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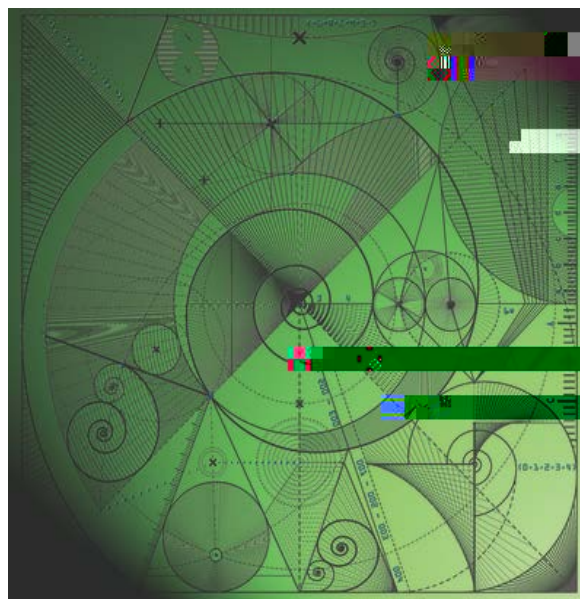
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Existence, Uniqueness and Structure of Second Order Absolute Minimisers

by

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EXISTENCE, UNIQUENESS AND STRUCTURE OF SECOND ORDER ABSOLUTE MINIMISERS

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded open $C^{1,1}$ set. In this paper we prove the existence of a unique second order absolute minimiser u_1 of the functional

$$E_1(u; \Omega) := \int_{\Omega} kF(\cdot; u) dx; \quad \Omega \text{ measurable};$$

with prescribed boundary conditions for u and Du on $\partial\Omega$ and under natural assumptions on F . We also show that u_1 is partially smooth and there exists a harmonic function $f_1 \in L^1(\Omega)$ such that

$$F(x; u_1(x)) = e_1 \operatorname{sgn} f_1(x)$$

for all $x \in \Omega$ with $f_1 \geq 0$ a.e., where e_1 is the minimum of the global energy.

1. Introduction

For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let also $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a real function that is $L^1(\Omega) \times B(R)$ -measurable, namely, measurable with respect to the product σ -algebra of the Lebesgue subsets of Ω with the Borel subsets of $\mathbb{R}$. In this paper we consider variational problems for second order supremal functionals of the form

$$(1.1) \quad E_1(u; \Omega) := \int_{\Omega} F(\cdot; u) dx; \quad \Omega \text{ measurable};$$

where the admissible functions u range over the (Frechet) Sobolev space

$$(1.2) \quad W^{2,1}_0(\Omega) := \left\{ u \in W^{2,p}_0(\Omega) : u \in L^1(\Omega) : 1 < p < \infty \right\}$$

The following is a natural notion of minimiser for variational problems of this type.

Definition 1 (Second order absolute minimiser)

Example. For $\Omega = (-1; 0) \cup (0; 1)$, consider the functional $E_1(u; \Omega) = k u^0_{L^1(\Omega)}$ and let $Q(x) := x \chi_{(-1; 0)}(x) - x(x-1) \chi_{(0; 1)}(x)$, $x \in \mathbb{R}$. Then Q is a global minimiser of $E_1(\cdot; \Omega)$ in $W^{2,1}_Q(\Omega)$ with $E_1(Q; \Omega) = 2$ and is also a second order absolute minimiser. However, for any function $\varphi \in C^1_c((-1; 0))$ with $0 < k^0_{L^1((-1; 0))} < 1$, the perturbation $Q + \varphi$ still satisfies $E_1(Q + \varphi; \Omega) = 2$ and lies in $W^{2,1}_Q(\Omega)$, but does not minimise $E_1(\cdot; (-1; 0))$ over $W^{2,1}_Q((-1; 0))$ because the only minimiser on $(-1; 0)$ with boundary data Q is the identity.

