While the literature on Tre tz nite elements for time-harmonic wave propagation problems is nowadays quite developed (see e.g. [4, 6, 11, 14, 23, 29, 32] for di erent approaches using Tre tz-type basis functions and e.g. [2, 15, 19, 20] for theoretical analyse 10 in Remark 2.1 (in one space dimension) and again in Remark 3.4 (in any space dimension).

Given a space domain = $(x_L; x_R)$ and a time domain $I = (0; T)$, we set $Q :=$ I.
We denote by $n_Q = (n_Q^x; n_Q^t)$ the outward pointing unit normal vector on @Q

We assume the electric permittivity " = "(x

3.1 Mesh and DG notation

We introduce a mesh $\mathcal{T}_h$ on Q, such that its elements are rectangles with sides parallel to the space and time axes, and all the discontinuities of the parameters ε and lie on interelement boundaries (note that the method described in this paper can be generalised to allow discontinuities lying inside the elements as in [25]). The mesh may have hanging nodes.

We denote with $\mathcal{F}_h = \bigcup_{K \in \mathcal{T}_h} \partial K$ the mesh skeleton and its subsets:

$$\begin{aligned}
\mathcal{F}_h^{hor} &:= \text{the union of the internal horizontal element sides } (t = \text{constant}); \\
\mathcal{F}_h^{ver} &:= \text{the union of the internal vertical element sides } (x = \text{constant}); \\
\mathcal{F}_h^{0} &:= [x_L, x_R] \times \{0\}; \\
\mathcal{F}_h^{T} &:= [x_L, x_R] \times \{T\}; \\
\mathcal{F}_h^{L} &:= \{x_L\} \times [0, T]; \\
\mathcal{F}_h^{R} &:= \{x_R\} \times [0, T].
\end{aligned}$$

We define the following broken Sobolev space:

$$H^1(\mathcal{T}_h) := \qquad (12)$$

where

a

$$a^{\text{wave}}_{\text{TDG}}(v_{hp},\sigma_{hp};w,\tau) = \ell^{\text{wave}}_{\text{TDG}}(w,\tau) \qquad \forall (w,\tau) \in \mathbf{V}_p(\mathcal{T}_h), \tag{11}$$

with

$$\begin{aligned}
a^{\text{wave}}_{\text{TDG}}(v_{hp},\sigma_{hp};w,\tau) &:= \int_{\mathcal{F}_h^{hor}} \left(c^{-2} v_{hp}^- [\![w]\!]_t + \sigma_{hp}^- \cdot [\![\tau]\!]_t \right) dx + \int_{\mathcal{F}_h^{T}} \left(c^{-2} v_{hp} w + \sigma_{hp} \cdot \tau \right) dx \\
&\quad + \int_{\mathcal{F}_h^{ver}} \left(\{\!\{v_{hp}\}\!\} [\![\tau]\!]_N + \{\!\{\sigma_{hp}\}\!\} \cdot [\![w]\!]_N + \alpha [\![v_{hp}]\!]_N \cdot [\![w]\!]_N + \beta [\![\sigma_{hp}]\!]_N [\![\tau]\!]_N \right) dS \\
&\quad + \int_{\mathcal{F}_h^{\partial}} \left(\sigma_{hp} \cdot \mathbf{n} + \alpha v_{hp} \right) w \, dS; \\
\ell^{\text{wave}}_{\text{TDG}}(w,\tau) &:= \int_{\mathcal{F}_h^{0}} \left(c^{-2} v_0 w + \sigma_0 \cdot \tau \right) dx + \int_{\mathcal{F}_h^{\partial}} g \left(\alpha w - \tau \cdot \mathbf{n} \right) dS;
\end{aligned}$$

where $\sigma_0 = -\nabla U(\cdot;0)$ and $v_0 = \frac{\partial U}{\partial t}(\cdot;0)$ are (given) initial data. Here, the jumps are defined as follows: $[\![w]\!]_t := (w^- - w^+)$ and $[\![\tau]\!]_t := (\tau^- - \tau^+)$ on horizontal faces, $[\![w]\!]_N := w_{|K_1} \mathbf{n}^x_{K_1} + w_{|K_2} \mathbf{n}^x_{K_2}$ and $[\![\tau]\!]_N := \tau_{|K_1} \cdot \mathbf{n}^x_{K_1} + \tau_{|K_2} \cdot \mathbf{n}^x_{K_2}$ on vertical faces.

