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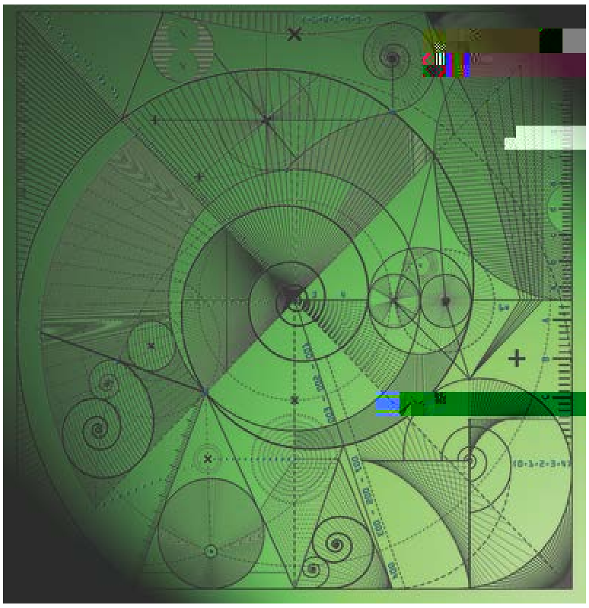

SOLUTIONS OF VECTORIAL HAMILTON-JACOBI EQUATIONS ARE RANK-ONE ABSOLUTE MINIMISERS IN L^1

NIKOS KATZOURAKIS

Abstract. Given the supremal functional $E_1(u;$

$\mathbb{R}^n$, the respective PDE is the 1 -Laplace equation

$$(1.3) \qquad {}_1 u := D u \quad Du : D^2 u = \sum_{i,j=1}^{n} D_i u \, D_j u \, D^2_{ij} u = 0 :$$

Despite the importance for applications and the deep analytical interest of the area, the vectorial case of N 2 remained largely unexplored until the early 2010s. In particular, not even the correct form of the respective PDE systems associated to L^1 variational problem was known. A notable exception is the early vectorial contributions [BJW1, BJW2] wherein (among other deep results) L^1 versions of lower semi-continuity and quasiconvexity were introduced and studied and the existence of Absolute Minimisers was established in some generality with H depending on u itself but for min f n; N g different sets of variations. In [K2] we proved the following variational characterisation in the class of classical solutions. A C^2 map $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is a solution to

$$\Delta_\infty u := Du \otimes Du : D^2u = 0 \tag{1.5}$$

if and only if it is a Rank-One Absolute Minimiser on Ω, namely when for all $D \Subset \Omega$, all scalar functions $g \in C^1_0(D)$ vanishing on ∂D and all directions $\xi \in \mathbb{R}^N$, u is a minimiser on D with respect to variations of the form $u + \xi g$ (Figure 1):

$$\|Du\|_{L^\infty(D)} \leq \|Du + \xi \otimes Dg\|_{L^\infty(D)}. \tag{1.6}$$

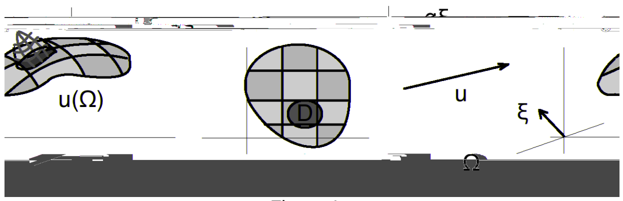

Figure 1.

Further, if $\mathrm{rk}(Du) \equiv$ const., u is a solution to

$$|Du|^2[Du]^\perp \Delta u = 0 \tag{1.7}$$

if and only if $u(\Omega)$ has ∞-Minimal Area, namely when for all $D \Subset \Omega$, all scalar functions $h \in C^1(\overline{D})$ (not vanishing on ∂D) and all vector fields $\nu \in C^1(D; \mathbb{R}^N)$ which are normal to $u(\Omega)$, u is a minimiser on D with respect to normal free variations of the form $u + h\nu$ (Figure 2):

$$\|Du\|_{L^\infty(D)} \leq \|Du + D(h\nu)\|_{L^\infty(D)}. \tag{1.8}$$

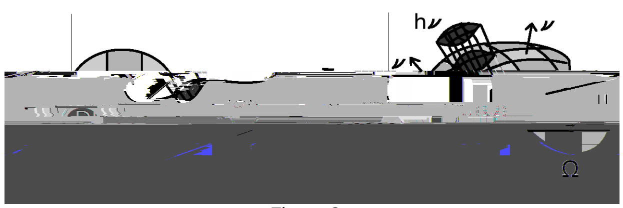

Figure 2.

We called a map ∞-Minimal with respect to functional $\|D(\cdot)\|_{L^\infty(\cdot)}$ when it is a Rank-One Absolute Minimiser on Ω and $u(\Omega)$ has ∞-Minimal Area.

