Department of Mathematics and Statistics

Preprint MPCS-2017-03

9 March 2017

The eigenvalue problem for the f-

(up to a multiplicative constant) and strictly positive (or negative) inside . Furthermore, as a straightforward application of Talenti's symmetrisation principle [T], which we recall in our second Appendix, a Faber-Krahn type inequality holds true: among all domains with fixed volume, the first eigenvalue is minimised by the ball up to perhaps rigid motions.

On the other hand, the clamped eigenvalue problem presents several interesting features already in the case of $p = 2$, which make its study a highly nontrivial matter. Indeed, the first eigenfunction might be sign-changing, even for relatively simple domains such as squares or elongated ellipses [Co]. Moreover, some domains admit more than one first eigenfunction, as shown in [CD]. However, if is a ball, the first eigenfunction is unique and strictly positive (see for instance [GGS, Theorem 3.7]). The Faber-Krahn inequality has been shown to hold true only in dimensions $n = 2$ [N] and $n = 3$ [AB], while it still remains a challenging open problem in higher dimensions. The limiting case $p = 1$ has been studied by the second author jointly with Ruf and Tarsi in [PRT1, PRT2], wherein results analogous to the case $p = 2$ were obtained. However, in the clamped case, positivity of the first eigenfunction in a ball and the Faber-Krahn inequality were shown to

Then, there exists a sequence of exponents $(p_\ell)_{\ell=1}^{\infty}$ tending to infinity, such that

$$(u_{p_\ell}, \lambda_{p_\ell}) \to (u_\infty, \lambda_\infty), \qquad \text{in } C^1(\overline{\Omega})$$

of L^∞ functionals and of their associated analogues of Euler-Lagrange equations is known as Calculus of Variations in L^∞. Variational problems for first order functionals

$$(1.11)\qquad E_\infty(u,\mathcal{O}) = \operatorname*{ess\,sup}_{x\in\mathcal{O}} H\big(x,u(x),Du(x)\big), \quad u\in W^{1,\infty}(\Omega),\ \mathcal{O}\in\mathcal{L}(\Omega),$$

together with the associated equations, first emerged in the work of Aronsson in the 1960s ([A1]–[A3]). The area is now well developed and the relevant bibliography is vast; for a pedagogical introduction accessible to non-experts, we refer to [K8] (see also [C]). The vectorial case of (1.11) for maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a rapidly developing contemporary topic which first emerged in the work of the first author in the early 2010s (see [K1]–[K7], [K9]–[K12] as well as the joint contribution with Abugirda, Ayanbayev, Croce, Kristensen, Manfredi, Moser, Pisante and Pryer [AK, AyK, KM, KMo, CKP, KK, KP, KP2, KP3]).

Higher order L^∞ variational problems have only very recently begun to be investigated and are still poorly understood. In the most recent paper [KP2], the first author jointly with Pryer considered second order variational problems and their relevant equations, focusing on functional of the form

$$E_\infty(u,\mathcal{O}) = \operatorname*{ess\,sup}_{x\in\mathcal{O}} H\big(D^2u(x)\big), \quad u\in W^{2,\infty}(\Omega),\ \mathcal{O}\in\mathcal{L}(\Omega).$$

In the model case of H being the Euclidean norm on $\mathbb{R}^{n\times n}$, the relevant PDE playing the role of the Euler-Lagrange equation is the so-called ∞-Polylaplacian

$$\Delta^2_\infty u := \big(D^2u\big)^{\otimes 3} : \big(D^3u\big)^{\otimes 2} = 0$$

which is a fully nonlinear equation of third order. Subsequently, in the joint paper with Moser [KMo] the case of dependence on second derivatives through the Laplacian was considered, focusing on the model case of so-called ∞-Bilaplacian:

$$(1.12)\qquad \Delta^2_\infty u := \big(\Delta u\, I\big)^{\otimes 3} : \big(D^3u\big)^{\otimes 2} = 0.$$

These papers were partly motivated by problems arising in applied disciplines like Data Assimilation in the geosciences, PDE-constrained optimisation, etc. (see e.g. [K9] and references therein), as well as by curvature minimisation problems arising in Riemannian geometry previously studied by Moser-Schwetlick [MS] and Sakellaris [S] which relate to the Yamabe problem. In particular, our Theorem 1 draws heavily on methods successfully deployed to relevant but different L^∞ settings in [MS, S, KMo].

