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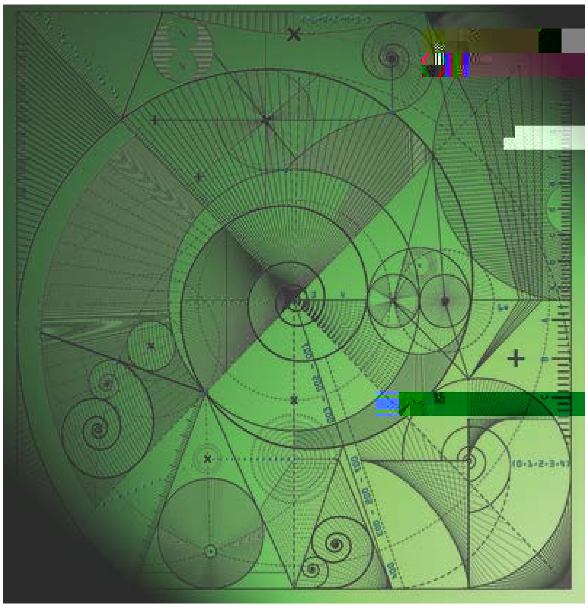

EXISTENCE OF DIM SOLUTIONS TO THE EQUATIONS OF VECTORIAL CALCULUS OF VARIATIONS IN L^1

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Abstract. In the very recent paper [K] we introduced a new duality-free theory of generalised solutions which applies to fully nonlinear PDE systems of any order. As one of our rst applications, we proved existence of vectorial solutions to the Dirichlet problem for the 1 -Laplace PDE system which is the analogue of the Euler-Lagrange equation for the functional $E_1(u;) = \|Du\|_{L^1()}$. Herein we prove existence of a solution $u : \ R \, ! \ R^N$ to the Dirichlet problem for the system arising from the functional $E_1(u;) = \|H(; u; u^0)\|_{L^1()}$. This is nontrivial even in the present 1 D case, since the equations are non-divergence, highly nonlinear, degenerate, do not have classical solutions and standard approaches do not work. We further give an explicit example arising in variational Data Assimilation to which our result apply.

Contents

1. Introduction

Calculus of Variations in L^1 has a long history and was pioneered by Aronsson in the 1960s [A1]-[A5]. In the simpler case of one space dimension, the basic object of study is the functional

$$(1.1) \qquad E_1(u;) := \left\| H(; u; u^0) \right\|_{L^1()} ; \quad u : \ R \, ! \ R^N ;$$

where $N \ \ 1$ and $H \ 2 \ C^2(\ \ R^N \ \ R^N)$ is a Hamiltonian function whose arguments will be denoted by $(x; \ ; P)$. Aronsson was the rst to note the locality problems associated to this functional and by introducing the appropriate minimality notion in L^1, proved the equivalence between the so-called Absolute Minimisers

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and classical solutions of the "Euler-Lagrange" equation which is associated to the functional. The higher dimensional analogue $H(\cdot, u, Du)$ when $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar function has also attracted considerable attention by the community, see e.g. [BEJ], [C] and for an elementary introduction [K8]. In particular, the Crandall-Ishii-Lions theory of Viscosity Solutions proved to be an indispensable tool in order to study the equations in L^∞ which are non-divergence, highly nonlinear and degenerate. Even in the simplest case where the Hamiltonian is the Euclidean norm, i.e. $H(P) = |P|^2$, in general the solutions are non-smooth and the corresponding PDE which is called ∞-Laplacian reads

$$\Delta_\infty := Du \otimes Du : D^2 u = \sum_{i,j=1}^{N} D_i u \, D_j u \, D^2_{ij} u = 0. \tag{1.2}$$

However, until the early 2010s, the theory was essentially restricted to the scalar case $N = 1$. The main reason for this limitation was the absense of an effective theory of generalised solutions for non-divergence PDE system which would allow to study the counterparts of (1.2) emerging in the vectorial case. In a series of recent papers [K1]-[K7], the author has initiated the study of the vector-valued case, which except for its intrinsic mathematical interest it is also of paramount importance for applications. The results in the aforementioned papers include the study of the analytic properties of classical solutions to the fundamental equations and their connection to the supremal functional. In the case of

$$E_\infty(u, \Omega) = \big\| |Du|^2 \big\|_{L^\infty(\Omega)} \tag{1.3}$$

applied to Lipschitz mappings $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ (where $|Du|$ denotes the Euclidean norm on $\mathbb{R}^{N \times n}$), the respective ∞-Laplace system is

$$\Delta_\infty u := \Big(Du \otimes Du + |Du|^2 [Du]^\perp \otimes I \Big) : D^2 u = 0. \tag{1.4}$$