In particular, the bilinear form $a_{TDG}(\ ;\ )$ is coercive in the space $T(T_h)$ with respect to the DG norm, with coercivity constant equal to 1.

Proof. Using the elementwise integration by parts in time and space

$$\sum_{K\in T_h}\iint_K \frac{\partial F}{\partial t}\,dx\,dt = \int_{F_h^{hor}} [\![F]\!]_t\,dx + \int_{F_h^{T}} F\,dx \quad \int_{F_h^{0}} F\,dx$$
$$\sum_{K\in T_h}\iint_K \frac{\partial F}{\partial x}\,dx\,dt = \int_{F_h^{ver}} [\![F]\!]_x\,dt + \int_{F_h^{R}} F\,dt \quad \int_{F_h^{L}} F\,dt \qquad \forall F\in W^{1,1}(T_h); \tag{14}$$

and the jump identity

$$v\ [\![v]\!]_t \quad \frac{1}{2}[\![v^2]\!]_t = \frac{1}{2}[\![v]\!]_t^2 \qquad \text{on } F_h^{hor}; \quad \forall v\in H^1(T_h); \tag{15}$$

we obtain the identity in the assertion:

$$a_{DG}(v_E; v_H; v_E; v_H) \overset{(7)}{=} \sum_{K\in T_h}\iint_K \left(\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon v_E^2 + \ v_H^2\right) + \frac{\partial}{\partial x}(v_E v_H)\right) dx\,dt$$
$$+ \int_{F_h^{hor}} (\epsilon v_E\ [\![v_E]\!]_t + \ v$$

Proof. To prove uniqueness, assume that $E_L = E_R = E_0 = H_0 = 0$. Proposition 4.2 implies $E_{hp} = H_{hp} = 0$. Existence follows from uniqueness. For (16), the triangle inequality gives

$$jjj(E;H) \quad (E_{hp};H_{hp})jjj_{DG} \quad jjj \quad (E;H) \quad (v_E;v_H)jjj_{DG} + jjj(E_{hp};H_{hp}) \quad (v_E;v_H)jjj_{DG} \qquad (17)$$

for all $(v_E;v_H)$ 2 $V_p(T_h)$. Since $(E_{hp};H_{hp}) \quad (v_E;v_H)$ 2 $V_p(T_h) \quad T(T_h)$, Proposition 4.2, consistency (which follows by construction and from the consistency of the numerical uxes), and Proposition 4.3 give

$$jjj(E_{hp};H_{hp}) \quad (v_E;v_H)jjj^2_{DG} = a_{TDG}(E \quad v_E;H \quad v_H;E_{hp} \quad v_E;H_{hp} \quad v_H$$
$$2jjj(E;H) \quad (v_E;v_H)jjj_{DG}$$

for $\ ;\ \in L^2(Q)$. More precisely, we will need a bound on the L^2 norm of the traces of v_E and v_H on horizontal and vertical segments in terms of the $L^2(Q)$ norm of $(\ ;\ )$:

$$
\begin{aligned}
&\left\| \varepsilon^{1/2} v_E \right\|^2_{L^2(F_h^{hor} \cup F_h^{T})} + \left\| {}^{1/2} v_H \right\|^2_{L^2(F_h^{hor} \cup F_h^{T})} \\
+ &\left\| {}^{1/2} v_E \right\|^2_{L^2(F_h^{ver})} + \left\| {}^{1/2} v_H \right\|^2_{L^2(F_h^{ver} \cup F_h^{L} \cup F_h^{R})} \\
&\quad \le C^2_{stab} \left(\left\| \varepsilon^{1/2} \right\|^2_{L^2(Q)} + \left\| {}^{1/2} \right\|^2_{L^2(Q)} \right) \qquad \forall (\ ;\ ) \in L^2(Q)^2; \qquad (19)
\end{aligned}
$$

for some $C_{stab} > 0$. We have inserted the numerical flux parameters within the third and fourth term on the left-hand side of (19) because this is what we need in the proof of Proposition 4.7 below; then the constant C_{stab} will also depend on and .