On the other hand, if $u \in W^{2,1}_Q(\Omega)$ minimises $E_1(\cdot; \Omega)$ uniquely in $W^{2,1}_Q(\Omega)$ with respect to its own boundary conditions, then we will show that u is actually the unique second order absolute minimiser. This is the situation that we will find in the main results of this paper. We first give a condition that guarantees that u is the unique minimiser for its boundary values on any subdomain. While it is not obvious that this condition can be met, we will subsequently prove that it is satisfied by exactly one function under given boundary conditions and very mild additional assumptions.

In the following, we will use the symbolisation sgn for the sign function, with the convention that $\text{sgn}(0) = 0$. We will assume also that

(1.3) Ω is a bounded connected open subset of $\mathbb{R}^n$, $n \geq 1$,

and

(1.4) $\begin{cases} F: \mathbb{R} \rightarrow \mathbb{R} \text{ is } L^1(\cdot) \text{ } B(\mathbb{R})\text{-measurable and for a.e.} \\ x \in \mathbb{R}, \quad \forall! F(x; \cdot) \text{ is strictly increasing with } F(x; 0) = 0. \end{cases}$

Our first main result therefore is:

Theorem 2 (Criterion for unique minimisers). Suppose (1.3)-(1.4) hold and consider (1.1) and a function $u \in W^{2,1}_Q(\Omega)$. If there exist a number $e \geq 0$ and a function $f \in L^1(\Omega)$ satisfying

(1.5) $f = 0; \quad \text{on } \partial\Omega;$

such that

(1.6) $F(\cdot; u) = e \text{sgn}(f); \quad \text{a.e. on } \Omega;$

then

$$E_1(u; \Omega) < E_1(u + \varphi; \Omega)$$

for any open $O \subset \Omega$ and any $\varphi \in W^{2,1}_Q(O)$ with $\varphi = 0$ on ∂O .

is by assumption strictly increasing, (1.6) is equivalent to the next representation formula for u :

$$u(x) = F(x; \cdot)^{-1} e \operatorname{sgn} f(x) \quad ; \quad \text{a.e. } x \in \mathbb{R}^2 :$$

Moreover, by using standard argument involving Green functions (see e.g. [GT, Ch. 2]), we could represent u in terms of $F; f; e; u|_{\partial\Omega}; D u|_{\partial\Omega}$.

Theorem 2 gives a connection between the variational problem and a PDE system of second order equations with a parameter consisting of (1.5) and (1.6). We will see later that under certain assumptions on F and the boundary data, the system has in fact a solution $(u; f; e)$ with $f \geq 0$ if $e > 0$. It then follows that $(u; e)$ is unique and the system is equivalent to unique global minimality under prescribed boundary data. We may think of (1.5) and (1.6) as a PDE formulation of the L^1 variational problem. There does exist, however, a more conventional analogue of the "Euler-Lagrange equation" for (1.1). This is the fully nonlinear PDE of third order

$$(1.7) \quad F(x; u) F(x; u) D_j F(x; u) j^2 = 0; \quad \text{on } \Omega :$$

A particular model case of (1.1)-(1.7) is what we call the " L^1 -Bilaplacian" and arises from the choice $F(x; \cdot) = \cdot$. Then, equation (1.6) becomes $u = e \operatorname{sgn}(f)$ and (1.7) becomes $u D_j (u^2) j^2 = 0$. Due to the particular structure of the functional (1.1), however, in this case (1.7) becomes redundant since all the structural information of second order absolute minimisers can be obtained directly from the L^1 variational problem.

For our existence result, we will assume that $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$(1.8) \quad \begin{aligned} & F \in C^2(\mathbb{R}); \\ & F(x; 0) = 0; \quad x \in \mathbb{R}^2; \\ & \exists c > 0 : \quad \begin{aligned} & c \leq F(x; \cdot) \leq \frac{1}{c}; \quad (x; \cdot) \in \mathbb{R}^2 \times \mathbb{R}; \\ & F(x; \cdot) F(x; \cdot) \leq \frac{1}{c}; \quad (x; \cdot) \in \mathbb{R}^2 \times \mathbb{R}; \end{aligned} \end{aligned}$$

where subscripts of F denote partial derivatives. The conditions of (1.8) imply that for any fixed $x \in \mathbb{R}^2$ the partial function $jF(x; \cdot)$ is level-convex on $\mathbb{R}$.

where $e_1 = E_1(u_1; \cdot)$. Further, $f_1 \geq 0$ if $e_1 > 0$.