In (1.4), $[Du(x)]^\perp$ denotes the orthogonal projection on the nullspace of the linear map $Du(x)^\top : \mathbb{R}^N \to \mathbb{R}^n$ and in index f -5.978ence, highly nonlin 8.20f 5.535 0 Td [(R)]TJ/F10sTJ/F8 9.9626

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a 2nd order ODE system which is quasilinear, non-divergence non-monotone and with discontinuous coefficients. Even in the scalar case of $N = 1$, it is known since the work of Aronsson that (1.5) in general does not have solutions any more regular than just $C^1(\Omega;\mathbb{R}^N)$ and their "weak" interpretation is an issue.

Motivated in part by the necessity to study the nonlinear systems arising in L^∞, in the very recent paper [K] the author proposed a new theory of generalised solutions which applies to fully nonlinear PDE systems. In addition, this theory allows to interpret merely measurable mappings $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ as solutions of PDE systems which may even be defined by discontinuous nonlinearities and can be of any order:

$$F\big(\cdot, u, Du, D^2u, \dots, D^pu\big) = 0, \quad F \text{ Carathéodory map}.$$

In the above $Du, D^2u, \dots, D^pu$ denote the derivatives of u of 1st, 2nd,... ,pth order respectively. Our approach is duality-free and bypasses the standard insufficiency of the theory of Distributions (and of weak solutions) to apply to even linear non-divergence equations with rough coefficients. The standing idea of the use of integration-by-parts in order to "pass derivatives to test functions" is replaced by a probabilistic description of the limiting behaviour of the difference quotients. This builds on the use of Young (Parameterised) measures over the compactification of the "state space", namely the space wherein the derivatives are valued. Background material on Young measures can be found e.g. in [P, FL, E, FG, M, V, CFV], but for the convenience of the reader we recall herein all the rudimentary properties we utilise in the paper.

The essential idea behind our new notion of solution is thoroughly explained later, but, at least for the case needed in this paper, it can be briefly motivated as follows. Assume that $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ is a strong a.e. solution of the system

$$F(\cdot, u, u', u'') = 0, \quad \text{on } \Omega. \tag{1.6}$$

We need a notion of solution which makes sense even if u is at most once differentiable. If u is twice differentiable, we have

$$F\Big(x, u(x), u'(x), \lim_{h\to 0} D^{1,h}u'(x)\Big) = 0,$$

for a.e. $x \in \Omega$, where $D^{1,h}$ stands for the difference quotient operator. By continuity, the limit commutes with the nonlinearity. Hence, we have

$$\lim_{h\to 0} F\Big(\cdot, u, u', D^{1,h}u'\Big) = 0,$$

a.e. on Ω. Note now that the above statement makes sense even if u is once differentiable, although as it stands does not look promising. In order to represent this limit in a convenient fashion, we embed the difference quotients $D^{1,h}u' : \Omega \longrightarrow \mathbb{R}^N$ into the probability-valued maps (see the next section for the precise definitions) from Ω to the Alexandroff compactification $\overline{\mathbb{R}}^N$ and consider instead

$$\delta_{D^{1,h}u'} : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^N\big).$$

By the weak* compactness of this space, regardless regularity of u there always exists a sequential limit $\mathcal{D}^2u$ of $\delta_{D^{1,h}u'}$ such that $\delta_{D^{1,h}u'} \overset{*}{\rightharpoonup} \mathcal{D}^2u$ as $h \to 0$. Then, it can be shown that for any "test function" $\Phi \in C^0_c\big(\mathbb{R}^N\big)$, we have

$$\int_{\overline{\mathbb{R}}^N} \Phi(\mathrm{X})\, F\Big(x, u(x), u'(x), \mathrm{X}\Big)\, d\big[\mathcal{D}^2u(x)\big](\mathrm{X}) = 0, \quad \text{a.e. } x \in \Omega.$$

We emphasise that $\mathcal{D}^2u$ always exists independently of the twice differentiability of u. If u'' happens to exist, then we have the extra information that $\mathcal{D}^2u = \delta_{u''}$ a.e. on Ω and we reduce to strong solutions. This above property essentially consists the notion of Dim Solutions (in the special case of once differentiable solutions of 2nd order ODE systems) and will be taken as principal in this work.