Proposition 4.7. Assume that the estimate (19) holds true for $(v_E; v_H)$ solution of problem (18). Then, for any Trefftz function $(w_E; w_H) \in T(T_h)$, the $L^2(Q)$ norm is bounded by the DG norm:

$$
\left(\left\| {}^{1/2} w_E \right\|^2_{L^2(Q)} + \left\| \varepsilon\, {}^{1/2} w_H \right\|^2_{L^2(Q)} \right)^{1/2} \le \sqrt{2}\, C_{stab} \, |||(w_E; w_H)|||_{DG};
$$

with C_{stab} as in (19).

Proof. Let $(v_E; v_H)$ be the solution of the auxiliary problem (18 we obtain the desired estimate. ☐

Recalling that the error $((E \quad E_{hp});(H \quad H_{hp}))$ 2 $T(T_h)$, and combining Proposition 4.7 and the quasi-optimality in DG norm proved in Theoremosition which in turns implies

$$E(t) = E(0) + \iint_{(0;t)} v_E + v_H \, d$$

$$\stackrel{(18)}{=} \frac{2A}{h^x} \int\int_K \quad {}^{1}v_E^2 + " \ {}^{1}v_H^2 + 2(x \quad x_K) \quad \frac{@}{@t}(v_E v_H) + \quad {}^{1}v_E \quad + " \ {}^{1}v_H \qquad dx\,dt$$

$$\frac{2A}{h^x} \int\int_K \quad 2 \quad {}^{1}v_E^2 + 2" \ {}^{1}v_H^2 + (x \quad x_K)^2 \quad {}^{1} \ {}^{2} + " \ {}^{1} \ {}^{2} \qquad dx\,dt$$

$$+ \frac{2A}{h^x} \int_{@K^{SN}} 2j(x \quad x_K) v_E v_H j \ dx$$

Using $2v_E v_H \quad (\ v_E^2 + {}^{1}v_H^2)$ with weight $\quad = "c = (\ c)^{\ 1}$, we have the bound

$$\sum_{K \, 2T_h} \quad {}^{1=2}v_E \quad {}^{2}_{L^2(@K^{WE})} + \quad {}^{1=2}v_H \quad {}^{2}_{L^2(@K^{WE})}$$

$$= \frac{4A}{h^x} \quad {}^{1=2}v_E \quad {}^{2}_{L^2(Q)} + \frac{4A}{h^x} \ " \ {}^{1=2}v_H \quad {}^{2}_{L^2(Q)} + \frac{Ah^x}{2} \quad {}^{1=2} \quad {}^{2}_{L^2(Q)} + \frac{Ah^x}{2} \ " \ {}^{1=2} \quad {}^{2}_{L^2(Q)}$$

$$+ 2A \int_{F_h^{hor} [F_h^T} c("v_E^2 + \ v_H^2)\,dx$$

$$\stackrel{(27);(26)}{} \ A \quad \frac{4T^2}{h^x} c_1^4 + \frac{h^x}{2} c_1^2 + 2c_1^3 \ N_{hor} \, e^T \qquad "^{1=2} \quad {}^{2}_{L^2(Q)} + \quad {}^{1=2} \quad {}^{2}_{L^2(Q)} \ ;$$

recalling that $c_1 = kck_{L^1(Q)}$.