(II) Let

$$u_1 := f_1^{-1}(f_0 g):$$

If $e_1 > 0$, then u_1 belongs to $C^3(\bar{\Omega} \setminus \Gamma)$ for any $\varepsilon \in (0, 1)$ and Γ is a Lebesgue nullset. If $e_1 = 0$, then u_1 is a harmonic function.

(III) For $p \geq 2$, let

$$e_p := \inf_{u \in W_{u_0}^{2,p}(\Omega)} E_p(u):$$

Then, for any p large enough there exists a global minimiser $u_p \in W_{u_0}^{2,p}(\Omega)$ of E_p satisfying $E_p(u_p) = e_p$. Moreover, $e_p \rightarrow e_1$ as $p \rightarrow \infty$ and there exists a subsequence $(p_k)_{k=1}^\infty$ such that $u_{p_k} \rightarrow u_1$ in the weak topology of $W^{2,1}(\Omega)$ as $k \rightarrow \infty$. In addition,

$$\begin{aligned} & u_{p_k} \rightarrow u_1; \quad \text{in } C^1(\bar{\Omega}); \\ & D^2 u_{p_k} \rightarrow D^2 u_1; \quad \text{in } L^q(\Omega; \mathbb{R}^{n \times n}) \text{ for all } q \in (1, 1^*); \\ & u_{p_k} \rightarrow u_1; \quad \text{a.e. on } \Omega \text{ and in } L^q(\Omega) \text{ for all } q \in (1, 1^*); \end{aligned}$$

as $k \rightarrow \infty$. Furthermore, $u_{p_k} \rightarrow u_1$ locally uniformly on $\bar{\Omega} \setminus \Gamma$ if $e_1 > 0$ and locally uniformly on $\bar{\Omega}$ if $e_1 = 0$.

(IV) Let

$$(1.9) \quad f_p := \frac{1}{e_p^{p-1}} F(\cdot; u_p)^{p-2} F(\cdot; u_p) F(\cdot; u_p)$$

if $e_p \neq 0$ and $f_p = 0$ if $e_p = 0$. Then, the harmonic function f_1 in (I) may be chosen such that $f_{p_k} \rightarrow f_1$ as $k \rightarrow \infty$ in the strong local topology of $C^1(\bar{\Omega})$.

By invoking Theorem 2, an immediate consequence is that the modes of convergence in Theorem 3(III) as $p \rightarrow \infty$ are actually full and not just subsequential. Also, known results on the regularity of nodal sets of solutions to elliptic equations [HS] imply that Γ is countably rectifiable, being equal to the union of countably many smooth submanifolds of $\bar{\Omega}$ and a set of vanishing $(n-2)$ -dimensional Hausdorff measure. This, however, uses only the fact that f_1 is harmonic and it seems plausible that the full statement in (I) could give further information.

The optimal regularity of Γ is an open question which we do not attempt to answer here. Certainly, full regularity of the set Γ cannot be expected as there are limitations: in [KP2] it was noted that Γ in general may not be a smooth submanifold, as for certain data u_0 the intersection of transversal lines in $\bar{\Omega}$ was observed in numerical experiments. In most cases the set Γ is necessarily non-empty and divides $\bar{\Omega}$ into two distinct parts, whilst the equations $F(\cdot; u_1) = e_1$ do not permit any solutions in $W_{u_0}^{2,1}(\Omega)$ for most boundary data, even if $n = 1$. Therefore, the Laplacian of u_1 will have a jump on Γ and, in terms of u_1 , no more regularity than $W^{2,1}(\Omega)$ can be expected (see the numerical and explicit solutions in [KP2]).