Perhaps the greatest difficulty associated to (1.1) and (1.4

In this paper we consider the obvious generalisation of the rank-one minimality notion of (1.6) adapted to the functional (1.1). To this end, we identify a large class of rank-one Absolute Minimisers: for any $c \geq 0$, every solution $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ to the vectorial Hamilton-Jacobi equation

$$H\big(x, Du(x)\big) = c, \quad x \in \Omega, \tag{1.9}$$

actually is a rank-one absolute minimiser. Namely, for any $\Omega' \Subset \Omega$, any $\phi \in W_0^{1,\infty}(\Omega')$ and any $\xi \in \mathbb{R}^N$, we have

$$\operatorname*{ess\,sup}_{x \in \Omega'} H\big(x, Du(x)\big) \leq \operatorname*{ess\,sup}_{x \in \Omega'} H\big(x, Du(x) + \xi \otimes D\phi(x)\big).$$

For the above implication to be true we need the solutions to be in $C^1(\Omega; \mathbb{R}^N)$ and not just in $W^{1,\infty}_{loc}(\Omega; \mathbb{R}^N)$. This is not a technical difficulty: it is well known even in the scalar case that if we allow only for 1 non-differentiability point, strong solutions of the Eikonal equation $|Du| = 1$ are not absolutely minimising for the L^∞ norm of the gradient (e.g. the cone function $x \mapsto |x|$). However, due to regularity results which available in the scalar case, it suffices to assume everywhere differentiability (see [CEG, CC]).

Our only hypothesis imposed on H is that for any $x \in \Omega$ the partial function $H(x, \cdot) : \mathbb{R}^{N \times n} \to \mathbb{R}$ is rank-one level-convex. This means that for any $t \geq 0$, the sublevel sets $\{H(x, \cdot) \leq t\}$ are rank-one convex sets in $\mathbb{R}^{N \times n}$. A set $C \subseteq \mathbb{R}^{N \times n}$ is called rank-one convex when for any matrices $A, B \in C$ with $\mathrm{rk}(A - B) \leq 1$, the convex combination $\lambda A + (1-\lambda)B$ is in C for any $0 \leq \lambda \leq 1$. An equivalent way to phrase the rank-one level-convexity of $H(x, \cdot)$ is via the inequality

$$H\big(x, \lambda A + (1-\lambda)B\big) \leq \max\big\{H(x, A), H(x, B)\big\}, \qquad x,$$

Theorem 1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $n, N \in \mathbb{N}$ and $H : \Omega \times \mathbb{R}^{N \times n} \to [0, \infty)$ a continuous function, such that for all $x \in \Omega$, $P \mapsto H(x, P)$ is rank-one level-convex, that is

$$H(x, \cdot) \quad t \ \text{ is a rank-one convex in } \mathbb{R}^{N \times n}, \text{ for all } t \quad 0,\ x \in \Omega.$$

Let $u \in C^1(\Omega; \mathbb{R}^N)$ be a solution to the vectorial Hamilton-Jacobi PDE

$$H(\cdot, Du) = c \quad \text{on } \Omega,$$

for some $c \quad 0$. Then, u is a rank-one Absolute Minimiser of the functional

$$E_\infty(u, \Omega') = \operatorname*{ess\,sup}_{x \in \Omega'} H\big(x, Du(x)\big), \quad \Omega' \Subset \Omega, \ u \in W^{1,\infty}_{loc}(\Omega; \mathbb{R}^N).$$

In addition, the following marginally stronger result holds true: for any $\Omega' \Subset \Omega$, any $\ \in W^{1,\infty}_0(\Omega')$ and any $\ \in \mathbb{R}^N$, we have

$$E_\infty(u, \Omega') \quad \inf_{B \in \mathcal{B}(\ , \Omega')} E_\infty\big(u + \ , B\big)$$

where $\mathcal{B}(\ , \Omega')$

We illustrate the idea by assuming rst in addition that 2 $W^{1;1}_u$ (^{0}; R^N) \ C^1(^{0}; R^N). In this case, the point x is a critical point of (u) and we have D (u) (x) = 0. Hence,

$$D(\quad u)(x) = [\ ]^> D(\quad u)(x) + [\ ]^? D(\quad u)(x)$$
$$= \quad D \quad (\quad u) \ (x) + D \ [\ ]^? (\quad u) \ (x)$$
$$= 0$$

because $[]^?$ $[\ ]^?$ u on ^{0}. Thus,

$$E_1 (u; \ ^0) = c = H (x; Du(x))$$
$$= H \ x; \quad D(\ u)(x) + [\ ]^? Du(x)$$

and hence

(2.2)
$$E_1 (u; \ ^0) = H \ x; \quad D(\quad)(x) + [\ ]^? D \ (x)$$
$$= H (x; D \ (x))$$
$$\text{ess sup}_{y 2 B \ (x)} H (y; D \ (y))$$
$$= E_1 \quad ; B \ (x)$$

for any B (x) B (u); ^{0} , whence the conclusion ensues.