This, together with (26), gives the bound (19) with constant C^2_{stab} as in (21). □

In case of a tensor product mesh with all elements having horizontal edges of length h^x and vertical edges of length $h^t = h^x = c$, the constant C_{stab} is proportional to $(h^x)^{1=2}$. We stress that we cannot expect a bound like (19) with C_{stab} independent of the meshwidth: indeed if the mesh is refined, say, uniformly, while the term in the brackets in the right-hand side of (19) is not modified, the left-hand side grows (consider e.g. the simple case $\ = \ ,$ $\ = 0,\ v_E = 0,\ v_H = t$).

One could attempt to derive the stability bound (19) by controlling with $(\ ;\ )$ either the $H^1(Q)$ or the Lx(singgth

$\frac{1}{2}^{Z}$

for all $(E;H);(v_E;v_H)$ 2 $T(T$

and denote their length by

$$h_D := x_1 - x_0 + c(t_1 - t_0). \tag{32}$$

Their relevance is the following: the restriction to D of the solution of a Maxwell initial value problem posed in $\mathbb{R} \times \mathbb{R}^+$ will depend only on the initial conditions posed on ${}^-_D \cup {}^+_D$; see Figure 1.

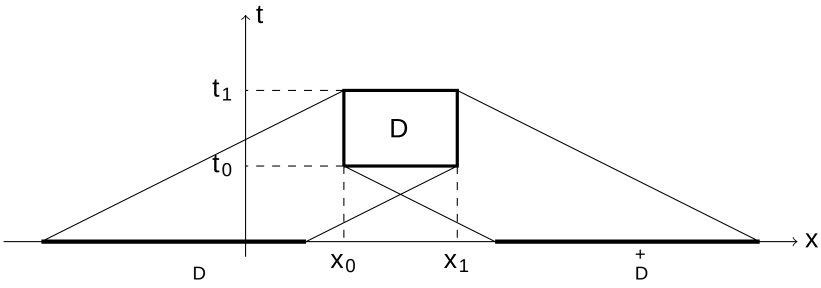

Figure 1: The intervals ${}^\pm_D$ in (31) corresponding to the space–time rectangle D.

Let $\alpha = (\alpha_x, \alpha_t) \in \mathbb{N}_0^2$ be a multi-index; for a sufficiently smooth function v, we define its anisotropic derivative $D_c^\alpha v$ as

$$D_c^\alpha v(x,t) := \frac{1}{c^{\alpha_t}} D^\alpha v(x,t) = \frac{1}{c^{\alpha_t}} \frac{\partial^{|\alpha|} v(x,t)}{\partial x^{\alpha_x} \partial t^{\alpha_t}}.$$

Note that, if u and w satisfy

$$u(x,t) = u_0(x - ct), \qquad w(x,t) = w_0(x + ct), \tag{33}$$

with u_0 and w_0 defined in ${}^-_D$ and ${}^+_D$, respectively, then

$$D_c^\alpha u(x,t) = (-1)^{\alpha_t} u_0^{(|\alpha|)}(x - ct),$$
$$D_c^\alpha w(x,t) = w_0^{(|\alpha|)}(x + ct).$$

We define the Sobolev spaces $W_c^{j,1}(D)$ and $H_c^j(D)$ as the spaces of functions whose D_c^α derivatives, $0 \le |\alpha| \le j$, belong to $L^1(D)$ and $L^2(D)$, respectively. We define the following seminorms:

$$|v|_{W_c^{j,1}(D)} := \sup_{|\alpha|=j} \|D_c^\alpha v\|_{L^1(D)}, \qquad |v|^2_{H_c^j(D)} := \sum_{|\alpha|=j} \|D_c^\alpha v\|^2_{L^2(D)}.$$

Note that for j W(D)(and L D) an

A similar result holds for $w(x;t) = w_0(x + ct)$, with $^{+}_{D}$ instead of $_{D}$.