Let us also note further that (I)–(IV) above have been obtained in [KP2] for $n = 1$ and in some other special cases (although were not stated in this explicit fashion), whilst the qualitative behaviour emerging here was observed numerically for $n = 2$ and $F(x; \cdot) = \cdot$.

When combined, Theorem 2 and Theorem 3 imply in particular the following.

Corollary 4. Under the hypotheses of Theorem 3, there exists a unique global minimiser u_1 of $E_1(\cdot; \cdot)$ in $W_{u_0}^{2,1}(\cdot)$, which is a second order absolute minimiser and a strong solution to the Dirichlet problem for (1.7):

$$\begin{aligned} & \Delta F(\cdot; u) F(\cdot; u) - D_j F(\cdot; u) j^2 = 0; & \text{in } \Omega; \\ & u = u_0; & \text{on } \partial\Omega; \\ & Du = D u_0; & \text{on } \partial\Omega; \end{aligned}$$

More precisely, u_1 is thrice differentiable a.e. on Ω and satisfies the PDE in the pointwise sense.

The study of supremal functionals and of their associated equations is known as Calculus of Variations in the space L^1 . Second order variational problems in L^1 have only relatively recently been studied and are still poorly understood. It is remarkable that for our specific problem, we obtain not just unique absolute minimisers, but also a fair amount of detailed information about their structures, with relatively simple means. On the other hand, our methods take advantage of the special structure of the problem and are unlikely to work in general, although they allow the following modest generalisation: all of the preceding results hold for the seemingly more general case where the Laplacian is replaced by the projection $A : D^2 u = \sum_{i,j} A_{ij} D_{ij}^2 u$ on a fixed positive symmetric matrix $A \in \mathbb{R}_+^{n \times n}$. This gives rise to the following functional:

$$E_1(u; \Omega) = \int_{L^1(\Omega)} F(\cdot; A : D^2 u) \, dx; \quad \Omega \text{ measurable};$$

However, this case can easily be reduced to the case we study herein via the change of variables $x \mapsto O^T x$ for a diagonal $n \times n$ matrix O and an orthogonal matrix $O \in O(n)$ arising from the spectral representation $A = O \Lambda O^T$.

Some of the techniques that underpin Theorem 3 have been successfully deployed to problems somewhat different to (1.1) (with dependence on u in addition to ∇u) [MS, S1], which suggests that further generalisation might be possible. In order to keep the presentation simple, however, we do not explore this possibility any further in this work.

We conclude this introduction by placing the L^1 problem we study herein into the wider context of Calculus of Variations. Variational problems for first order functionals of the form

$$(1.10) \quad (u; \Omega) \mapsto \int_{\Omega} H(x; \nabla u(x)) \, dx; \quad u \in W^{1,1}(\Omega); \quad \Omega \text{ measurable};$$

together with the associated PDEs, first arose in the work of Aronsson in the 1960s [A1][A3]. The first order case is very well developed and the relevant bibliography is very extensive. For a pedagogical introduction to the theme which is accessible to non-experts, we refer to the monograph [K8] (see also [C]). The vectorial case of (1.10) for maps

with dependence on pure second derivatives only. Some preliminary investigations (relevant to the second order case of energy density $H(\cdot; u; u^0; u^{0g})$ when $n = 1$) had previously been performed via different methods by Aronsson and Aronsson-Barron in [A4, AB].

Apart from the intrinsic mathematical interest, the motivation to study higher order L^1 minimisation problems comes from several diverse areas. In applied disciplines like Data Assimilation in the geosciences, PDE-constrained optimisation in aeronautics, etc. (see e.g. the model problem in [K9] and references therein, as well as the classical monograph [L]), a prevalent underlying problem is the construction of approximate solutions to second order ill-posed PDE problems. For instance, in the modelling of aquifers, one needs to solve a Poisson equation $\Delta u = f$ coupled with a pointwise constraint of the form $K(u) = k$ for given functions $f; K; k$. By minimising the error function $\int |u - f|^2 + \int K(u) - k^2$ in L^1 , one can obtain uniformly (absolutely) best approximations.