In the light of the above general L^∞ framework, we see the quantities $\Lambda^H_\infty(\Omega)$ and $\Lambda^C_\infty(\Omega)$ as the first eigenvalues of the ∞-Bilaplacian under the respective (hinged or clamped) boundary conditions and the parametric system (1.9) as the analogue of the constrained Euler-Lagrange equations for the minimisation problems (1.3)-(1.4). However, there does exist a more conventional PDE arising in the formal limit of the Dirichlet problems (1.7)-(1.8) as $p\to\infty$: by exploiting the relation

$$\Delta^2_p u = (p-1)|\Delta u|^{p-2}\Delta^2 u + (p-1)(p-2)|\Delta u|^{p-4}\Delta u|D(\Delta u)|^2$$

and performing similar computations as in [JLM], one can see that any putative ∞-eigenfunction u_∞ has to satisfy

$$\min\big\{|\Delta u| - \Lambda_\infty |u|,\ \Delta^2_\infty u\big\} = 0,$$

where Δ^2_∞ is the ∞-Bilaplacian given by (1.12). Notwithstanding, this is merely a formal claim, since we can not expect the solutions to be classical, and, to the best of our knowledge, there does not exist any analogue of the theory of viscosity solutions for the higher order problem at hand which is equally stable under limiting processes. However, this is not an issue because for the particular problem herein, the method of L^p-approximations constructs second order ∞-eigenfunctions with finer structure. This renders the direct study of the formal third order PDE redundant, whilst we obtain also a selection principle of the numerous possible ∞-eigenfunctions realising the infima in (1.3)-(1.4). A similar phenomenon has already arisen in the paper [KMo], wherein the authors proved existence and uniqueness of (absolute) minimisers to $u \mapsto \|\Delta u\|_{L^\infty(\Omega)}$ by solving the parametric system

$$\text{(1.13)} \qquad \begin{cases} \Delta u_\infty = \Lambda_\infty \,\mathrm{sgn}(f_\infty) & \text{a.e. in } \Omega, \\ \Delta f_\infty = 0 & \text{a.e. in } \Omega, \end{cases}$$

for any given prescribed boundary values $u_\infty = g$ and $Du_\infty = Dg$ on $\partial\Omega$. In (1.13), "sgn" is the usual single-valued sign function. In particular, (1.13) implies that $|\Delta u_\infty| = \Lambda_\infty$ a.e. in Ω and any such u_∞ is the unique minimising ∞-Biharmonic function solving (1.12) in the appropriate sense of $\mathcal{D}$-solutions, a new theory of generalised solutions for fully nonlinear systems recently introduced in [K9, K10]; The fact that u_∞ solves (1.12) if it solves (1.13) can be readily seen formally by recasting (1.12) as $\Delta u\,|D(|\Delta u|^2)|^2 = 0$.

2. Existence, structure and L^p-approximation to the eigenproblem for the ∞-Bilaplacian

Let $\Omega \Subset \mathbb{R}^n$ be a given domain with $C^{1,1}$ boundary $\partial\Omega$. In this section we establish Theorem 1. Its proof consists of several lemmas and, as in the statement, we tackle both cases simultaneously. To this end, it suffices to consider only the case of hinged boundary conditions, because if we obtain the desired existence-compactness-approximation conclusion by requiring the weaker condition "$u = 0$ on $\partial\Omega$" for the L^p approximating sequences of eigenfunctions, then it most certainly holds under the stronger requirement "$u = |Du| = 0$ on $\partial\Omega$" of clamped boundary conditions. Also, the putative limit eigenfunction u_∞ will be in the respective space because

$$W^{2,\infty}(\Omega) \supseteq W^{2,\infty}_H(\Omega) \supseteq W^{2,\infty}_C(\Omega)$$

and the hinged/clamped functional spaces are closed in their super-space

$$W^{2,\infty}(\Omega) := \bigcap_{1<p<\infty} \Big\{ u \in W^{2,p}(\Omega) \,:\, \Delta u \in L^\infty(\Omega) \Big\}.$$

