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INDUCED RANDOM β -TRANSFORMATION

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Abstract. In this article we study the first return map defined on the switch region induced by

$$K : [0; \frac{1}{1}] \rightarrow [0; \frac{1}{1}] \text{ defined by}$$

$$K(x) = \begin{cases} (T_0 x); & \text{if } 0 \leq x < \frac{1}{2}; \\ (T_{-1} x); & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

k	k	k	k
1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$1+2^{-1}=2$
2	1:7549::	1:8393::	1:8546::
3	1:8668::	1:9276::	1:9305::
4	1:9332::	1:9660::	1:9666::
5	1:9672::	1:9836::	1:9837::

Figure 1. Tables of values for k , k and k

and let k denote the k -th multinacci number. Recall that the k -th multinacci number is the unique root of

$$x^{k+1} - x^k - x^{k-1} - \dots - x - 1 = 0$$

contained in $(1; 2)$.

Theorem 1.1. Let $\beta \in (\frac{1}{k}; \frac{1}{k}]$ for some $k \geq 2$; then for any $\alpha \in \mathbb{R}$ and $(j_i) \in \mathbb{R}^{\mathbb{N}}$ there exists $x \in S$ such that $r_i(\alpha; x) = j_i$, for $i = 1; 2; \dots$. Moreover, if $\beta \in (\frac{1}{k}; \frac{1}{k}]$ for all $k \geq 2$; then there exists $\alpha \in \mathbb{R}$ and $(j_i) \in \mathbb{R}^{\mathbb{N}}$ such that no $x \in S$ satisfies $r_i(\alpha; x) = j_i$, for $i = 1; 2; \dots$.

As we will see, the algebraic properties of k and k correspond naturally to conditions on the orbit of 1 and its reflection $\frac{1}{1-\beta} - 1$. These points determine completely the dynamics of the greedy map G and lazy map L respectively, and hence it is not surprising that these points play a crucial role in our situation as well. For values of β lying outside of the intervals $(\frac{1}{k}; \frac{1}{k}]$ it is natural to ask whether the following weaker condition is satisfied: given $(j_i) \in \mathbb{R}^{\mathbb{N}}$ does there exist $\alpha \in \mathbb{R}$ and $x \in S$ such that $r_i(\alpha; x) = j_i$ for $i = 1; 2; \dots$. Let k denote the unique root of the equation

$$2x^{k+1} - 4x^k + 1 = 0$$

contained in $(1; 2)$:

Theorem 1.2. Let $\beta \in (\frac{1}{k}; \frac{1}{k}]$ for some $k \geq 1$, then for any sequence $(j_i) \in \mathbb{R}^{\mathbb{N}}$ there exists $\alpha \in \mathbb{R}$ and $x \in S$ such that $r_i(\alpha; x) = j_i$ for $i = 1; 2; \dots$.

If β satisfies the hypothesis of Theorem 1.2 then the orbit of 1 and $\frac{1}{1-\beta} - 1$ satisfy a cross over property. This cross over property is sufficient to prove Theorem 1.2. Note that $k < k < k$ for each $k \geq 1$. We include a tables of values for k , k and k in Figure 1.

The second half of this paper is concerned with the maps J_0 and U_1 . Before we state our results it is necessary to make a definition. Given a closed interval $[a; b]$, we call a map $T : [a; b] \rightarrow [a; b]$ a generalized Lüroth series transformation (abbreviated to GLST) if there exists a countable set of bounded subintervals I_n ($I_n = (l_n; r_n)$; $l_n; r_n = 1$

This Lüroth expansion (a_n) can be seen to be generated by the map $T : [0, 1] \rightarrow [0, 1]$ where

$$T(x) = \begin{cases} n(n+1)x - n; & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}] \\ 0; & \text{if } x = 0 \end{cases}$$

GLST's were introduced in [2]. Our definition is slightly different to that appearing in this paper but all of the main results translate over into our context. Namely if $T : [a, b] \rightarrow [a, b]$ is a GLST then the normalised Lebesgue measure on $[a, b]$ is a T -invariant ergodic measure. Our main result for the maps $U_{\frac{1}{2}}$ and $U_{\frac{1}{3}}$ is the following theorem.