Minimisation problems in L^1 similar to the above have also been studied in the context of differential geometry and in questions related to the Yamabe problem. In particular, the second author, together with Schwetlick [MS], and subsequently Sakellaris [S1] considered the problem of minimising the scalar curvature of a Riemannian metric on a given manifold and in a given conformal class. When formulated in terms of differential operators, this gives rise to a functional similar in structure to (1.1). This work uses different boundary conditions, however, and no attempt is made to prove uniqueness or find second order absolute minimisers. Nevertheless, some of the tools in the proofs of the above results originate in the above quoted papers.

We close with some remarks about generalised solutions to the equations governing the "extremals" of Calculus of Variations in L^1 . In the scalar first order case, the theory of viscosity solutions of Crandall-Ishii-Lions (see [CIL, C, K8]) proved to be an apt framework within which the generally non-smooth solutions to the so-called Aronsson equation, which is a second order PDE, can be studied rigorously. However, viscosity solutions are of purely scalar nature and fail to work in either the vectorial or the higher order case (where we have either a second order system with discontinuous coefficients, or a fully nonlinear third order PDE). In the recent papers [K9, K10] a new theory of generalised solutions has been introduced which is based on a probabilistic representation of derivatives which do not exist classically and in the papers [AK, AyK, CKP, K11, K12, K13, KP, KP2, KP3] several results have been obtained in this framework. However, in the setting of the present paper, the particular structure of the problem at hand allows to prove directly existence of strong solutions to the fully nonlinear PDE (1.7).

2. Proofs

In this section we establish the proofs of Theorems 2 and 3 and of Corollary 4.

Proof of Theorem 2. Fix $W \in W_0^{2,1}(\Omega)$ with $W \geq 0$ on Ω . Since f is a harmonic function in $L^1(\Omega)$, it follows that

$$(2.1) \quad \Delta f = 0:$$

We set

$$g := f^{-1}(f \circ g):$$

By standard results on the nodal set of solutions to elliptic equations [HS] and the connectedness of Ω , it follows that if $f \not\equiv 0$ then Ω is a Lebesgue nullset and if $f \equiv 0$ then $\Omega = \emptyset$.

Let us first consider the case $f \not\equiv 0$. Note that u cannot vanish almost everywhere on Ω (as this would imply that $u \equiv 0$ by uniqueness of solutions of the Dirichlet problem for the Laplace equation). Therefore, we deduce that $f \not\equiv 0$ on a subset of positive Lebesgue measure in Ω . Hence, (2.1) implies that there exist measurable sets $\Omega_\pm$ with $L^n(\Omega_\pm) > 0$ such that

$$f > 0; \quad \text{a.e. on } \Omega_+;$$

where L^n

which establishes the desired bound. Finally, (2.5) is a consequence of (1.8), which gives $\|jF(x; \cdot)\|_{L^p(\Omega)} \leq c\|j\|$, and of (2.3), which gives $\|cj\|_{L^p(\Omega)} \leq c^2\|jF(x; \cdot)\|_{L^p(\Omega)}$.

We will construct the solutions to our problem by approximation with minimisers of L^p functionals. Therefore, we need to understand the behaviour of the latter.

