker(M\,^0[0])\ \ V$ by setting M

Since $e^{\log(|x|)S}$ is orthogonal and $|OA| = |A|$ for any $A, O \in \mathbb{R}^{n \times n}$ with O being orthogonal, we have

$$|Du(x)|^2 =$$

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