Now we return to the general case of 2 $W^{1;1}_u$ (^{0}; R^N). We extend by u on n ^{0} and consider the sets

(2.3)
$$_k := \begin{cases} x \ 2 \ \ ^0 : \text{dist}(x; @ \ ^0) > \dfrac{d_0}{k} \ ; & k \ 2 \ N; \\ ; ; & k = 0 ; \end{cases}$$

where $d_0 > 0$ is a constant small enough so that $_1$ 6= ; . We set

(2.4)
$$V_k := \ _k \ n \ \overline{_{k \ 1}}; \quad k \ 2 \ N$$

and consider a partition of unity ($_k)^1_{k=1}$ C^1_c (^{0}) over ^{0} so that 0) :(398 w 0 0 m 22.295 0 l S Q BT /

= k

j j = 1, we have

$$k \quad " \quad k_{C(\overline{V_l})} = \sup_{V_l} \; X^1_{k=1} \; k \quad "=k \,) \quad X^1_{k=1} \; k \; !$$

$$\text{(2.7)} \qquad \sup_{V_l} \; X^1_{k=1} \; k \quad "=k$$

$$= \sup_{V_l} \; X^{l+1}_{k=l\;\;1} \; k \quad "=k$$

$$3 \max_{k=l\;\;1;l;l+1} \quad "=k \quad {}_{C(\overline{\;\;0})};$$

whilst, for l = 1 we similarly have

$$\text{(2.8)} \qquad k \quad " \quad k_{C(\overline{V_1})} \quad 2 \max_{k=1;2} \quad "=k \quad {}_{C(\overline{\;\;0})}:$$

By the standard properties of molli ers, we have that the function

$$\text{(2.9)} \qquad !(t) := \sup_{0< \;<t} \quad {}_{C(\overline{\;\;0})}; \quad 0 < t < d_{\;0};$$

is an increasing continuous modulus of continuity with ! $(0^+) = 0$. By (2.7)-(2.9), we have that

$$\text{(2.10)} \qquad k \quad " \quad k_{C(\overline{V_l})} \quad \begin{cases} 3! \; \dfrac{"}{l \quad 1} \; ; & l \quad 2; \\ 2!\,("); & l = 1: \end{cases}$$

Since the C^1 regularity of " is obvious (because u by assumption is such and $[\;]^?$ $[\;]^?$ u), the claim has been established.

Note now that since u 2 $W^{1;1}_0$ (0; R^N), the set B (

our continuity assumption and the $W^{1;1}$ regularity of imply that there exists a positive increasing modulus of continuity $!_1$ with $!_1(0^+) = 0$ such that on the ball $B_{=2}(x_0)$ we have

$$
\begin{aligned}
H(\ ; D^{"}) = H\ \ ; \ \ \sum_{k=1}^{1}\ _k\ D(\ \ )^{"=k} \\
+ \sum_{k=1}^{1} D\ _k\ (\ \ )^{"=k} + [\ \ ]^? D \\
\text{(2.13)} \qquad H\ \ ; \ \ \sum_{k=1}^{1}\ _k\ D(\ \ )^{"=k} + [\ \ ]^? D \\
+\ !_1 \sum_{k=1}^{1} D\ _k\ (\ \ )^{"=k}\ ) \\
=: A + B:
\end{aligned}
$$

By further restricting $" < \ = 2$, we may arrange

$$
\text{(2.14)} \qquad \bigcup_{x \, 2 \, B_{=2}(x)} B_{"}(x) \quad B\ (x_0)
$$

and by (2.4)-(2.5), there exists $K(\ ) \ 2\ N$ such that

$$
\text{(2.15)} \qquad B\ (x_0) \quad \bigcup_{k=1;:::;K(\ )} \overline{V_k}:
$$

