

Speaking about the system (1.1), we would like to mention that the system (1.1) is called the "1-Laplacian" and it arises as a sort of Euler-Lagrange PDE of vectorial variational problems in L^1 for the supremal functional

$$(1.4) \quad E_1(u; O) := kH(Du)k_{L^1(O)}; \quad u \in W_{loc}^{1,1}(\cdot; \mathbb{R}^N); \quad O \in \mathbb{R}^N;$$

when the Hamiltonian (the non-negative function $H \in C^2(\mathbb{R}^N \times \mathbb{R}^n)$) is chosen to be $H(Du) = \frac{1}{2} |Du|^2$, with $|\cdot|$ is the Euclidean norm on the space $\mathbb{R}^N \times \mathbb{R}^n$. the 1-Laplacian is a special case of the system

$$(1.5) \quad \Delta_1 u := -H$$

Interestingly, even when the operator Δ_1 is applied to C^1 maps, which may even be solutions, (1.1) may have discontinuous coefficients. This further difficulty of the vectorial case is not present in the scalar case. As an example consider

$$(1.8) \quad u(x; y) := e^{ix} - e^{iy}; \quad u : \mathbb{R}^2 \rightarrow \mathbb{R}^2:$$

Katzourakis has showed in [11] that even though (1.8) is a smooth solution of the Δ_1 -Laplacian near the origin, we still have the coefficient $jDu j^2 |Du|^2$ of (1.1) is discontinuous. This is because when the dimension of the image changes, the projection $|Du|^2$ "jumps". More precisely, for (1.8) the domain splits to three components according to the $\text{rk}(Du)$, the "2D phase Ω_2 ", whereon u is essentially 2D, the "1D phase Ω_1 ", whereon u is essentially 1D (which is empty for (1.8)) and

- (i) On Ω_2 we have $\text{rank}(Du) = 2$, and the map $u: \Omega_2 \rightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$(1.10) \quad |Du|^2 = C^2 > 0;$$

The constant C may vary on different connected components of Ω_2 .

- (ii) On Ω_1 we have $\text{rank}(Du) = 1$ and the map $u: \Omega_1 \rightarrow \mathbb{R}^N$ is given by an essentially scalar 1-Harmonic function $f: \Omega_1 \rightarrow \mathbb{R}$:

$$(1.11) \quad u = a + f; \quad \Omega_1 \cap f = 0; \quad a \in \mathbb{R}^N; \quad \Omega_2 \subset S^{N-1};$$

The vectors a_i may vary on different connected components of Ω_1 .

- (iii) On S , $|Du|^2$ is constant and $\text{rank}(Du) = 1$. Moreover if $S = \Omega_1 \setminus \Omega_2$ (that is if both the 1D and 2D phases coexist) then $u: S \rightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$(1.12) \quad u = a + f; \quad |Df|^2 = C^2 > 0; \quad a \in \mathbb{R}^N; \quad \Omega_2 \subset S^{N-1};$$

The main result of this paper is to generalise these results to higher dimension $N \geq n \geq 2$. The principle result in this paper is the following extension of theorem 1.1:

Theorem 1.2 (Phase separation of n -dimensional 1-Harmonic mappings).

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u: \Omega \rightarrow \mathbb{R}^N$, $N \geq n \geq 2$, be an 1-Harmonic map in $C^2(\Omega; \mathbb{R}^N)$, that is a solution to the 1-Laplace system (1.1). Then, there exist disjoint open sets Ω_r , $r=1, \dots, n$, and a closed nowhere dense set S such that $\Omega = \bigcup_{i=1}^n \Omega_i \cup S$ such that:

- (i) On Ω_n we have $\text{rank}(Du) = n$ and the map $u: \Omega_n \rightarrow \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$(1.13) \quad |Du|^2 = C^2 > 0;$$

The constant C may vary on different connected components of Ω_n .

- (ii) On Ω_r we have $\text{rank}(Du) = r$, where r is integer in $\{2, 3, 4, \dots, (n-1)\}$, and $|Du(t; x)|$ is constant along trajectories of the parametric gradient flow of $u(t; x)$

$$(1.14) \quad \begin{aligned} \frac{d}{dt} u(t; x) &= -\nabla |Du(t; x)|^2; \quad t \in (0; \infty); \quad u(0; x) = x; \\ u(0; x) &= x; \end{aligned}$$

where $\Omega_r \subset S^{N-1}$, and $\nabla |Du(t; x)|^2 \neq 0$.

- (iii) On Ω_1 we have $\text{rank}(Du) = 1$ and the map $u: \Omega_1 \rightarrow \mathbb{R}^N$ is given by an essentially scalar 1-Harmonic function $f: \Omega_1 \rightarrow \mathbb{R}$:

$$(1.15) \quad u = a + f; \quad \Omega_1 \cap f = 0; \quad a \in \mathbb{R}^N; \quad \Omega_2 \subset S^{N-1};$$

The vectors a_i may vary on different connected components of Ω_1 .

- (iv) On S , when $S = \Omega_p \setminus \Omega_q = \emptyset$ for all p and q such that $2 \leq p < q \leq n-1$, then we have that $|Du|^2$ is constant and $\text{rank}(Du) = 1$. Moreover on

$$\Omega_1 \setminus \Omega_n \subset S;$$

(when both 1D and n D phases coexist), we have that $u: S \rightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$(1.16) \quad u = a + f; \quad |Df|^2 = C^2 > 0; \quad a \in \mathbb{R}^N; \quad \Omega_2 \subset S^{N-1};$$

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Lemma 2.3 (cf. [14]).

Let $u: \mathbb{R}^n \rightarrow \mathbb{R}^N$ be in $u \in C^2(\mathbb{R}^n; \mathbb{R}^N)$. Consider the gradient flow

$$(2.6) \quad \begin{aligned} \frac{\partial}{\partial t} u(t; x) &= - \frac{\sum_j |Du_j|^2}{\sum_j |Du_j|^2} Du(t; x); \quad t \geq 0 \\ u(0; x) &= x; \end{aligned}$$

for $x \in \mathbb{R}^n$, $\sum_j |Du_j|^2 \leq n \|Du\|^2$

u must coincide and hence, we obtain $u = f + a$ for $\gamma \in S^{N-1}$; $a \in \mathbb{R}^N$ and $f \in C^2(\gamma; \mathbb{R})$ where γ and a may vary on different connected components of γ_1 . The theorem follows.

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Department of Mathematics, College of Science, University of Basra, Basra, Iraq
 AND Department of Mathematics and Statistics, University of Reading, Whiteknights,,
 PO Box 220, Reading RG6 6AX, UK
 E-mail address : h.a.h.abugirda@pgr.reading.ac.uk