For technical convenience in the proof we modify our notation slightly, as follows: for $p \in [1,\infty]$, we consider the normalised L^p norm with respect to the probability measure $\mathcal{L}^n_\Omega = \mathcal{L}^n / \mathcal{L}^n(\Omega) \in \mathcal{P}(\Omega)$, that is

$$\text{(2.1)} \qquad \|f\|_{L^p(\Omega,\mathcal{L}^n_\Omega)} := \begin{cases} \left(\fint_\Omega |f|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \|f\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

and, given a fixed $p \in (1,\infty)$, we also consider the constrained variational problem of finding $u_p \in W^{2,p} \cap W^{1,p}_0(\Omega)$ with $\|u_p\|_{L^p(\Omega,\mathcal{L}^n_\Omega)} = 1$ such that

$$\text{(2.2)} \qquad \|\Delta u_p\|_{L^p(\Omega,\mathcal{L}^n_\Omega)} = \Lambda_p,$$

where

$$\Lambda_p := \inf\Big\{ \|\Delta v\|_{L^p(\Omega)} \, : \, v \in W^{2,p} \cap W^{1,p}_0(\Omega),\ \|v\|_{L^p(\Omega)} = 1 \Big\}. \tag{2.3}$$

By standard weak compactness, lower semicontinuity and Lagrange multiplier arguments (see e.g. the relevant arguments for the Laplacian in [E]), one easily sees that for any $p \in (1,\infty)$ there indeed exists a desired minimiser u_p of (2.2)-(2.3) which solves weakly the Dirichlet problem

$$\begin{cases} \Delta^2_p u_p = (\Lambda_p)^p |u_p|^{p-2} u_p & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Note that we refrain from stating the natural boundary condition "$\Delta u_p = 0$ on $\partial\Omega$" which is also satisfied weakly in the hinged case only, because we do not utilise it in any way in the foregoing reasoning which applies to both cases.

We begin with the next lemma.

Lemma 3. Let $\{(u_p, \Lambda_p) : 1 < p < \infty\}$ be the family of eigenfunctions and eigen-

Since @ is of class $C^{1;1}$, by the Calderon-Zygmund global L^k-estimate (see e.g. [GT, Lemma 9.17, p. 242]), it follows that there exists a constant $C = C(k;) > 0$ such that

(2.6) $$\|u_p\|_{W^{2;k}( )} \quad C(k;) \, \| \quad u_p\|_{L^k(\; ; \;)}:$$

By (2.2),(2.3) and Hölder inequality, for any $p \quad k$ we have

(2.7) $$\| \quad u_p\|_{L^k(\; ; \;)} \quad \| \quad u_p\|_{L^p(\; ; \;)} = \quad {}_p$$

and hence by (2.5)-(2.6) we infer that

(2.8) $$\|u_p\|_{W^{2;k}( )} \quad \frac{2C(k;) \, \| \quad \|_{L^1( )}}{\| \; \|_{L^1( )}}; \quad p \quad k:$$

By (2.8), the sequence $(u_p)_1^1$ is bounded in $W^{2;k}( )$ for any k 2 N. By passing to a further subsequence if necessary, by Morrey's theorem and a standard weak compactness diagonal argument there exists

$$u_1 \; 2 \bigcap_{1<p< \, 1} W^{2;p}( ) \; \backslash \; C^1(\overline{\ })$$

such that u_p ! u_1 strongly in $C^1(\overline{\ })$ and $D^2 u_p \; * \; D^2 u_1$ weakly in

Proof of Lemma 4. By (1.1) we have that $W^{2,\infty}_H(\Omega) \subseteq W^{2,p}\cap W^{1,p}_0(\Omega)$ for all $p \in (1,\infty)$. Hence, by (2.2)-(2.3) and minimality, we have

$$\Lambda_p \leq \|\Delta v\|_{L^p(\Omega;\mathcal{L}^n)}, \quad \forall\, v \in W^{2,\infty}_H(\Omega) : \|v\|_{L^p(\Omega;\mathcal{L}^n)} = 1.$$