Theorem 1.3. There exists a set $M \subset (1, 2)$ of Hausdorff dimension 1 and Lebesgue measure zero such that:

- (1) If $\alpha \in M$ then both $U_{\frac{1}{2}}$ and $U_{\frac{1}{3}}$ are GLSTs.
- (2) If $\alpha \notin M$ then both $U_{\frac{1}{2}}$ and $U_{\frac{1}{3}}$ are not GLSTs.

What is more we can describe the set M explicitly.

Before we move on to our proofs of Theorems 1.1, 1.2 and 1.3 we provide a worked example. Namely we consider the case where $\alpha = \frac{1+\sqrt{5}}{2}$. This case exhibits some of the important features of our later proofs.

Example 1.1. When $\alpha = \frac{1+\sqrt{5}}{2}$ then $S = [\frac{1}{2}, 1]$. Let $C_j = \{i \in \mathbb{N} : i_1 = j\}$, $j = 0, 1$, then for any $i \in C_0$, $r_1(i; 1) = 1$, and $r_1(i; \frac{1}{2}) = 1$, while for any $i \in C_1$, we have $r_1(i; 1) = 1$ and $r_1(i; \frac{1}{2}) = 1$. If $x \in (\frac{1}{2}, 1)$, then $r_1(i; x) \geq 2$ for all $i \in \mathbb{N}$.

Let

$$(1) \quad B_i^0 := \{x \in S : U_{\frac{1}{2}}(x) = (T_1^{i-1} T_0)(x)\}$$

and

$$(2) \quad B_i^1 := \{x \in S : U_{\frac{1}{3}}(x) = (T_0^{i-1} T_1)(x)\}$$

where $i \geq 2$. A simple calculation shows that

$$(3) \quad B_i^0 = \bigcup_{n=2}^{i-1} \left(\frac{1}{n}, \frac{1}{n-1} \right] = (T_1^{i-1} T_0)(\frac{1}{2}, 1]$$

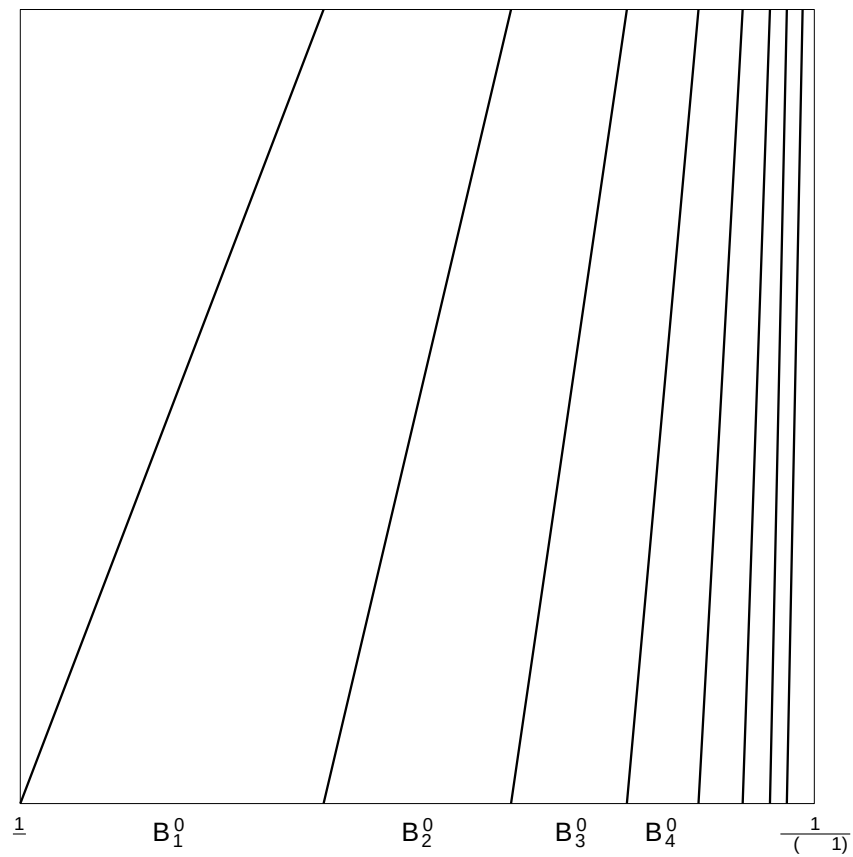


Figure 2. The graph of $U_{;0}$ when $\frac{p}{2} = \frac{1+\sqrt{5}}{2}$

Where in the above B_i^0 denotes the characteristic function on B_i^0 : Note that the result stated above holds with $r_j((0)^1; x)$ replaced with $r_j((1)^1; x)$:

By Theorem 1.1 we know that there exist 2 and $(j_i)_{i=1}^{2 \cdot R_{\frac{1+\sqrt{p}}{2}}^N}$ for which no x satisfies $r_i(!; x) = j_i$ for

following properties are important consequences of the above. First of all it is straightforward to see that for $2 \in (k; k]$ we have $R = f(k+1; k)$

every $x \in S$ is eventually mapped outside of S . This implies that equation (11) cannot hold and we have proved our result.

Combining Propositions 2.1, 2.2 and 2.3 we conclude Theorem 1.1.

2.2. Proof of Theorem 1.2. We now prove Theorem 1.2 our proof is similar to Theorem 1.1 in that we make use of a nested interval construction. However, with our proof we do not explicitly construct the desired I ; we can only show existence, as such our proof takes on an added degree of abstraction.

Let us start by examining the consequences of $2 \in (\beta_k; \beta_k]$ for some $k \geq 1$: For $k \geq 2$ we ignore the intervals $(\beta_k; \beta_k]$ as their proof is covered by Theorem 1.1. For $k=1$ in the remaining parameter space the following inclusions hold

$$(12) \quad (T_1^k - T_0) \frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2} \left(\frac{1}{1 - \beta_k} \right)^i$$

numbers, thus M has Hausdorff dimension 1 and Lebesgue measure zero. It is worth noting that if $2 \leq f \leq 2(1; 2) : U_{\beta; 0}(\frac{1}{2}) \leq f \leq \frac{1}{(f-1)}$ then $\text{card}(M) = \infty$. The important observation to make from the definition of M is that the following statement holds

$2 \in M \setminus \emptyset$ and $\frac{1}{f}$ and $\frac{1}{(f-1)}$ are never mapped into the interior of S :

This property will be sufficient to prove that both $U_{\beta; 0}$ and $U_{\beta; 1}$ are GLSTs. Our proof of Theorem 1.3 is split over the following propositions.

Proposition 3.1. If $f \notin M$ then $U_{\beta; 0}$ and $U_{\beta; 1}$ are not GLSTs.

Proof. If $f =$

We begin with the most simple case, we assume that $U_0(x) = (T_1^{n_1} T_0)(x)$, i.e. $T_0(x) \in D_{n_1}$. Importantly, since $n_1 \geq M$ we know that $1 \notin D_{n_1}^0$. Thus $T_0(S) \setminus D_{n_1} = [1; \frac{1}{1}] \setminus D_{n_1} = D_{n_1}$. Therefore $T_0^{-1}(D_{n_1}) \subseteq S$ and any y in this interval satisfies $U_0(y) = (T_1^{n_1} T_0)(y)$. This implies that

$$(21) \quad \forall y \in S : U_0(x) = (T_1^{n_1} T_0)(y)g = (T_1^{n_1} T_0)^{-1}(S):$$

It remains to show that equation (20) holds in the general case. Obviously

$$(22) \quad \forall y \in S : U_0(y) = (T_{l_i}^{n_i} T_1^{n_1} T_0)(y)g = (T_{l_i}^{n_i} T_1^{n_1} T_0)^{-1}(S):$$

So we have to show that the opposite inclusion holds, for this we examine the formula for U_0 more closely. We assume $U_0(x) = (T_{l_i}^{n_i} T_1^{n_1} T_0)(x)$ for some $i \geq 2$. Since $i \geq 2$ we have $T_0(x)$ is contained in a connected component of $[1; \frac{1}{1}] \cap [\frac{1}{n+1}; 1] \cap D_n$. Let us denote this interval by I_1 . We also let

$$E := T_0^{-n} \left[\frac{1}{1} \right]; T_0^{-n} \left[\frac{1}{(1)} \right]; T_1^{-n} \left[\frac{1}{1} \right]; T_1^{-n} \left[\frac{1}{(1)} \right]; G^n(1); G^n \left[\frac{1}{1} \right]; \dots : n \geq 0 :$$

Here G is the greedy map defined earlier. Since $n_1 \geq M$ no element of E is contained in the interior of a C_n , a D_n , or S .