Proof. For the $W_c^{j;1}$ (D)-seminorms in (i), we have

$$juj_{W_c^{j;1}(D)} = \sup_{j\;\;j=j} kD_c\,uk_{L^1(D)} = \sup_{j\;\;j=j} \; u_0^{(j\;\;j)}(x \quad ct) \;_{L^1(D)} = ju_0j_{W^{j;1}(\;_D)} :$$

For the bound of $ju_0j^2_{H^j(\;_D)}$ in (ii), we have

$$ju_0j^2_{H^j(\;_D)} = \int_{D} \; u_0^{(j)}(z) \;^2 dz \qquad _D \sup_{z2\;_D} \; u_0^{(j)}(z) \;^2 = \quad _D \sup_{j\;\;j=j} \; \sup_{(x;t)2D} jD_c\,u(x;t)j^2$$

$$= h_D \; juj^2_{W_c^{j;1}(D)} :$$

Considd [(2)1.169416 0 Td [(1)996864 Tf 13.6801 0 Td [())2.56329].15739]TJ /R14880-1.4,75T618 (

This suggests a construction of discrete subspaces of $T(D)$: given $p \in \mathbb{N}_0$ and two sets of $p+1$ linearly independent functions $\Phi^- = \{\varphi_0, \dots, \varphi_p\} \subset C^m(\Gamma_D^-)$ and $\Phi^+ = \{\varphi_0^+, \dots, \varphi_p^+\} \subset C^m(\overline{\Gamma_D^+})$ we define the space

$$V_p(D) := \operatorname{span}\left\{ \left(\frac{\varphi_0(x-ct)}{2\epsilon^{1/2}}, \frac{\varphi_0(x-ct)}{2\mu^{1/2}}\right), \dots, \left(\frac{\varphi_p(x-ct)}{2\epsilon^{1/2}}, \frac{\varphi_p(x-ct)}{2\mu^{1/2}}\right), \left(\frac{\varphi_0^+(x+ct)}{2\epsilon^{1/2}}, -\frac{\varphi_0^+(x+ct)}{2\mu^{1/2}}\right), \dots, \left(\frac{\varphi_p^+(x+ct)}{2\epsilon^{1/2}}, -\frac{\varphi_p^+(x+ct)}{2\mu^{1/2}}\right) \right\}$$

which is a subspace of $T(D) \cap C^m(\overline{D})^2$ with dimension $2(p+1)$.

By virtue of Proposition 5.1, the approximation properties of $V_p(D)$ in $T(D)$ only depend on the approximation properties of the one-dimensional functions $\{\varphi_0, \dots, \varphi_p\}$: for all $(E, H) \in T(D) \cap W_c^{j,1}(D)^2$, defining u, w, u_0 and w_0 from (E, H) using (35) and (33),

$$\begin{aligned}
&\inf_{(E_{hp}, H_{hp}) \in V_p(D)} \left\| \epsilon^{1/2}(E - E_{hp}) \right\|_{W_c^{j,1}(D)} + \left\| \mu^{1/2}(H - H_{hp}) \right\|_{W_c^{j,1}(D)} \qquad (36) \\
&\overset{(35)}{=} \inf_{\substack{u_{0,p} \in \operatorname{span}\{\varphi_0, \dots, \varphi_p\}, \\ w_{0,p} \in \operatorname{span}\{\varphi_0^+, \dots, \varphi_p^+\}}} \frac{1}{2} \left| u(x,t) - u_{0,p}(x-ct) + w(x,t) - w_{0,p}(x+ct) \right|_{W_c^{j,1}(D)} \\
&\qquad + \frac{1}{2} \left| u(x,t) - u_{0,p}(x-ct) - w(x,t) + w_{0,p}(x+ct) \right|_{W_c^{j,1}(D)} \\
&\overset{\text{Prop. 5.1 (i)}}{\le} \inf_{u_{0,p} \in \operatorname{span}\{\varphi_0, \dots, \varphi_p\}} \left| u_0 - u_{0,p} \right|_{W^{j,1}(\Gamma_D^-)} + \inf_{w_{0,p} \in \operatorname{span}\{\varphi_0^+, \dots, \varphi_p^+\}} \left| w_0 - w_{0,p} \right|_{W^{j,1}(\Gamma_D^+)};
\end{aligned}$$

while for all $(E, H) \in T(D) \cap H_c^j(D)^2$

$$\inf_{(E_{hp}, H_{hp}) \in V_p(D)} \left\| \epsilon^{1/2}(E - E_{hp}) \right\|_{D)} \quad {}^{D2}(H \quad E_{hp}) \; w_{D)} \qquad (36)$$