By fixing v and letting $p \to \infty$, by Lemma 3 we obtain

$$\|\Delta u_\infty\|_{L^\infty(\Omega)} = \hat{\Lambda}_\infty \leq \|\Delta v\|_{L^\infty(\Omega)}, \quad \forall\, v \in W^{2,\infty}_H(\Omega) : \|v\|_{L^\infty(\Omega)} = 1.$$

By taking infimum over all such v, we deduce the equality $\hat{\Lambda}_\infty = \Lambda_\infty$, as claimed. Finally, recall that we already know $0 \leq \Lambda_\infty < \infty$. Suppose for the sake of contradiction that $\Lambda_\infty = 0$. Then, the constraint $\|u_\infty\|_{L^\infty(\Omega)} = 1$ contradicts the uniqueness of solutions to the Dirichlet problem for the Laplace equation because $\Delta u_\infty = 0$ in Ω and $u_\infty = 0$ on $\partial\Omega$. The lemma has been established.

Next, we prepare towards the construction of the function $f_\infty \in L^1(\Omega) \cap BV_{loc}(\Omega)$ and the signed measure $\mu_\infty \in \mathcal{M}(\Omega)$ associated with the ∞-eigenpair $(u_\infty, \Lambda_\infty)$ which was constructed in Lemmas 3-4 above.

Lemma 5. *Let $(u_p)_1^\infty$ be the subsequence of the L^p minimisers (satisfying for each p the equalities (2.2)-(2.3) and solving the Dirichlet problem (2.4)) along which the conclusion of Lemmas 3-4 hold. We define the measurable functions $f_p, g_p : \Omega \to \mathbb{R}$ by*

$$f_p := \frac{|\Delta u_p|^{p-2}\Delta u_p}{(\Lambda_p)^p}, \tag{2.11}$$

$$g_p := |u_p|^{p-2}u_p. \tag{2.12}$$

Then, we have

$$\Delta f_p = g_p \quad \text{in } D'(\Omega), \tag{2.13}$$

and if $p' = p/(p-1)$, we also have

$$\|f_p\|_{L^{p'}(\Omega;\mathcal{L}^n)} = \frac{1}{\Lambda_p}, \tag{2.14}$$

$$\|g_p\|_{L^{p'}(\Omega;\mathcal{L}^n)} = 1. \tag{2.15}$$

Proof of Lemma 5. The proof is elementary, but we provide it anyway for the sake of completeness. Let f_p, g_p be given by (2.11)-(2.12). We begin by noting that (2.13) is a consequence of (2.4) and the definitions. For (2.14), by (2.1)-(2.3) we have

$$\begin{aligned}
\|f_p\|_{L^{p'}(\Omega;\mathcal{L}^n)} &= \frac{1}{(\Lambda_p)^p}\left(\int_\Omega \Big| |\Delta u_p|^{p-2}\Delta u_p \Big|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \frac{1}{(\Lambda_p)^p}\left(\int_\Omega |\Delta u_p|^{p}\right)^{\frac{p-1}{p}} \\
&= \frac{1}{(\Lambda_p)^p}(\Lambda_p)^{p-1} \\
&= \frac{1}{\Lambda_p}
\end{aligned}$$

and similarly, in view of (2.3) we have

$$\begin{aligned}\|g_p\|_{L^{p'}(\Omega;\mathbb{R})} &= \left(\int_\Omega \left(|u_p|^{p-2}u_p\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\ &= \left(\int_\Omega |u_p|^p\right)^{\frac{p-1}{p}}\\ &= 1.\end{aligned}$$

The lemma ensues.

Lemma 6. In the setting of Lemma 5, there exist a function $f_\infty \in L^1(\Omega) \cap BV_{loc}(\Omega)$ and a signed Radon measure $\mu_\infty \in \mathcal{M}(\Omega)$ associated with the ∞-eigenpair $(u_\infty, \Lambda_\infty)$ such that

$$\begin{aligned} f_p &\longrightarrow f_\infty, \quad \text{in } L^q_{loc}(\Omega) \text{ for all } q \in \Big[1, \frac{n}{n-1}\Big), \\ f_p &\overset{*}{\rightharpoonup} f_\infty, \quad \text{in } BV_{loc}(\Omega), \\ g_p\mathcal{L}^n &\overset{*}{\rightharpoonup} \mu_\infty, \quad \text{in } \mathcal{M}(\Omega), \end{aligned}$$