Importantly $I_1 = (a_1; b_1)$ where $a_1, b_1 \in E$. In this case either

$$(a_1; b_1) = \left[1; T_1^{-n_1} \left[\frac{1}{1} \right] \right] \text{ or } (a_1; b_1) = \left[T_1^{-(n_1-1)} \left[\frac{1}{(1)} \right]; T_1^{-n_1} \left[\frac{1}{1} \right] \right] :$$

Therefore

$$T_1^k(I_1) \setminus S = \emptyset; \text{ for } 1 \leq k \leq n_1 - 1 \text{ and } T_1^{n_1}(I_1) \subseteq \left[\frac{2}{1}; \frac{1}{1} \right] :$$

The endpoints of $T_1^{n_1}(I_1)$ are elements of E and are therefore not contained in the interior of any C_n . Either $(T_1^{n_1} T_0)(x) \in C_n$ for some n or maybe $(T_1^{n_1} T_0)(x) \in T_1^{n_1}(I_1) \cap [\frac{1}{n+1}; 1] \cap C_n$. If $(T_1^{n_1} T_0)(x) \in T_1^{n_1}(I_1) \cap [\frac{1}{n+1}; 1] \cap C_n$ then let the connected component it is contained in be denoted by I_2 . Let $I_2 = (a_2; b_2)$ then again $a_2, b_2 \in E$. In which case

$$(23) \quad T_0^k(I_2) \setminus S = \emptyset; \text{ for } 1 \leq k \leq n_2 - 1 \text{ and } T_0^{n_2}(I_2) \subseteq \left[\frac{1}{(1)}; 1 \right]$$

The endpoints of $T_0^{n_2}(I_2)$ are again contained in E and therefore do not intersect the interior of any D_n . The point x has either been mapped into aD_n or is contained in a connected component of $T_0^{n_2}(I_2) \cap [\frac{1}{n+1}; 1] \cap D_n$. If it is contained in a connected component of $T_0^{n_2}(I_2) \cap [\frac{1}{n+1}; 1] \cap D_n$ then we repeat the previous steps. Eventually x is mapped into either C_{n_i} or D_{n_i} and our algorithm terminates. Without loss of generality we assume x is eventually mapped into D_{n_i} . The above algorithm yields a finite sequence of intervals $(I_j)_{j=1}^i$ which satisfy the following properties:

- (1) $I_1 \subseteq T_0(S)$;
- (2) For $1 \leq j \leq i-1$

$$T_{l_j}^k(I_{n_j}) \setminus S = \emptyset; \text{ for } 1 \leq k \leq n_j$$
- (3) For $1 \leq j \leq i-2$ we have $l_{j+1} = T_{l_j}^{n_j}(I_j)$
- (4)
$$D_{n_i} = T_{l_i}^{n_i-1}(I_{i-1}):$$

Where in the above $l_j = 0$ if j is even and $l_j = 1$ if j is odd. These properties have the following consequences:

- (5) $(T_{l_i}^{n_i})^{-1}(S) \subseteq I_{i-1}$
- (6) For $1 \leq j \leq i-1$

$$(T_{l_i}^{n_i} T_{l_j}^k)^{-1}(S) \setminus S = \emptyset; \text{ for } 1 \leq k \leq n_j :$$
- (7) For $1 \leq j \leq i-1$

$$(T_{l_i}^{n_i} T_{l_j}^{n_j})^{-1}(S) \subseteq I_{n_j}$$
- (8)
$$(T_{l_i}^{n_i} T_1^{n_1} T_0)^{-1}(S) \subseteq S:$$

Property (8) states that $(T_{\beta_i}^{n_i} \circ T_1^{n_1} \circ T_0)^{-1}(S) \subset S$. Moreover, properties (5), (6) and (7) imply that every $y \in (T_{\beta_i}^{n_i} \circ T_1^{n_1} \circ T_0)^{-1}(S)$ satisfies $U_{\beta_i}(y) = (T_{\beta_i}^{n_i} \circ T_1^{n_1} \circ T_0)(y)$.