As a first application of this new approach, in the paper [K] among other things we proved existence of Dim solutions to the Dirichlet problem for (1.4) when $n = N$. Herein we consider the relevant but different question of existence of Dim solutions to the Dirichlet problem for the ODE system (1.5). This is a non-trivial task even in the 1D case. In fact, it is not possible to be done in the generality of (1.1), (1.5) without structural conditions on H. The most important of these is that the Hamiltonian has to be radial in P. This means that there exist mappings

$$
\begin{aligned}
h &: \Omega \times \mathbb{R}^N \times [0,\infty) \longrightarrow \mathbb{R},\\
V &: \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N,
\end{aligned}
$$

such that H can be written as

$$
H(x,\eta,P) = h\Big(x,\eta,\frac{1}{2}|P - V(x,\eta)|^2\Big). \tag{1.7}
$$

Unfortunately this condition is necessary, since as we proved in [K3] it is (roughly) both necessary and sufficient for the ODE system to be degenerate elliptic. Under the assumption (1.7), the system (1.5) becomes

$$
\left\{
\begin{aligned}
(h_p)^2\big(u' - V\big)^{\otimes 2}\big(u'' - V_\eta u' - V_x\big) = &-h_p\big(h_x + h_\eta u'(u' - V)\big)\\
&+ h_p|u' - V|^2[u' - V]^\perp\big(h_\eta - h_p(u' - V)^\top V_\eta\big)
\end{aligned}
\right. \tag{1.8}
$$

for a.e. $x \in \Omega$, where $D^{1,h}$ stands for the standard difference quotient operator. But since F is a Carathéodory map, the limit commutes with the coefficients:

$$\lim_{h\to 0} F\Big(\cdot, u, u', D^{1,h}u'\Big) = 0, \tag{2.3}$$

a.e. on Ω. Note now that this statement makes sense if u is once differentiable and the limit when taken outside may exist even when it may not exist when it is put back inside. However, the above as it stands does not look useful since we need a way to represent this limit. Going back to (2.2), we observe that u is a strong solution of (2.2) if and only it satisfies

$$\int_{\mathbb{R}^N} \Phi(X)\, F(\cdot, u, u', X)\, d[\delta_{u''}](X) = 0, \quad \text{a.e. on } \Omega,$$

for any compactly supported "test" function $\Phi \in C^0_c\big(\mathbb{R}^N\big)$. This gives the idea that we can embed the difference quotient map $D^{1,h}u' : \Omega \to \mathbb{R}^N$ into the spaces of Young measures and consider instead the Dirac measure evaluated at the difference quotients of u':

$$\delta_{D^{1,h}u'} : \Omega \to \mathscr{P}\big(\overline{\mathbb{R}}^N\big).$$

We use the Alexandroff compactification $\overline{\mathbb{R}}^N$ instead of $\mathbb{R}^N$ and attach the point at ∞

u as the subsequential limits D^2u of the 1st di erence quotients of the derivative u^0 in the space of Young measures Y ; $\overline{\ }$

a Carathéodory map. Consider the ODE system

$$F\quad ; u; u', u'' = 0; \quad \text{on} \quad : \tag{2.5}$$

We say that the $W^{1;1}_{loc}(\ ;\mathbb{R}^N)$ map $u : \quad \mathbb{R} \to \quad \mathbb{R}^N$ is a Dim Solution of (2.5) when there exist a Dim 2nd derivative D^2u

[Co]). The latter approaches present some serious inconsistencies when applied to differential equations.

3. Derivation of the fundamental equations in L^∞

In this section we formally derive the fundamental equations associated to variational problems for the supremal functional

$$E_\infty(u,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} H\big(x,u(x),u'(x)\big),$$

when $H \in C^2(\Omega\times\mathbb{R}^N\times\mathbb{R}^N)$ is a nonnegative Hamiltonian, $\Omega \subseteq \mathbb{R}$ is open and $N \geq 1$. The next derivation has been performed in [K3] in the general higher-dimensional case, but we include it here because it provides further insights since the method of proof makes the foregoing formal calculations rigorous. For the sake of completeness of the exposition, we give all the steps of the derivation.