Following the second route, we prove simple approximation bounds in $H_c^1(Q)$, for a general rectangle

6 Convergence rates

We now derive the convergence rates of the Trefftz-DG method with polynomial approximating spaces

$$V_p(T_h) = \ (v_E; v_H) \in L^2(Q)^2 : \ (v_E; v_H)|_K \ \text{are as in (38) with} \ p = p_K \ : \qquad (43)$$

The two main ingredients are the quasi-optimality results investigated in section 4 and the best approximation bounds proved in section 5. To combine them, we need to control the DG^+ norm (12) of the approximation error with its $H^1_c(Q)$ norm, weighted with " and , to be able to use the bound (40). To this purpose, we define the following parameters:

$_K$:= max "

If the bound (19) holds true for the solution of the auxiliary problem (18), we also have the following bound in $L^2(Q)$:

$$\left(\left\| \varepsilon^{1/2}(E - E_{hp}) \right\|^2_{L^2(Q)} + \left\| \mu^{1/2}(H - H_{hp}) \right\|^2_{L^2(Q)} \right)^{1/2}$$

$$\leq \frac{12\sqrt{2}}{\sqrt{c}} C_{stab} \sum_{K \in T_h} \left(6\left(c + \frac{h^x_K}{h^t_K}\right) + 8\,{}_K \left(1 + c\frac{h^t_K}{h^x_K}\right) \right)^{1/2} (e/2)^{\frac{s_K^2}{p_K}} \frac{\left(h^x_K + c h^t_K\right)^{s_K + \frac{3}{2}}}{p_K^{s_K}}$$

$$\left(\left\| \varepsilon^{1/2} E \right\|_{W_c^{s_K+1,1}(K)} + \left\| \mu^{1/2} H \right\|_{W_c^{s_K+1,1}(K)} \right). \qquad (47)$$

with C_{stab} from (19).

Proof. Given an element $K \in T_h$, we denote by ∂K^N, ∂K^S, ∂K^W, and ∂K^E its North, South, West and East sides, respectively, North pointing in the positive time direction, and set $\partial K^{WE} := \partial K^W \cup \partial K^E$.

For all $(v_E; v_H) \in H^1(T_h)^2$, expanding the DG+ norm(a)-5@H98 -3.6 Td [(:=)-[(+)-0.569-6.

S(5 Tf 5.4 -192(e).298(a)-5.8 0 3/9(h)v

where the last inequality follows noting that $f(x) = (1 - x)^{\frac{1}{x} - 1}(1 + x)^{-\frac{1}{x} - 1} 4^x e^{2 - 2x} \geq 1$ for all $0 < x < 1$ (which in turn can be verified by checking the convexity of $\log f$ and its limit values for $x \to 0$ and 1).

The estimate in

(What we actually need is only that $e u_0$ and $e w_0$ are analytic in a su ciently large complex neighbourhood of the nite segments $_Q$ and ^+_Q, respectively.)

For every mesh element K as above, we x h_K := length($_K$). The complex ellipses with foci at the extrema of $_K$ and sum of the semiaxes equal to σ + p

Figure 3 shows convergence of the h-version for degree zero through three. Solid lines correspond to results obtained with the Trefftz basis whereas the dashed lines were obtained using the non-Trefftz basis. Uniform mesh step sizes are applied by reducing h_x and h_t simultaneously. The Trefftz method exhibits optimal algebraic convergence rates h^{p+1}. However, in the non-Trefftz case, the results seem to suggest an odd-even pattern of the convergence rates, with convergence being suboptimal for odd degrees (by one order). Numerical odd-even effects in the convergence rates of DG methods have also been reported, e.g. in [18, section 6.5], although it has been shown in [17] that in some situations this might

$$:= \max\left\{ (\epsilon\mu)^{-1} + (\ \ )^{-1}_{L^1(F_h^{ver})};\ (\ \ )^{-1}_{L^1(F_h^L \cup F_h^R)} \right\}.$$

Proof. We assume that $\jmath$ and m are continuous in Q; the general case will follow by a density argument.