Proposition 6. Suppose that $F \in C^2(\mathbb{R}^n)$ satisfies (1.8). Then for any $p > c^3 + 1$ there exists a minimiser u_p of E_p over the space $W_{u_0}^{2,p}(\Omega)$. Moreover, u_p is a weak solution to the Dirichlet problem for the Euler-Lagrange equation associated with the functional E_p :

$$\begin{aligned} & \Delta u_p - F_p(x; u_p) = 0; \quad \text{in } \Omega; \\ & u_p = u_0; \quad \text{on } \partial\Omega; \\ & Du_p = Du_0; \quad \text{on } \partial\Omega. \end{aligned}$$

Furthermore, there exist a (global) minimiser u_1 of the functional $E_1(x; \cdot)$ over the space $W_{u_0}^{2,1}(\Omega)$ such that $E_p(u_p) \rightarrow E_1(u_1; \cdot)$ as $p \rightarrow 1$. Also, there exists a subsequence $(p_k)_{k=1}^\infty$ such that

$$\begin{aligned} u_{p_k} & \rightarrow u_1; \quad \text{in } C^1(\bar{\Omega}); \\ D^2 u_{p_k} & \rightharpoonup^* D^2 u_1; \quad \text{in } L^q(\Omega; \mathbb{R}^{n \times n}); \text{ for all } q \in (1; 1^*); \end{aligned}$$

as $k \rightarrow \infty$.

Proof of Proposition 6. By (2.3) and (2.4) of Lemma 5, for $p > c^3 + 1$ the functional E_p is convex in $W_{u_0}^{2,p}(\Omega)$ and

$$E_p(u) \leq c(L^n(\Omega))^{1/p} \|u\|_{L^p(\Omega)};$$

for any $u \in W_{u_0}^{2,p}(\Omega)$. Since $u - u_0 \in W_0^{2,p}(\Omega)$, by the Calderon-Zygmund L^p estimates (e.g. [GT, GM]) and the Poincaré inequality we have a positive constant $c_0 = c_0(p)$

and hence $(u_p)_p$ is bounded in $L^k(\Omega)$. By the previous arguments and (2.6), we conclude that $(u_p)_p$ is bounded in $W_{u_0}^{2;k}(\Omega)$ for any $k \geq 2$. By a standard diagonal argument, weak compactness and the Morrey theorem, there exists

$$u_1 \in \bigcap_{1 < k < \infty} W_{u_0}^{2;1}(\Omega)$$

such that the desired convergences hold true along a subsequence as $p \rightarrow \infty$. When we pass to the limit as $p \rightarrow \infty$ in (2.7), the weak lower semicontinuity of the L^k norm implies

$$\|u_1\|_{L^k(\Omega)}^k \leq \frac{(L^n(\Omega))^{1-k}}{c} E_1(u_0; \Omega).$$

Letting $k \rightarrow \infty$ we obtain $u_1 \in L^1(\Omega)$. Thus, $u_1 \in W_{u_0}^{2;1}(\Omega)$, as desired. It remains to show the convergence of E_1 .

In other words, given any boundary condition $u_0 \in W^{2,1}_0(\Omega)$ such that u_0 is continuous up to the boundary, we can find another function w in the same space with the same boundary data, the Laplacian of which is as small as desired in a neighbourhood of the boundary.

Proof of Lemma 7. Since $\partial\Omega$ is $C^{1,1}$ -regular, the result of the appendix establishes that the distance function $\text{dist}(\cdot; \partial\Omega)$ belongs to $C^{1,1}(\overline{\Omega_{2r_0}})$ for some $r_0 > 0$ small enough.

Let d be an extension of $\text{dist}(\cdot; \partial\Omega)$ from Ω_{r_0} to $\mathbb{R}^n$ which is in the space $W^{2,1}_0(\mathbb{R}^n)$. Extend u_0 and d by zero on $\mathbb{R}^n \setminus \Omega$. Let $(\eta_\epsilon)_{\epsilon>0} \subset C_c^1(\mathbb{R}^n)$ be a standard mollifying family (as e.g. in [E]). We set

$$(2.9) \quad v := u_0 - \frac{d^2}{2} \quad u_0 :$$

Then $v \in W^{2,1}_0(\Omega)$ since $d = 0$ on $\partial\Omega$ and

$$Dv = Du_0 - d \frac{d}{2} D \quad u_0 + \quad u_0 Dd ;$$

$$D^2v = D^2u_0 - Dd \cdot Dd \quad u_0 - d \frac{d}{2} D^2 \quad u_0$$

$$+ D \quad u_0 \cdot Dd + Dd \cdot D \quad u_0 + \quad u_0 D^2d :$$

By using that

$$\text{tr}(Dd \cdot Dd) = |Dd|^2 = 1 \quad \text{on } \Omega_{r_0};$$

for $0 < r < r_0$ we deduce

$$\|k \cdot v\|_{L^1} \leq \epsilon \quad 4[(+)]TJ/F1 9.9626 T2281.1Tf 6.048 Td [15set$$