We begin by noting that since the functional is not Gateaux differentiable, we can not obtain the equations by taking variations as in the integral case. Instad, we obtain the equations in the limit of the Euler-Lagrange equations related to the L^m integral functional

$$E_m(u,\Omega) := \int_\Omega \Big(H\big(x,u(x),u'(x)\big)\Big)^m dx, \tag{3.1}$$

as $m\to\infty$. Here we suppose that $m \geq 2$ and u is a map $u : \Omega \subseteq \mathbb{R} \to \mathbb{R}^N$. The Euler-Lagrange equation of (3.1) is the ODE system

$$\Big(H^{m-1}(\cdot,u,u')H_P(\cdot,u,u')\Big)' = H^{m-1}(\cdot,u,u')H_\eta(\cdot,u,u'). \tag{3.2}$$

Evidently, the subscripts denote derivatives with respect to the respective argument. By distrbuting derivatives and normalising, the equation (3.2) gives

$$\text{(3.3)}\quad H(\cdot,u,u')'\,H_P(\cdot,u,u') + \frac{H(\cdot,u,u')}{m-1}$$

By expansion of derivatives, we further get

$$H_P \quad H_P^{>} u^{00} + H^{>} u^{0} + H_x \quad + \; H[H_P$$

correspoding to both the L^m functional and the L^1 functional for the speci c form of H as in (3.11). We rst di erentiate (3.11) and for the sake of brevity we suppress the arguments

$$
\begin{aligned}
H_P &= h_p(P \quad V) \ ; \\
H &= h \qquad h_p(P \quad V) \ V \quad ; \\
H_x &= h_x \quad h_p(P \quad V) \ V_x \ ; \\
H_{P\ P} &= h_{pp}(P \quad V) \ (P \quad V) \ + h_p
\end{aligned}
$$

and

$$
\begin{aligned}
H_{P\ x} &= \quad h_p V_x \ + \ h_{px} \quad h_{pp}(P \quad V) \ V_x \ (P \quad V) \ ; \\
H_P &= \quad h_p V \quad + (P \quad V) \ h_p \qquad h_{pp}(P \quad V) \ V \quad :
\end{aligned}
$$

Then, by using the identity

$$h_p(P \quad V)^{>} = [P \quad V]^{>} ; \tag{3.15}$$

which is a consequence of the assumption we have made that $h_p > 0$ and by grouping terms, the equation (3.7) after a calculation gives

$$\Bigg(\frac{h[u^0 \quad V]^{>}}{m \quad 1} \ h_p I + V$$

adding it to (3.16), we obtain

(3.19)

$$\begin{cases} \dfrac{h\ h_p + h_{pp}|u^0 \quad V|^2\ [u^0 \quad V]^>}{m \quad 1} + (\ h_p)^2|u^0 \quad V|^2 I \quad u^{00} \quad V\, u^0 \quad V_x \\ = \dfrac{h[u^0 \quad V]^>}{m \quad 1} \quad h \ + h_p(u^0 \quad V)^> V \ + \ h_p \quad u + h_{px}\ (u^0 \quad V) \\ \qquad h_p\ h_x + h \quad u^0\ (u^0 \quad V) \\ \quad + h_p|u^0 \quad V|^2[u^0 \quad V]^? \ h \quad h_p(u^0 \quad V)^> V \ : \end{cases}$$

The ODE system (3.19

and the L^1 system (3.20) arising from the supremal functional (3.14) can be written as

$$(3.24)\quad h_p^2 \; ;u;\frac{1}{2}\,u^0 \quad V(\;;u)^{2} \quad u^0 \quad V(\;;u)^{2} \Big[u^{00} \quad V(\;;u)^{0} \Big] = F^1(\;;u;u^0)$$

where F^1; f^1; A^1 are given by (3.21), (3.22).

3.2. A model of Data Assimilation in Meteorology. (This subsection is a result of the discussions with J. Bröcker, who we would like to thank.) Suppose R is a bounded interval and let us choose the Hamiltonian

$$H\;(x;\;;P) := 1 + \frac{1}{2}\,k(x) \quad K(\;)^{2} + \frac{1}{2}\,P \quad V(x;\;)^{2};$$

where $N;M \in \mathbb{N}$ and

$$k : \quad R \to R^M;$$
$$K : R^N \to R^M;$$
$$V : \quad R^N \to R^N$$

are all C^1 mappings. In the notation of the previous subsection, H corresponds to the choice of

$$h\;(x;\;;p) := 1 + \frac{1}{2}\,k(x) \quad R$$

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where $C > 0$ depends only on the assumptions and the length of Ω. In addition, there is a subsequence $(m_k)_1^\infty$ and $u^\infty \in W_b^{1;\infty}(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} u^m \longrightarrow u^\infty, & \text{in } C^0(\overline{\Omega};\mathbb{R}^N); \\ u^{m\,\prime} \overset{*}{\rightharpoonup} u^{\infty\,\prime}, & \text{in } L^p(\Omega;\mathbb{R}^N); \text{ for any } p > 2; \end{cases}$$

along $m_k \to \infty$, and also

$$\|u^\infty\|_{W^{1;\infty}(\Omega)} \le C; \tag{4.4}$$

where the constant C depends only on the assumptions, b and Ω.