This implies that for any $x\ 2\ B\ (x_0)$,

$$
\text{(2.16)} \qquad \sum_{1}^{1}\ _k(x) = \sum_{1}^{K(\ )+1}\ _k(x) = 1
$$

forming a convex combination. We now recall for immediate use right below the following Jensen-like inequality for level-convex functions (see e.g. [BJW1, BJW2]): for any probability measure on an open set U R^n and any -measurable function $f : U \quad R^n\ !\ \ [0; 1\ )$, we have

$$
\text{(2.17)} \qquad \int_U f(x)\, d\ (x) \qquad \operatorname{ess\,sup}_{x \, 2 \, U}\ f(x)\ ;
$$

when $: R^n\ !\ \ R$ is any continuous level-convex function. Further, by our rank-one level-convexity assumption on H and if is as above, for any $x\ 2$ and $2\ R^N$ with $j\ j = 1$, the function

$$
\text{(2.18)} \qquad (\ p) := H\ \ x;\ \ \ p + [\ \ ]^? D\ \ (x)\ \ ; \quad p\ 2\ R^n;
$$

is level-convex. Indeed, given $p; q\ 2\ R^n$ and $t \quad 0$ with $(\ p);\ (\ q) \quad t$, we set

$$
\begin{cases}
P := \quad p + [\ \ ]^? D\ \ (x); \\
Q := \quad q + [\ \ ]^? D\ \ (x):
\end{cases}
$$

Then, $P \quad Q = \quad (p \quad q)$ and hence $\mathrm{rk}(P \quad Q) \quad 1$. Moreover, $H(x; P) = (\ p) \quad t$ and $H(x; Q) = (\ q) \quad t$ which gives

$$
\ p + (1 \quad)q\ = H\ \ x;\ P + (1 \quad)Q \qquad t
$$

for any 2 [0; 1], as desired.

Now, by using (2.4)-(2.5), (2.14)-(2.16) and the level-convexity of the function of (2.18), for any x 2 $B_{=2}(x_0)$ we have the estimate

$$A(x) = H\left(x;\ \left[\sum_{k=1}^{K(\ )+1} {}_k(x)\ \ D(\ \ )^{"=k}(x)\right] + [\ ]^? D\ (x)\right)$$

(2.19)
$$= \left(\sum_{k=1}^{K(\ )+1} {}_k(x)\ \ D(\ \ )^{"=k}(x)\right)$$

$$\max_{k=1;:::;K(\ )+1} D(\ \ )^{"=k}(x)$$

$$= \max_{k=1;:::;K(\ )+1} \left(\int_{B_{"=k}(x)} D(\ \ )(y)\ ^{"=k}(jx \ \ yj)\, dy\right) :$$

Since for any x and "; k, the map

$$:= \ ^{"=k}(jx \ \ j\ )\, L^n$$

is a probability measure on the ball $B_{"=k}(x)$ which is absolutely continuous with respect to the Lebesgue measure L^n, in view of (2.17), (2.19) gives

(2.20)
$$A(x) \quad \max_{k=1;:::;K(\ )+1} \left(\operatorname*{ess\,sup}_{y 2 B_{"=k}(x)} D(\ \ )(y)\right)$$

$$= \max_{k=1;:::;K(\ )+1} \left(\operatorname*{ess\,sup}_{y 2 B_{"=k}(x)} H\ \ x;\ \ D(\ \ )(y) + [\ ]^? D\ (x)\right)$$

$$\operatorname*{ess\,sup}_{y 2 B_{"}(x)} H\ \ x;\ \ D(\ \ )(y) + [\ ]^? D\ (x) :$$

By the continuity of H and Du, there is a positive increasing modulus of continuity $!_2$ with $!_2(0^+) = 0$ such that

$$\begin{cases} H(x; P) \quad H(y; Q) & !_2\ \ jx \ \ yj + jP \ \ Qj ; \\ Du(x) \quad Du(y) & !_2\ \ jx \ \ yj ; \end{cases}$$

for all x; y 2 B (x_0) and jPj; jQj k D $k_{L^1(\ \ ^0)}$ + 1. By using that $[\ ]^?$ $[\ ]^?$ u on ^{0}, (2.20) and the above give

$$A(x) \quad \operatorname*{ess\,sup}_{y 2 B_{"}(x)} H\ \ x;\ [\ ]^> D\ \ (y) + [\ ]^? D\ \ (x)$$

$$\operatorname*{ess\,sup}_{y 2 B_{"}(x)} \left(H\ \ y;\ [\ ]\right.$$

>88(of)]TJ/F11 9.9626 Tf 89.385 0 Td [(H)]TJ/F8 9.9626 Tf 11.961 0 Td [(and)-288(D)]TJ/F11 9.9626 Tf 26.532 0 T

By (2.14), (2.21) gives

$$
\begin{aligned}
A(x) &\quad \operatorname*{ess\,sup}_{y \in B_{\varepsilon}(x)} H\big(y, D\,(y)\big) + \sup_{y \in B_{\varepsilon}(x)} \omega_2\big(|x - y| + |Du(y) - Du(x)|\big) \\
&\quad \operatorname*{ess\,sup}_{y \in B\,(x_0)} H\big(y; D\,(y)\big) + \omega_2\big(\varepsilon + \omega_2(\varepsilon)\big) ;j
\end{aligned}
$$

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