Proof of Theorem 3. Let us begin by setting

$$e_1 := \inf \{ E_1(u; \cdot) : u \in W_{u_0}^{2,1}(\cdot) \} :$$

If $e_1 = 0$, then everything in Theorem 3

r of the boundary ∂ . Since $u_p - w \in W_0^{2,p}(\Omega)$, it is an admissible test function and by (2.12), integration by parts gives

$$\int_{\Omega} f_p (u_p - w) = 0:$$

Hence, by the above together with (1.9), (1.8), (2.2) and (2.5), we obtain

$$\begin{aligned} \int_{\Omega} f_p w &= - \int_{\Omega} f_p u_p \\ &= - \frac{1}{e_p^{p-1}} \int_{\Omega} F(\cdot; u_p)^{p-2} F(\cdot; u_p) F(\cdot; u_p) u_p \\ &= - \frac{1}{e_p^{p-1}} \end{aligned}$$

for $x \in \Omega_{n-1}$. On any compact set $K \subset \Omega_{n-1}$, we have the uniform convergence $f_p \rightarrow f_1$ as $p \rightarrow \infty$, whereas F is uniformly bounded from above and below by (1.8). Hence, by restricting ourselves along the subsequence and letting $p \rightarrow \infty$ we obtain uniform convergence of the right-hand side of (2.17) to $\Delta u_1(x; e_1 \operatorname{sgn}(f_1))$ on K . But since we already know that $u_p \rightharpoonup^* u_1$ weakly in $L^2(\cdot)$, it follows that

$$(2.18) \quad \Delta u_1(x) = \Delta u_1(x; e_1 \operatorname{sgn}(f_1)) \quad ; \quad x \in K.$$

As a consequence,

$$\Delta u_1(x) = \Delta u_1(x; e_1 \operatorname{sgn}(f_1)) \quad ; \quad x \in K.$$

Now let us recall that $L^2(\Omega_{n-1}) = 0$. This is a consequence of general regularity results for nodal sets of solution to elliptic equations [HS]. The statement of item (I) then follows.

In order to prove item (II), we note that (2.18) implies that $u_1 \in C^2(\Omega_{n-1})$. The desired statement then follows from standard Schauder theory [GT].

For item (III), first recall the subsequential convergence

$$(u_2)_{p_k} \rightarrow u_1$$

The C^2 regularity of the distance function for a C^2 boundary ∂ is a classical result, see e.g. [GT, Appendix 14.6]. On the other hand, the case of C^1 regularity of the distance function when the boundary ∂ is C^1 holds under the extra hypothesis that the distance is realised at one point; see e.g. [F].

In order to prove the desired $C^{1,1}$ regularity of the distance function near ∂ when the boundary itself is a $C^{1,1}$ manifold (which we utilised in Lemma 7), we first note the following fact: suppose that $r > 0$ is such that $1/r$ is larger than the essential supremum of the curvature of ∂ . If $x \in \mathbb{R}^n$ and $y \in \partial$ with $|x - y| < r$

Now note that

$$Dd(x^0; t) = N(x^0)$$

whenever $t > 0$ is small enough. Hence if π denotes the projection onto $\mathbb{R}^{n-1} \times \{0\}$, then we obtain the formula

$$Dd = N \quad \text{near } x_0.$$

The right-hand side is of class $C^{0,1}$, and thus d is of class $C^{1,1}$ near x_0 . A compactness argument then proves the above statement.

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