Proof of Lemma 4.3. Step 1. We begin by recording some elementary inequalities we will use in the sequel. For any $t \ge 0$, $0 < \lambda < 1$ and $\varepsilon > 0$, Young's inequality gives

$$t^\lambda \le \varepsilon t + \left(\frac{1}{\varepsilon}\right)^{\frac{\lambda}{1-\lambda}}(1-\lambda). \tag{4.5}$$

Moreover, for any $P, V \in \mathbb{R}^N$ and $0 < \delta < 1$, we also have

$$(1-\delta)|P|^2 \le |P - V|^2 + \frac{1}{\delta}|V|^2. \tag{4.6}$$

Finally, for any $u \in W^{1;2m}(\Omega;\mathbb{R}^N)$, we have the following Poincaré inequality which is uniform in $m \in \mathbb{N}$:

$$\|u\|_{L^{2m}(\Omega)} \le 2(|\Omega|+1)\left(\|u'\|_{L^{2m}(\Omega)} + \max_{\partial\Omega}|u|\right). \tag{4.7}$$

Indeed, in order to see (4.7), suppose u is smooth and since $|u(x) - u(y)| \le \int_\Omega |u'|$, for $y \in \partial\Omega$ we have

$$\begin{aligned} |u(x)|^{2m} &\le \left(\int_\Omega |u'| + \max_{\partial\Omega}|u|\right)^{2m} \\ &\le 2^{2m-1}\left[\left(\int_\Omega |u'|\right)^{2m} + \max_{\partial\Omega}|u|^{2m}\right] \\ &\le (2(|\Omega|+1))^{2m-1}\left(\int_\Omega |u'|^{2m} + \max_{\partial\Omega}|u|^{2m}\right); \end{aligned}$$

which leads to (4.7).

Step 2. We now show that the functional E_m is weakly lower semicontinuous in $W^{1;2m}(\Omega;\mathbb{R}^N)$. Indeed, by setting

$$F(x;\eta;P) := h^m\left(x;\eta;\frac{1}{2}\left|P - V(x;\eta)\right|^2\right); \tag{4.8}$$

we have for the hessian with respect to P that (we suppress arguments again)

$$F_{PP} = mh^{m-2}\left\{hh_p I + \left(hh_{pp} + (m-1)(h_p)^2\right)(P-V)\otimes(P-V)\right\}.$$

By our assumptions on h and since the projection $[P-V]^\perp$ satisfies $[P-V]^\perp \le I$, we obtain the matrix inequality

$$\begin{aligned} F_{PP} &\ge mh^{m-2}\left\{hh_p I + hh_{pp}(P-V)\otimes(P-V)\right\} \\ &\ge mh^{m-1}\left(h_p I + (c_0 - h_p)[P-V]^\perp\right) \\ &\ge mc_0 I. \end{aligned}$$

Since F is convex in P and nonegative, the conclusion follows by standard lower semicontinuity results (see e.g. [D, GM]).

Step 3. Now we use Steps 1, 2 and derive the energy estimate, which will guarantee the coercivity of E_m. By our assumptions on h and the mean value theorem, there is a $\hat{p}\ 2\ [0; p]$ such that

$$h(x; \ ; p) = h_p(x; \ ; \hat{p})p + h(x; \ ; 0) \quad c_0 p + 1:$$

Hence, by using (4.6) the above gives

$$h \ x; \ ; \frac{1}{2} P \quad V(x; \ ) \ ^2 \qquad \frac{c_0}{2} P \quad V(x; \ ) \ ^2 \tag{4.9}$$

$$\frac{c_0}{2}(1 \quad)jPj^2 \qquad \frac{c_0}{2} jV(x; \ )j^2:$$

Then, by our assumption on V and (4.5), (4.6), for $\ > \ 0$ small we have

$$h \ x; \ ; \frac{1}{2} P \quad V(x; \ ) \ ^2 \qquad \frac{c_0}{2}(1 \quad)jPj^2 \qquad \frac{1}{2c_0}(1 + j \ j \ )^2$$

$$\frac{c_0}{2}(1 \quad)jPj^2 \qquad \frac{}{c_0} j \ j^2 \qquad C(\ ; \ ; \quad);$$

where $C(\ ; \ ; \quad)$ denotes a constant depending only on these numbers. We now select

$$:= \frac{1}{2}; \qquad := 2c_0"; \quad " > 0;$$

to nd

$$h \ x; \ ; \frac{1}{2} P \quad V(x; \ ) \ ^2 \qquad \frac{c_0}{4} jPj^2 \qquad "j \ j^2 \qquad C("; \quad):$$