First, we extend the initial problem to the entire space $\mathbb{R}$. Define $\widetilde{v}_E; \widetilde{v}_H; \widetilde{\jmath}; \widetilde{m}$ in $\mathbb{R}\times\mathbb{R}^+$ as the $2(x_R - x_L)$-periodic functions in x that satisfy $\widetilde{v}_E|_Q = v_E; \widetilde{v}_H|_Q = v_H; \widetilde{\jmath}|_Q = \jmath; \widetilde{m}|_Q = m$ and such that $\widetilde{v}_E$ and $\widetilde{m}$ are odd around x_L (and consequently also around x_R), and $\widetilde{v}_H$ and $\widetilde{\jmath}$ are even around the same points, i.e.

$$\widetilde{v}_E(x_L + x; t) = -\widetilde{v}_E(x_L - x; t); \qquad \widetilde{v}_H(x_L + x; t) = \widetilde{v}_H(x_L - x; t);$$
$$\widetilde{\jmath}(x_L + x; t) = \widetilde{\jmath}(x_L - x; t); \qquad \widetilde{m}(x_L + x; t) = -\widetilde{m}(x_L - x; t); \qquad \forall (x; t) \in \mathbb{R}\times\mathbb{R}^+.$$

(Note that the absolute values are $(x_R - x_L)$-periodic in x.) Since time derivatives preserve parities and space derivatives swap them, the extended functions $\widetilde{v}_E$ and $\widetilde{v}_H$ are continuous and satisfy the extended initial problem

$$\begin{aligned}
&\frac{\partial \widetilde{v}_E}{\partial x} + \frac{\partial(\mu \widetilde{v}_H)}{\partial t} = \widetilde{m} && \text{in } \mathbb{R}\times\mathbb{R}^+;\\
&\frac{\partial \widetilde{v}_H}{\partial x} + \frac{\partial(\epsilon \widetilde{v}_E)}{\partial t} = \widetilde{\jmath} && \text{in } \mathbb{R}\times\mathbb{R}^+;\\
&\widetilde{v}_E(\cdot; 0) = 0; \quad \widetilde{v}_H(\cdot; 0) = 0 && \text{on } \mathbb{R}.
\end{aligned}$$

Second, we split the right- and the left-propagating components. Define

$$u := \epsilon^{1/2}\widetilde{v}_E + \mu^{1/2}\widetilde{v}_H; \quad w := \epsilon^{1/2}\widetilde{v}_E - \mu^{1/2}\widetilde{v}_H; \qquad \text{so that} \qquad \widetilde{v}_E = \frac{u+w}{2\epsilon^{1/2}}; \quad \widetilde{v}_H = \frac{u-w}{2\mu^{1/2}}.$$

They satisfy the inhomogeneous transport equations in $\mathbb{R}\times\mathbb{R}^+$

$$\frac{\partial u}{\partial x} + \frac{\partial(c^{-1}u)}{\partial t} = \epsilon^{1/2}\widetilde{m} + \mu^{1/2}\widetilde{\jmath} =: f; \qquad \frac{\partial w}{\partial x} - \frac{\partial(c^{-1}w)}{\partial t} = \epsilon^{1/2}\widetilde{m} - \mu^{1/2}\widetilde{\jmath} =: g;$$

recalling that $(\epsilon\mu)^{1/2} = c^{-1}$, so they can be written explicitly with the following representation formula (e.g. [9, section 2.1.2, equation (5)], recall that from the assumptions made in the proof, f and g are piecewise continuous)

$$u(x; t) = \int_0^t c f\big(x + c(s - t); s\big)\, ds; \qquad w(x; t) = \int_0^t c g\big(x - c(s - t); s\big)\, ds.$$