along perhaps a further subsequence as $p_j \to \infty$. Moreover, f_∞ is a distributional solution to the Poisson equation with right hand side μ_∞:

$$-\Delta f_\infty = \mu_\infty \quad \text{in } \mathcal{D}'(\Omega).$$

Proof of Lemma 6. By Lemmas 3 and 6, we have that the sequences $(f_p)_1^\infty$, $(g_p)_1^\infty$ are uniformly bounded in $L^1(\Omega)$ and for each p along a subsequence they satisfy

$$-\Delta f_p = g_p \quad \text{in } \mathcal{D}'(\Omega).$$

By Lemma 17 and Corollary 18 in our first Appendix, we have that $(f_p)_1^\infty$ is bounded in $L^{n/(n-1)}_{loc}(\Omega) \cap BV_{loc}(\Omega)$ and there exists a limit function f_∞ such that the desired modes of convergence hold true. Moreover, since the absolutely continuous measures $(g_p\mathcal{L}^n)_1^\infty \subseteq \mathcal{M}(\Omega)$ have bounded total variation, there exists a signed Radon measure μ_∞ such that the desired weak* convergence holds true as well. By passing to the weak* limit in (2.13) as $p \to \infty$ along an appropriate subsequence, we obtain $-\Delta f_\infty = \mu_\infty$ on Ω in the sense of distributions.

It remains to show that $f_\infty \in L^1(\Omega)$. Indeed, fix a compact set $K \Subset \Omega$ with positive measure. Since $f_p \to f_\infty$ as $p \to \infty$ strongly in $L^1_{loc}(\Omega)$ and $(f_p)_1^\infty$ is bounded in $L^1(\Omega)$, by (2.14) and (2.1) we have

$$\|f_\infty\|_{L^1(K)} = \lim_{p\to\infty} \|f_p\|_{L^1(K)} \leq \limsup_{p\to\infty} \|f_p\|_{L^1(\Omega)} \leq \frac{\mathcal{L}^n(\Omega)}{\Lambda_\infty}.$$

We conclude by invoking the upper continuity properties of the measure $\|f_\infty\|_{L^1(\cdot)}$ on Ω.

Now we show the validity of the desired differential inclusion which the ∞-eigenpair $(u_\infty, \Lambda_\infty)$ satisfies.

Lemma 7. Let the quadruple $(u_\infty, \Lambda_\infty, f_\infty, \mu_\infty)$ be as in Lemmas 3-6. Then, we have

$$u_\infty(x) = \Lambda_\infty \frac{f_\infty(x)}{|f_\infty(x)|}, \quad \text{a.e. } x \in \Omega \setminus \{f_\infty = 0\}.$$

Proof of Lemma 7. By (2.11

Then, the ∞-eigenpair $(u_\infty, \Lambda_\infty)$ satisfies

$$\Delta u_\infty(x) \in \Lambda_\infty \operatorname{Sgn} f_\infty(x) \quad \text{a.e. } x \in \Omega.$$

We complete the proof of Theorem 1 by showing that in the case of hinged boundary condition, the differential inclusion reduces to just the Poisson equation with constant right hand side. This result reconciles with the more general findings on (absolute) minimisers of second order functionals in Calculus of Variations in L^∞ in the papers ([MS, S, KP2, KMo]).

Proposition 9. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Then $u_\infty \in W^{2,\infty}_C(\Omega)$ is a minimiser for $\Lambda^H_\infty(\Omega)$ if and only if it is a multiple of the solution to

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

In particular, u_∞ is strictly positive (or strictly negative) in Ω, and unique up to a nonzero multiplicative constant.

Note that for this last part of the proof of the theorem, we do not need any boundary regularity.