Hence, for any $m\ 2\ N$

Step 4. We nally show existence of minimisers and convergence. We have the a priori energy bounds

$$\inf \left\{ E_m(v;)^{\frac{1}{2m}} : v\, 2\, W_b^{1;2m}(; R^N) \right\} \quad E_m(b;)^{\frac{1}{2m}}$$

$$\left(\int h^m \left(; b; \frac{1}{2} b^0 \quad V(; b) \right)^2 \right)^{\frac{1}{2m}}$$

$$j \quad j^{\frac{1}{2m}} \left\| h \left(; b; \frac{1}{2} b^0 \quad V(; b) \right)^2 \right\|^{\frac{1}{2}}_{L^1( )}$$

and

$$E_m(v;) \quad 0; \quad v\, 2\, W_b^{1;2m}(; R^N):$$

Proof of Lemma 4.4. Let F be given by (4.8). Then, by suppressing once again the arguments of $h; h_p; h$, we have that

$$F_P(x; \ ; P\ ) = mh^{m\ \ 1}h_p\ \ P\ \ \ V(x;\ \ )\ ;$$
$$F\ (x; \ ; P\ ) = mh^{m\ \ 1}h_p\ \ h\ \ \ \ h_p\ P\ \ \ V(x;\ \ )^{>} V\ (x;\ \ )$$

and the system (4.12) can be written compactly as

$$(4.13) \qquad F_P(\ ; u; u^0)^{\ 0} = F\ (\ ; u; u^0):$$

By our assumption on h, we have

$$h(x;\ ; p\ )\ \ \ \ h(x;\ ;\ 0) + \max_{0\ \ p\ \ p} h_p(x;\ ;\ p)\, p\ \ \ \ C(j\ \ j)(1 +\ \ p):$$

Hence,

$$h^{m\ \ 1}\ x;\ ;\ \frac{1}{2} P\ \ \ V(x;\ \ )^{2}\ \ \ \ C(j\ \ j)\ \ 1 +\ \ P\ \ \ V(x;\ \ )^{2\ \ m\ \ 1}$$
$$C(j\ \ j)\ \ 1 +\ \ P\ \ \ V(x;\ \ )^{2m\ \ 2}\ :$$

Further, by our assumptions on h and V , we have the bounds

$$(4.14) \qquad \begin{aligned} F_P(x;\ ; P\ )\ \ \ \ & C(j\ \ j)\ 1 + jPj^{2m\ \ 2}\ \ P\ \ \ V(x;\ \ ) \\ & C(j\ \ j)\ 1 + jPj^{2m\ \ 2}\ \ jPj + C(j\ \ j) \\ & C(j\ \ j)\ 1 + jPj^{2m\ \ 1}\ ; \end{aligned}$$

and

$$(4.15) \qquad \begin{aligned} F\ (x;\ ; P\ )\ \ \ \ & C(j\ \ j)\ 1 + jPj^{2m\ \ 2}\ \overset{h}{\ } C(j\ \ j)\ \ 1 + \ P\ \ \ V(x;\ \ )^{2} \\ & \underset{i}{\ } + C(j\ \ j)\ P\ \ \ V(x;\ \ ) \\ & C(j\ \ j)\ 1 + jPj^{2m\ \ 2}\ C(j\ \ j)\ jPj^{2} + 1 \\ & C(j\ \ j)\ 1 + jPj^{2m}\ : \end{aligned}$$

By standard results (see e.g. [D]), these bounds imply that the functional is Gateaux di erentiable on $W_b^{1;2m}(\ \ ; R^N)$ and the lemma follows.

Now we show that the weak solutions u^m of the respective Euler-Lagrange equations actually are smooth solutions. This will imply that the formal calculations of the previous section in the derivation of (3.19) make rigorous sense.

Lemma 4.5 (C^2 regularity). Let u^m be the sequence of minimisers of the Lemma 4.5, m 2. Then, each u^m is a classical solution in $C^2(\ \ ; R^N)$ of the Euler-Lagrange equation (4.12), and hence of the expanded form (3.19) of the same equation.

