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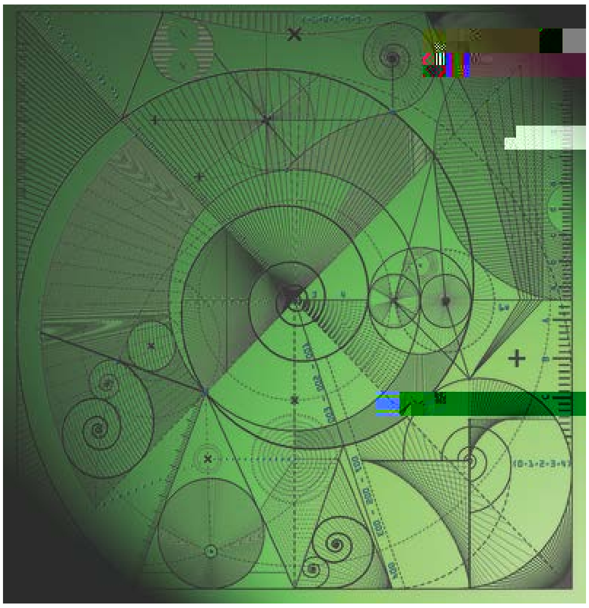

# D-SOLUTIONS TO THE SYSTEM OF VECTORIAL CALCULUS OF VARIATIONS IN $L^\infty$ VIA THE BAIRE CATEGORY METHOD FOR THE SINGULAR VALUES

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Abstract. For $H \in C^2(\mathbb{R}^{N\times n})$ and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$, consider the system

(1) $\Big(H_P \otimes H_P + H[H_P]^\perp H_{PP}\Big)(Du) : D^2u = 0.$

The PDE system (1) is associated to the supremal functional

(2) $E_\infty(u,\Omega') = \big\|H(Du)\big\|_{L^\infty(\Omega')}, \quad u \in W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N), \quad \Omega' \Subset \Omega,$

and first arose in recent work of the 2nd author as the analogue of the Euler-Lagrange equation. Herein we employ the Dacorogna-Marcellini Baire Category method to construct $\mathcal{D}$-solutions to the Dirichlet problem for (1), an apt notion of generalised solutions recently proposed for fully nonlinear systems. Our $\mathcal{D}$-solutions to (1) are $W^{1,\infty}$-submersions corresponding to "critical points" of (2) and are obtained without any convexity hypotheses. Along the way we establish a result of independent interest by proving existence of strong solutions to the singular value problem for general dimensions $n \neq N$.

## 1. Introduction

Let $H \in C^2(\mathbb{R}^{N\times n})$ be a given function and $\Omega \subseteq \mathbb{R}^n$ a given open set, $n, N \in \mathbb{N}$. In this paper we are interested in the problem of existence of appropriately defined generalised solutions with given Dirichlet boundary conditions to the second order PDE system

$$A_\infty u := \Big(H_P \otimes H_P + H[H_P]^\perp H_{PP}\Big)(Du) : D^2u = 0. \tag{1.1}$$

In the above, the subscript $P$ denotes the derivative of $H$ with respect to its matrix variable, while

$$Du(x) = \big(D_i u_\alpha(x)\big)_{i=1,\dots,n}^{\alpha=1,\dots,N} \in \mathbb{R}^{N\times n},$$
$$D^2u(x) = \big(D^2_{ij} u_\alpha(x)\big)_{i,j=1,\dots,n}^{\alpha=1,\dots,N} \in \mathbb{R}^{N\times n^2}_s,$$

denote respectively the gradient matrix and the hessian tensor of (smooth) maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$. The notation "$[H_P]^\perp$" symbolises the orthogonal projection on the orthogonal complement of the range of the linear map $H_P(P) : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[H_P(P)]^\perp := \mathrm{Proj}_{R(H_P(P))^\perp}. \tag{1.2}$$

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In index form, (1.1) reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\left(H_{P_{\alpha i}}(Du)\,H_{P_{\beta j}}(Du) + H(Du)\sum_{\gamma=1}^{N}[H_P(Du)]^{\perp}_{\alpha\gamma}\,H_{P_{\gamma i}P_{\beta j}}(Du)\right)D^2_{ij}u_\beta = 0,$$