Proof of Proposition 9. Let u_∞ be a minimiser realising the infimum in (1.3). By a rescaling, we may assume that $\|\Delta u_\infty\|_{L^\infty(\Omega)} = 1$ and by replacing u_∞ by $-u_\infty$, we may assume that

$$\|u_\infty\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega u_\infty.$$

Set $g := -\Delta u_\infty$ and suppose for the sake of contradiction that $g \not\equiv 1$ on Ω, keeping in mind that $-1 \leq g \leq 1$ a.e. on Ω. To this end, let v be the solution of (2.19). We have that

$$\begin{cases} -\Delta(v - u_\infty) = 1 - g & \text{in } \Omega, \\ v - u_\infty = 0, & \text{on } \partial\Omega, \end{cases}$$

and $1 - g \geq 0$ in Ω with $1 - g > 0$ on a subset of positive measure. By the strong maximum principle we infer that $u_\infty < v$ in Ω, and therefore

$$\|u_\infty\|_{L^\infty(\Omega)} < \|v\|_{L^\infty(\Omega)}$$

because the supremum is attained inside Ω. This leads to the contradiction to minimality, unique up to a

3. The Faber-Krahn inequality for the ∞-Bilaplacian and ∞-eigenpairs in the case of the ball

In this section we establish the proof of Theorem 2 in the case of hinged and clamped boundary conditions, whilst we also calculate the eigenvalues and the eigenfunctions in the case that the domain is a Euclidean ball.

The case of hinged boundary conditions. We begin with the simpler case of hinged boundary conditions. In this section we will be using the symbolisation ω_n for the volume of the unit ball in $\mathbb{R}^n$, whilst $\mathbb{B}_R$ will stand for the open ball in $\mathbb{R}^n$ of radius $R > 0$, allowing ourselves the convenient flexibility to mean either centred at the origin, or at any other point. The meaning will be clear from the context and in any case the invariance of the ∞-eigenvalue problem under rigid motions will not entail any ramifications.

Proposition 10 (The Faber-Krahn inequality in the hinged case)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary and let $\mathbb{B}_R$ be a ball with radius*

$$R := \left(\frac{\mathcal{L}^n(\Omega)}{\omega_n}\right)^{1/n}$$

namely, such that $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathbb{B}_R)$. Let $\lambda_1^H(\Omega)$ be given by (1.3). Then,

$$\lambda_1^H(\Omega) \geq \lambda_1^H(\mathbb{B}_R),$$

and equality holds if and only if Ω coincides with the ball $\mathbb{B}_R$ up to a rigid motion in $\mathbb{R}^n$.

Proof of Proposition 10. By a rescaling argument, we may assume without loss of generality that $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathbb{B}_1) = \omega_n$. Let u be a positive minimiser for $\lambda_1^H(\Omega)$. By Talenti's symmetrisation principle (see e.g. Kesavan [Ke1, Theorem 3.1.1]), if v is the solution of the problem

$$\begin{cases} -\Delta v = 1 & \text{in } \mathbb{B}_1, \\ v = 0 & \text{on } \partial \mathbb{B}_1, \end{cases}$$

we obtain that $0 \leq u^* \leq v$ in $\mathbb{B}_1$, where u^* is the Schwarz symmetrisation of u. Therefore, we deduce that

$$\|u\|_{L^\infty(\Omega)} = \|u^*\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)},$$

which implies $\lambda_1^H(\Omega) \geq \lambda_1^H(\mathbb{B}_1)$. By the results of [Ke2], it follows that equality holds if and only if Ω coincides with $\mathbb{B}_1$, up to rigid motions.

The next lemma, which is a direct consequence of Proposition 9 of the previous section, completes the picture in the case of hinged boundary conditions.

Corollary 11 (The ∞-eigenpairs in the hinged case)**.** *Let $\mathbb{B}_R$ be the ball of radius R in $\mathbb{R}^n$ centred at the origin. Then every minimiser is a nonzero multiple of the function defined as*

$$u_\infty(x) := \frac{1}{2n}(R^2 - |x|^2)$$

and we also have

$$\lambda_1^H(\mathbb{B}_R) = \frac{2n}{R^2}.$$

The case of clamped boundary conditions. We continue with the more complex case of clamped boundary conditions. Let us begin by noting that, if $u \in W^{2,\infty}_C(\Omega)$, then

$$\int_\Omega \Delta u = 0,$$

as a consequence of the Gauss-Green theorem. Nonetheless, the converse is not true in general for a function $u \in W^{2,\infty}(\Omega)$ (satisfying $u = 0$ on $\partial\Omega$), unless Ω is a ball B_R and u is radially symmetric. In this case,

$$0 = \int_{B_R} \Delta u = \int_{\partial B_R} Du \cdot \nu \, d\mathcal{H}^{n-1} = u'(R)\,\mathcal{H}^{n-1}(\partial B_R)$$

which implies that $u'(R) = 0$ and hence indeed $u \in W^{2,\infty}_C(\Omega)$ as claimed. In the above argument, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-Hausdorff measure restricted to $\partial\Omega$ and ν the outwards pointing normal vector field on $\partial\Omega$.