We first bound the L^2 norm of u and w on horizontal and vertical segments with the data $f; g$; from the triangle inequality $(\widetilde{v}_E; \widetilde{v}_H)$ will be bounded by $\widetilde{\jmath}$ and $\widetilde{m}$, and the bound for v_E and v_H will follow. For all $0 \le t \le T$

$$\begin{aligned}
\|u(\cdot; t)\|^2_{L^2(\Omega)} &= \int_{x_L}^{x_R} \left(\int_0^t c f\big(x + c(s - t); s\big)\, ds\right)^2 dx\\
&\le t c^2 \int_{x_L}^{x_R} \int_0^t \left(f\big(x + c(s - t); s\big)\right)^2 ds\, dx\\
&= t c^2 \int_0^t \int_{x_L + c(s-t)}^{x_R + c(s-t)} \big(f(y; s)\big)^2 dy\, ds\\
&\le 2 t c^2 \int_0^t \int_{x_L + c(s-t)}^{x_R + c(s-t)} \left(\epsilon|\widetilde{m}(y; s)|^2 + \mu|\widetilde{\jmath}(y; s)|^2\right) dy\, ds\\
&= 2 t c^2 \left(\left\|\epsilon^{1/2} m\right\|^2_{L^2(\Omega\times(0;t))} + \left\|\mu^{1/2} \jmath\right\|^2_{L^2(\Omega\times(0;t))}\right)
\end{aligned}$$

(the last equality follows from the symmetries of $\widetilde{\jmath}$ and $\widetilde{m}$ which ensure the equality of their L^2 norms on the rectangle $(x_L; x_R) \times (0; t)$ and on the parallelogram with vertices $(x_L - ct; 0); (x_R - ct; 0); (x_R; t); (x_L; t)$). Similarly, for all $x \in \Omega$

$$\|u(x; \cdot)\|^2_{L^2(0;T)} = \int_0^T \left(\int_0^t c f\big(x + c(s - t); s\big)\, ds\right)^2 dt$$

$$c^2 \int_0^T \int_0^t t \; f\left(x + c(s - t); s\right)^2 \, ds \, dt$$

$$= c^2 \int_0^T \int_s^T t \; f\left(x + c(s - t); s\right)^2 \, dt \, ds$$

$$= c^2 \int_0^T \int_{x - c(T - s)}^{x} \quad x \quad y$$

[8] H. Egger, F. Kretzschmar, S. M. Schnepp, I. Tsukerman, and T. Weiland, Transparent boundary conditions in a Discontinuous Galerkin Tre tz method , arXiv preprint arXiv:1410.1899, (2014).

[9] L. C. Evans, Partial Di erential Equations , Graduate Studies in Mathematics, American Mathematical Society, Providence, 3rd ed., 2002.

[10] R. S. Falk and G. R. Richter, Explicit nite element methods for symmetric hyperbolic equations, SIAM J. Numer. Anal., 36 (1999), pp. 935{952.

[11] C. Farhat, I. Harari, and U. Hetmaniuk, A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime, Comput. Methods Appl. Mech. Eng., 192 (2003), pp. 1389{1419.

[12] D. French and T. Peterson, A continuous space-time nite element method for the wave equation, Math. Comp., 65 (1996), pp. 491{506.

[13]

[27] A. Maciąg and J. Wauer, Solution of the two-dimensional wave equation by using wave polynomials, J. Engrg. Math., 51 (2005), pp. 339{350.

[28] P. Monk and G. R. Richter, A discontinuous Galerkin method for linear symmetric hyperbolic systems in inhomogeneous media, J. Sci. Comput., 22/23 (2005), pp. 443{477.

[29] P. Monk and D. Wang, A least squares method for the Helmholtz equation, Comput. Methods Appl. Mech. Eng., 175 (1999), pp. 121{136.

[30] S. Petersen, C. Farhat, and R. Tezaur, A space-time discontinuous Galerkin method for the solution of the wave equation in the time domai