Proof of Lemma 4.5. Fix m 2 and let us drop the superscripts and denote u^m by just u. The rst step is to prove higher local integrability and then bound the di erence quotients of u^0 in L^2. Let us x q 2 N and 2 $C^1_c(\ )$ with 0 1. We set:

$$(4.16) \qquad (x) := \ \ (x) \int_{\inf}^{x} u^0\ \ \ V(\ ; u)^{\ q}\ u^0\ \ \ V(\ ; u)\ ;\ \ \ x\ 2\ \ :$$

Then, $2\ W^{1;1}_c(\ ;\mathbb{R}^N)$ and

$$ {}^0(x) = {}^2(x)\ u^0(x) \quad V\ x;u(x) \quad {}^q\ u^0(x) \quad V\ x;u(x) $$
$$ + {}^0(x) \int_{\inf}^{x} u^0 \quad V(\ ;u)\ {}^q\ u^0 \quad V(\ ;u)\ ; $$

for a.e. $x\ 2$. Suppose now that $q \quad 2m \quad 1$. Then, since $u^0\ 2\ L^{2m}(\ ;\mathbb{R}^N)$, we have that $2\ W^{1;2m}_c(\ ;\mathbb{R}^N)$. By inserting the test function in the weak formulation of the system (4.13) (i.e. (4.12)) and by suppressing again the arguments for the sake of brevity, we have

$$ \int \Big(h^m\ {}^1 h_p(u^0 \quad V) \quad {}^2|u^0 \quad V|^q(u^0 \quad V) + {}^0 \int_{\inf} |u^0 \quad V|^q(u^0 \quad V) \Big) $$
$$ + \int \Big(h^m\ {}^1 h_p \quad h \quad h_p(u^0 \quad V)^{>} V \int_{\inf} |u^0 \quad V|^q(u^0 \quad V) \Big) = 0: $$

By our assumptions on $h;V$, we have that $h_p \quad c_0$ and $2h \quad c_0|u^0 \quad V|^2$. By using the bounds (4.14), (4.15) (when F is given by (4.8)) that $\quad 1$ and the elementary inequalities

$$ \int_{\inf}^{x} |f| \quad \int |f|; \quad x\ 2 \quad ; \quad f\ 2\ L^1(\ )\ ; $$
$$ t^{2m\ \ 1} \quad t^{2m} + 1\ ; \quad t \quad 0; $$

we have

$$ \int {}^2|u^0 \quad V|^{2m+q} \quad C \int |u^0 \quad V|^{q+1} \Big(\int |\ {}^0|\ h^m\ {}^1 h_p |u^0 \quad V| + $$
$$ + \int h^m\ {}^1 h_p\ h \quad h_p(u^0 \quad V)^{>} V \Big) $$

which gives

$$ \int {}^2|u^0 \quad V|^{2m+q} \quad C\ \|u\|_{L^1(\ )} \int |u^0 \quad V|^{q+1} $$
$$ \int \Big\{ |\ {}^0|\ 1 + |u^0 \quad V|^{2m\ \ 1} + \quad 1 + |u^0 \quad V|^{2m} \Big\} : $$

Hence, we have obtained

$$ \int {}^2|u^0 \quad V|^{2m+q} \quad C\ \|u\|_{L^1(\ )} \int |u^0 \quad V|^{q+1} \int 1 + |u^0 \quad V|^{2m}: \tag{4.17} $$

In view of the estimate (4.17), by taking $q + 1 = 2m$ we have that $u^0 \quad V\ 2$ $L^{4m\ \ 1}_{loc}(\ ;\mathbb{R}^N)$. Hence, we can go back and choose $q + 1 = 4m \quad 1$ and then the test function satisfies $2\ W^{1;4m\ \ 1}_c(\ ;\mathbb{R}^N)$ which makes it admissible and

The next step is to prove that $D^{1;t}u^0$ is bounded in L^2_{loc}. The idea is classical, but we provide the arguments for the sake of completeness. To this end, we test in the weak formulation of (4.13) against di erence quotients of the form

$$:= \quad D^{1; \quad t} \quad {}^2 D^{1;t} u \;; \qquad 2\ C^1_c(\ ) \;; \quad D^{1;t}u(x) = \frac{u(x+t) \quad u(x)}{t}; \quad t \neq 0:$$