$\alpha = 1,\ldots,N$. Our general notation is either self-explanatory or a convex combination of standard symbolisations as e.g. in [E, D, EG, DM2]. The system (1.1) is the analogue of the Euler-Lagrange equation when one considers nonstandard vectorial variational problems in the space $L^\infty$ for the supremal functional

$$E_\infty(u,\Omega') := \big\|H(Du)\big\|_{L^\infty(\Omega')},\quad u\in W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N),\quad \Omega'\Subset\Omega \tag{1.3}$$

and first arose in recent work of the second author ([K1]). Calculus of Variations in $L^\infty$, as the field is known today, was initiated by G. Aronsson in the 1960s who studied the scalar case $N=1$ quite systematically ([A1]-[A7]). Since then the area has been developed marvellously due to both the intrinsic mathematical interest and the importance for applications. In particular, the theory of Viscosity Solutions of Crandall-Ishii-Lions played a fundamental role in the study of the generally singular solutions to the scalar version of (1.1). When $N=1$, the respective single equation simplifies to

$$A_\infty u = H_P(Du)\otimes H_P(Du):D^2u = 0 \tag{1.4}$$

and is known as the "Aronsson equation". For a pedagogical introduction to the scalar case with numerous references see e.g. [K7, C]. See also [Pi] for a comparison between the Viscosity Solutions approach and the Baire Cathegory one.

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case of rankrk($Du$) 2, the appropriate minimality notion connecting (1.1) to (1.3) is not the obvious extension of Aronsson's scalar notion of Absolute Minimisers (see [K2, K10, K13]).

Perhaps the greatest difficulty associated to the study of (1.1) is that it is quasi-linear, non-divergence and non-monotone and all standard approaches in order to define generalised solutions based on integration-by-parts or on the maximum principle seem to fail. Motivated partly by the systems arising in Calculus of Variations in $L^1$, the second author has recently proposed in [K8] a new efficient theory of of Du in the Young measures valued into the sphere $\overline{\mathbb{R}}_s^{N \times n^2}$:

$$D^{1,h_m} Du \overset{*}{\rightharpoonup} D^2u, \quad \text{in } \mathscr{Y}\big(\Omega; \overline{\mathbb{R}}_s^{N \times n^2}\big), \quad \text{as } m \to \infty.$$

Our main ingredient for the solvability of (1.11) is a result of independent interest about the solvability of the following fully non-linear system, usually referred to as the prescribed singular value problem

$$
\begin{cases} \sigma_i(Du) = 1, & \text{a.e. on } \Omega,\ i = 1, \dots, n \wedge N, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{1.13}
$$

In (1.13), $\{\sigma_1(Du), \dots, \sigma_{n\wedge N}(Du)\}$ denotes the set of singular values of the matrix $Du$, namely the eigenvalues of the matrix $(Du^\top Du)^{1/2}$ in increasing order and we symbolise $n \wedge N = \min\{n, N\}$.

Systems of PDEs involving singular values, mostly related to non-convex problems in Calculus of Variations, have been considered by several authors (cf. for instance [Cr, BCR, DR, DPR]). In particular the problem of finding sufficient conditions on the boundary datum $g$ in order to get existence of solutions to problem (1.13) has been addressed, in the special case $n = N$, in [DM1, DM2, DT]. To the best of our knowledge no results are known for the case $n \neq N$. Accordingly, we establish the following result.

**Theorem 5.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Assume that $g \in A_{pw}(\overline{\Omega}; \mathbb{R}^N)$ is such that $\sigma_{n\wedge N}(Dg) < 1$, a.e. on $\Omega$. Then, there exists an infinite set of solutions $u \in W^{1,\infty}_g(\Omega; \mathbb{R}^N)$ to the system (1.13).

The proof of the previous theorem can be obtained as an application of the general existence theory for differential inclusion via the Baire Category method (cf. [PD]), in the same spirit as in the $N = n$ case. It relies on the characterisation of the rank-one convex envelope of the set of matrices

$$
E := \Big\{ Q \in \mathbb{R}^{N\times n} : \sigma_i(Q) = 1,\ i = 1, \dots, N \wedge n \Big\},
$$

which is

$$
\mathrm{Rco}E = \Big\{ Q \in \mathbb{R}^{N\times n} : \sigma_{N\wedge n}(Q) \leq 1 \Big\},
$$

as proved in Theorem 11.

In order to address the question of the existence of solutions to the problem (1.11) with $g \in W^{1,\infty}(\Omega; \mathbb{R}^N)$, some comments on the admissible regularity of the boundary datum in Theorem 5 are in order. Indeed, the piecewise affinity of the datum $g$ can be weakened to Lipschitz continuity if we restrict slightly the bound on the norm $\sigma_{n\wedge N}(Dg)$. This result, precisely stated in Corollary 6 that follows, is a simple consequence of the convexity of the rank-one convex hull of $E$ (cf. Theorem 11) and of the approximation result proved in [DM2, Corollary 10.21].

**Corollary 6.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Assume that $g \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ is such that for some $\delta > 0$ we have $\sigma_{n\wedge N}(Dg) \leq 1 - \delta$, a.e. on $\Omega$. Then there exists a infinite set of solutions $u \in W^{1,\infty}_g(\Omega; \mathbb{R}^N)$ to the system (1.13).