Before proving the Faber-Krahn inequality, we need some technical preparation which is the content of the next lemma.

Lemma 12. *Let $R \in (0,1]$, and $B_R \subseteq \mathbb{R}^n$ be the ball of radius R centred at the origin. Let f be defined on B_1 as*

$$f(x) := \begin{cases} 1, & \text{for } |x| \leq 2^{-\frac{1}{n}}, \\ -1, & \text{for } 2^{-\frac{1}{n}} < |x| < 1, \end{cases}$$

and let f_R be the restriction of f to B_R. Let w_R be the solution to the problem

$$\begin{cases} -\Delta w_R = f_R & \text{in } B_R, \\ w_R = 0 & \text{on } \partial B_R. \end{cases}$$

Then, when $n = 2$, w_R is given by

$$w_R(x) = \frac{1}{4}(R^2 - |x|^2)$$

if $R \leq 2^{-\frac{1}{2}}$, and

$$w_R(x) = \begin{cases} \frac{1}{4} - \frac{R^2}{4} + \frac{\ln R}{2} + \frac{\ln 2}{4} - \frac{|x|^2}{4}, & \text{for } |x| \leq 2^{-\frac{1}{2}} \\ \frac{|x|^2}{4} - \frac{\ln|x|}{2} - \frac{R^2}{4} + \frac{\ln R}{2}, & \text{for } 2^{-\frac{1}{2}} < |x| < R, \end{cases}$$

otherwise. If $n \geq 3$, w_R is given by

$$w_R(x) = \frac{1}{2n}(R^2 - |x|^2)$$

if $R \leq 2^{-\frac{1}{n}}$, and

$$w_R(x) = \begin{cases} \frac{2^{-\frac{2}{n}}}{n} - \frac{R^2}{2n} - \frac{R^{2-n}}{n(n-2)} + \frac{2^{1-\frac{2}{n}}}{n(n-2)} & \text{for } |x \end{cases}$$

(ii) w_R is radially symmetric and radially decreasing;
(iii) for $R = 1$, w_1 belongs to $W^{2;1}_C(B_1)$;
(iv) the function $R \mapsto \|w_R\|_{L^\infty()}$, defined on $(0;1]$, attains a strict maximum for $R = 1$.

The proof of this result is a computation exercise on the use of derivatives in polar coordinates and therefore we refrain from providing the tedious details of it. Now we have:

Proposition 13 (The Faber-Krahn inequality in the clamped case). Let $\mathbb{R}^n$ be a bounded domain with $C^{1;1}$ boundary and let B_R be a ball with radius

$$R := \left(\frac{\mathcal{L}^n()}{\omega_n}\right)^{1=n}$$

namely, such that $\mathcal{L}^n() = \mathcal{L}^n(B_R)$. Let $^C_1()$ be given by (1.4). Then,

$$^C_1() \quad ^C_1(B_R);$$

and equality holds if and only if coincides with the ball B_R up to a rigid motion in $\mathbb{R}^n$.

Proof of Proposition 13. Without loss of generality, we may assume that $\mathcal{L}^n() = \mathcal{L}^n(B_1) = \omega_n$. Let u be a minimiser realising the infimum in (1.4) for , rescaled in a way that $\| u\|_{L()}$

where v^+ is the positive part of v. Let Ω^+ be the open set $\{v > 0\}$ and suppose that $\mathcal{L}^n(\Omega^+) = \omega_n R^n$. Clearly, we have that $R \in (0, \infty]$. By Talenti's symmetrisation principle ([Ke1, Theorem 3.1.1], recalled in our second Appendix), if $\tilde{w}_R$ is the solution of the problem