Let F be given by (4.8). Then, for $" > 0$ and t small, we have

$$I := \int D^{1;t} \; F_P(\ ; u; u^0) \qquad {}^2 D^{1;t}u^0 + 2 \quad {}^0 D^{1;t}u$$

$$(4.18) \qquad \int F\ (\ ; u; u^0) \quad jD^{1;t}uj + \ {}^2 jD^{1;t}D^{1;t}uj$$

$$K \int F\ (\ ; u; u^0)^{\ 2} + \int ju^0j^2 \ + " \int {}^2 jD^{1;t}u^0j^2:$$

for some constant $K > 0$ independent of t. By using the inequality $F_{PP} \quad c_0 I$ and the identity

$$D^{1;t} \; F_P(\ ; u; u^0) \;(x)$$

$$= \int_0^1 \Big(F_{PP} \quad ; u\,(x+t) + (1 \quad)u(x);\ u^0(x+t) + (1 \quad)u^0(x) \;\; D^{1;t}u^0(x)$$

$$+ \; F_P \quad ; u\,(x+t) + (1 \quad)u(x);\ u^0(x+t) + (1 \quad)u^0(x) \;\; D^{1;t}u(x)$$

$$+ \; F_{P x_{1;t}} \quad ; u\,(x+t) + (1 \quad)u(x);\ u^0(x+t) + (1 \quad)u^0(x) \Big) \; d$$

(where $F_{P x_{1;t}}$ denotes di erence quotient with respect to the x variable), we have the bound the bound

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By passing to the limit as $r \to 0$ in (4.26), the Lebesgue differentiation theorem implies the inequality (4.25) is valid a.e. on Ω. We now set

$$\Omega^\infty := \left\{ x \in \Omega \ : \ v^\infty(x) > 0 \right\}.$$

By the continuity of v^∞, Ω^∞ is open in Ω, the set $\Omega \setminus \Omega^\infty$ is closed in Ω and

$$\Omega \setminus \Omega^\infty = \left\{ x \in \Omega \ : \ v^\infty(x) = 0 \right\}.$$

By (4.25), we have

$$\left| u^{\infty\prime} - V(\cdot, u^\infty) \right| = 0, \quad \text{a.e. on } \Omega \setminus \Omega^\infty. \tag{4.27}$$

On the other hand, since $v^m \to v^\infty$ in $C^0(\overline{\Omega})$, for any $\Omega' \Subset \Omega^\infty$, there is a $\varepsilon_0 > 0$ and an $m(\Omega') \in \mathbb{N}$ such that for all $m \geq m(\Omega')$, we have

$$v^m \geq \varepsilon_0 \quad \text{on } \Omega'. \tag{4.28}$$

By (4.28), (4.26) and (4.24), we have
(4.29)

$$\left| u^{m\prime\prime} - V(\cdot, u^m)' \right| \leq \frac{3}{(c_0 \varepsilon_0)^2} \left(\frac{f^\infty(\cdot, u^m, u^{m\prime})}{m-1} + F^\infty(\cdot, u^m, u^{m\prime}) \right), \quad \text{on } \Omega'.$$

By (4.29) and (4.20) we have that $u^{m\prime\prime}$ is bounded in $L^p_{loc}(\Omega^\infty; \mathbb{R}^N)$. Hence, we have that

$$\begin{cases} u^m \to u^\infty, & \text{in } C^0(\Omega^\infty; \mathbb{R}^N), \\ u^{m\prime} \rightharpoonup u^{\infty\prime}, & \text{in } L^p_{loc}(\Omega^\infty; \mathbb{R}^N), \text{ for all } p > 2, \\ u^{m\prime\prime} \rightharpoonup u^{\infty\prime\prime}, & \text{in } L^p_{loc}(\Omega^\infty; \mathbb{R}^N), \text{ for all } p > 2. \end{cases}$$

Thus, by passing to the limit in the ODE system (3.19) as $m \to \infty$ along a subsequence, we have that the restriction of u^∞ over the open set Ω^∞ is a strong a.e. solution of (3.20) on Ω^∞. By bootstapping in the equation, we have that actually $u^\infty \in C^2(\Omega^\infty; \mathbb{R}^N)$. On the other hand, we have that

$$\left| u^{\infty\prime} - V(\cdot, u^\infty) \right| = 0, \quad \text{a.e. on } \Omega \setminus \Omega^\infty.$$

Hence, if the set $\Omega \setminus \Omega^\infty$ has non-trivial topological interior, by differentiating the relation $u^{\infty\prime} = V(\cdot, u^\infty)$ we a.e. on u

Then, by Proposition 2.10 we have that u^1 is Dim solution on ^{1}, since it is a strong solution on this subdomain. Consequently, for any test function 2 $C^0_c(R^N)$ we have

[CFV] C. Castaing, P. R. de Fitte, M. Valadier, *Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory*, Mathematics and Its Applications, Kluwer Academic Publishers, 2004.

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