The rest of the paper is organised as follows. In the next section we recall some known results about Young measures valued into spheres. In Section 3 we provide the proof of the existence of solutions for the prescribed singular value problem and in the last section we prove existence and geometric properties of $\mathcal{D}$-solutions to the problem (1.10).

## 2. Young measures valued into spheres

Here we collect some basic material taken from [K8] which can be found in different guises and in greater generality e.g. in [CFV, FG]. Let $E \subseteq \mathbb{R}^n$ be measurable and consider the $L^1$ space of strongly measurable maps valued in the continuous functions over the sphere $\overline{\mathbb{R}}^{N\times n^2}_s$ (for details on these spaces see e.g. [Ed, FL]):

$$L^1\Big(E; C\big(\overline{\mathbb{R}}^{N\times n^2}_s\big)\Big).$$

This space consists of Carathéodory functions $\Phi : E \times \overline{\mathbb{R}}^{N\times n^2}_s \to \mathbb{R}$ for which

$$\|\Phi\|_{L^1(E;C(K))} = \int_E \big\|\Phi(x;\cdot)\big\|_{C(\overline{\mathbb{R}}^{N\times n^2}_s)}\, dx < \infty.$$

The dual of this (separable) Banach space is

$$L^\infty_{w^*}\Big(E; \mathcal{M}\big(\overline{\mathbb{R}}^{N\times n^2}_s\big)\Big) = \Big(L^1\Big(E; \mathcal{M}\big(\overline{\mathbb{R}}^{N\times n^2}_s\big)\Big)\Big)^*.$$

The dual space above consists of measure-valued maps $x \mapsto \vartheta(x)$ which are weakly* measurable, that is, for any fixed $\Phi \in C\big(\overline{\mathbb{R}}^{N\times n^2}_s\big)$, the function

$$E \ni x \mapsto \int_{\overline{\mathbb{R}}^{N\times n^2}_s} \Phi(X)\, d[\vartheta(x)](X)$$

Ed,

## 3. The prescribed singular value problem

In this section we prove Theorem 5 and Corollary 6. To this aim we start by recalling for the convenience of the reader some well known results about generalised convex hulls of sets of matrices (for further details we refer to the books [DM2] and [D]).

For $Q \in \mathbb{R}^{N\times n}$ we set

$$T(Q) := \left(Q; \operatorname{adj}_2 Q; \dots ; \operatorname{adj}_{N\wedge n} Q\right) \in \mathbb{R}^{\tau(N,n)},$$

where $\operatorname{adj}_s Q$ stands for the matrix of all $s\times s$ subdeterminants of the matrix $Q$, $1 \le s \le N\wedge n = \min\{N,n\}$ and

$$\tau(N,n) := \sum_{s=1}^{N\wedge n} \binom{N}{s}\binom{n}{s}, \qquad \binom{N}{s} = \frac{N!}{s!\,(N-s)!}.$$

**Definition 7.** Consider a function $f:\mathbb{R}^{N\times n}\to\mathbb{R}\cup\{+\infty\}$.

(1) $f$ is said to be *polyconvex* if there exists a convex function $g:\mathbb{R}^{\tau(N,n)}\to \mathbb{R}\cup\{+\infty\}$ such that $f(Q)=g(T(Q))$.

(2) $f$ is said to be *rank one convex* if

$$f\left(\lambda Q + (1-\lambda)R\right) \le \lambda f(Q) + (1-\lambda) f(R)$$

for every $\lambda\in[0,1]$ and every $Q,R\in\mathbb{R}^{N\times n}$ with $\operatorname{rk}(R-Q)=1$.

It is well known that if a function is polyconvex, then it is rank one convex. Next we recall the corresponding notions of convexity for sets.

**Definition 8.** Let $E$ be a subset of $\mathbb{R}^{N\times n}$. conwe) one 9 2 1] anRdorelf(Ra)sets4(1rywe)R RR

(1) We say that $E$ is *polyconvex* if there exists a convex s[(De nition)-383(8.3f28(v)2g8(e)-292(rp)51(olyc)

Analogous representations to (32

such that $(Z)_{n\wedge N;n\wedge N} = 1$ and the other entries of $Z$ are null. Then $Q + "Z$ would

## 4. D-solutions to the PDE system arising in vectorial Calculus of Variations in $L^1$

In this section we establish the proofs of Theorem 3 and of Proposition 4 by utilising Corollary 6

Now we complete the proof of the proposition. By our assumptions on H (see Theorem 3), for any u as above we have

$$H(Du) = h\big(Du\,Du^{\top}\big) = h\big(c^2 I\big)$$

and by splitting the identity as

$$c^2 I = [\,cI\,|\,O\,]\,[\,cI\,|\,O\,]^{\top}$$

where $[cI\,|\,O] \in \mathbb{R}^{N \times N} \quad \mathbb{R}^{N}$

for a.e. x 2 . Since D u 2 L[1] (

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