$$\begin{cases} -\Delta \tilde{w}_R = \tilde{f} & \text{in } B_R, \\ \tilde{w}_R = 0 & \text{on } \partial B_R, \end{cases}$$

then

$$\|v^+\|_{L^\infty(\Omega)} \leq \|\tilde{w}_R\|_{L^\infty(\Omega)}.$$

Let f_R and w_R be the functions defined =

is given by $u(x) = w(x=R)$, where w

(b) By the obtained estimate, the difference quotients $(D^{1;h} u \mathcal{L}^n)_{h \neq 0}$ of u have bounded total variation in the space of Radon measures $\mathcal{M}(\Omega')$ and hence by well known arguments

$$D^{1;h} u \mathcal{L}^n \overset{*}{\rightharpoonup} [Du] \quad \text{in } \mathcal{M}_{loc}(\Omega; \mathbb{R}^n);$$

as $h \to 0$. The estimate follows from the weak* lower semi-continuity of the total variation norm and the Sobolev inequality in the BV-space ([EG, Ch. 5]). The lemma ensues.

Proof of Corollary 18. The result is an immediate consequence of the Fréchet-Kolmogorov strong compactness theorem (see e.g. [B, Ch. 4]), the Vitali convergence theorem ([FL, Ch. 2]) via an equi-integrability argument similar to that employed in Lemma 7 and standard results on the weak* compactness of the spaces of BV functions and Radon measures ([EG, Ch. 5]).

5. Appendix: Some useful results

In this appendix we collect some useful results which have been utilised earlier in the paper. Some of the results are well-known,Er8 e( )]TJ/F8 9.96260pg6dT7ut60pg6dfor4(of)-414(ces)]T

In particular, by the above result it follows that

$$\|u\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega^*)} .$$

Further, by a result of Kesavan [Ke2], equality $\|u\|_{L^\infty(\Omega)} = \|v\|_{L^\infty(\Omega^*)}$ holds true if and only if $\Omega = \Omega^*$, and f is radially symmetric.

The Bathtub principle. In our proofs we have also used the following simple measure-theoretic fact, whose proof is a special case of a more general result (see [LL, Theorem 1.14]).

Proposition 20. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $f \in L^1(\Omega)$ a function such that, for every $t \in \mathbb{R}$, the level set $\{f = t\}$ is a Lebesgue null set. Let $a, b, \ell \in \mathbb{R}$ be fixed and such that $a|\Omega| \leq \ell \leq b|\Omega|$, and consider the set of functions*

$$C := \left\{ g \in L^1(\Omega) : a \leq g \leq b \text{ in } \Omega, \; \int_\Omega g(x)\, dx = \ell \right\}.$$

Then the supremum in the maximisation problem

$$\sup_{g \in C} \int_\Omega f(x) g(x)\, dx$$

is attained at a function

KK. N. Katzourakis, J. Kristensen, Strong convergence of derivatives of p-Polyharmonic mappings as p ! 1 , manuscript in preparation.

KM. N. Katzourakis, J. Manfredi, Remarks on the Validity of the Maximum Principle for the 1 -Laplacian , Le Matematiche, Vol. LXXI (2016) Fasc. I, 63 - 74, DOI: 10.4418/2016.71.1.5.

KMo. N. Katzourakis, R. Moser, Existence, uniqueness and structure of second order absolute minimisers , ArXiv preprint, http://arxiv.org/pdf/1701.03348.pdf .

KP. N. Katzourakis, T. Pryer, On the numerical approximation of 1 -Harmonic mappings , Nonlinear Di erential Equations & Applications 23 (6), 1-23 (2016).

KP2. N. Katzourakis, T. Pryer, 2 nd order L^1 variational problems and the 1 -Polylaplacian , ArXiv preprint, http://arxiv.org/pdf/1605.07880.pdf .

KP3. N. Katzourakis, T. Pryer, On the numerical approximation of 1 -Biharmonic and p-Biharmonic functions , ArXiv preprint, https://arxiv.org/pdf/1701.07415.pdf .

Ke1. S. Kesavan, Symmetrisations and applications , World Scienti c (2006).

Ke2. S. Kesavan, Some remarks Mose7415.pdf