Department of Mathematics and Statistics

Weighted -rate dominating sets in social networks

Danica Vukadinović Greetham[1], Anush Poghosyan[1], and Nathaniel Charlton[1,2]

[1] Centre for the Mathematics of Human Behaviour
Department of Mathematics and Statistics
University of Reading, UK

In health-related behaviour change context, for an intervention to work at the individual level, it is often of the utmost importance that a support network exist (see e.g. [8]). In this way an individual is surrounded with social support. Also, a support network needs to have a major influence on the individual, as possible negative influences also come from her/his social network (for example in interventions aimed at addictive behaviours).

For these reasons, one often needs to find a set of nodes/individuals such that all other or indeed all individuals are connected to that set. In graph theory such a set is called a dominating set and a problem of finding a dominating set of minimal cardinality is NP complete [7]. The notion was generalised introducing k-domination where each node needs to have at least k neighbours in the dominating set, and domination where $0 <$ 1, where each node not in the dominating set needs at least 100 percentage of neighbours in the dominating set [12]), and -rate domination [6] where each node (including ones in the dominating set) needs to have at least 100 percentage of neighbours in the dominating set. Again, finding minimum cardinalities of and -rate dominating sets is NP-complete.

Here, we introduce -rate dominating sets problems on weighted networks. Why weighted networks? It might be that the "best" candidates (from structural perspective) for dominating sets are not feasible for different reasons: they cannot be a part of intervention because they do not have desired attributes, or they do not have time to invest into intervention. We want to overcome this assigning a cost to be part of intervention to each node. Thus, our goal is to find a most cost effective set that we can control or dominate network from. Note that here we do not model negative influences that come from a social network, but just require at least 100 percents of neighbours to be in the support network.

In the next section we give preliminaries and formally define the problem. In Section 2 an overview of the previous work is given. In Section 3 theoretical upper bound on weighted -rate dominating set is given which leads to simple randomised rounding algorithm using linear programming formulation of the problem. In Section 4 we analyse the results obtained from the algorithm's application on a Twitter network and generated graphs and compare them for non-weighted case with the existing algorithm in [5] for -rate domination. We conclude in the Section 5.

1 Preliminaries

In this section we introduce the notation and definitions that we use throughout this paper.

A graph or undirected graph G is an ordered pair $G = (V, E)$ where V is a set, elements of which are called vertices or nodes, and E is a set of unordered pairs of distinct vertices called edges. If G is a graph of order n, then $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices in G, d_v denotes the degree of v, and $d_v = d_v + 1$. Let $N(v)$ denote the neighbourhood of a vertex v. Also, let $N(V) = \cup_{v \in V} N(v)$ and $N[V] = N(V) \cup V$. Then $d_v = |N[v]|$. Denote by (G) and (G) the minimum and maximum degrees of vertices of G, respectively. Put $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

A set D is called a *dominating set* if every vertex not in D is adjacent to one or more vertices in D. The minimum cardinality of a dominating set of G is the domination number $\gamma(G)$.

Let α be a real number satisfying $0 < \alpha \leq 1$. A set $X \subseteq V(G)$ is called an α-dominating set of G if $|N(v) \cap X| \geq \alpha d_v$ for every vertex $v \in V(G) \setminus X$, i.e. v is adjacent to at least $\lceil \alpha d_v \rceil$ vertices of X. The minimum cardinality of an α-dominating set of G is called the α-domination number $\gamma_\alpha(G)$. It is easy to see that $\gamma(G) \leq \gamma_\alpha(G)$, and $\gamma(G) = \gamma_\alpha(G)$ if α is sufficiently close to 0. A set $X \subseteq V(G)$ is considered an α-rate dominating set of G if for any vertex $v \in V(G)$, $|N[v] \cap X| \geq \alpha d_v$. The minimum cardinality of an α-rate dominating set of G is called the α-rate domination number $\gamma_{\times\alpha}(G)$. It is easy to see that $\gamma_\alpha(G) \leq \gamma_{\times\alpha}(G)$.

Now we consider the vertex-weighted graphs. These are finite and undirected graphs with no loops and multiple edges in which each vertex has been assigned a weight. Let w_v be the weight (cost) of each vertex v of graph G. Let $\gamma_w(G)$ denote a minimum weight of a dominating set X of G and let $\gamma_{\times\alpha;w}$ denote a minimum weight of an α-rate dominating set D. Finding an α-rate dominating set D of G such that $\sum_{v \in D} w_v$ is minimised is the main problem studied in this paper.

2 Previous work

Variants of domination have been studied extensively and have various applications for real life problems. Smaller number of studies in domination parameters consider weighted graphs in particular.

The minimum weighted dominating set problem is one of the classic NP-hard optimisation problems in graph theory. Zou et al. [20] studied the minimum-weighted dominating set and the minimum-weighted connected dominating set problems on a node-weighted unit disk graph and devised approximation algorithms for these problems with performance ratios of $5 + \varepsilon$ and $4 + \varepsilon$ respectively. In [18] Polynomial Time Approximation Scheme (PTAS) was generalised for weighted case in polynomial growth bounded graphs with bounded degree constraint. A variant of the weighted dominating set problem - the weighted minimum independent k-domination (WMIkD) problem was studied by Yen in [19]. An algorithm linear in the number of vertices of the input graph for the WMIkD problem on trees is presented.

Discussing a more general domination set problem [3], where the direct connections are replaced with shortest paths corresponding to some measure defined on the vertices of a graph, the authors give an approximation algorithm for the vertex-weighted version. Using randomised rounding they prove the approximation ratio of $O(\log \Delta)$ for their randomised algorithm, where Δ is the maximum cardinality of the sets of vertices that can be dominated by any single vertex, or in our case a maximum degree of the vertices in the graph.

In [2], the maximum spanning star forest problem is discussed, which is the complement problem of domination set. A 071-approximation algorithm for this problem is given, and for vertex-weighted case a :64-approximation algorithm is presented.

The -domination was introduced by Dunbar et al. in [12]. Introduced by Zverovich et al.[6] the concept of -rate domination can be considered as a particular case of an dominating set in the same graph. Note that both the and -rate domination problems are known to be NP-complete. Thus it is of importance to determine bounds for and -rate domination numbers and various similar parameters. In [5] and [6] the authors explicitly provide new upper bounds and randomised algorithms for nding the and -rate domination sets in terms of a parameter and graph vertex degrees on undirected simple nite graphs by using probabilistic constructions. Their algorithm is bounded by:

$$(G) \quad \left(1 \quad \frac{\hat{b}}{(1+\hat{b})^{1+1/\hat{b}} \, \hat{d}^{1/\hat{b}}}\right) n; \qquad (1)$$

where $\hat{d}$ is a closed -degree of G and $\hat{b} = \lfloor (1 \quad) \rfloor + 1$.

Studies of the propagation of in uence in the context of social networks carried out by Wang et al. in [16] resulted in introducing new variants of domination such as the positive in uence dominating set (PIDS) and total positive in uence dominating set (TPIDS). From the de nitions given in [16] it is easy to see that PIDS and TPIDS problems are equivalent to -dominating and -rate dominating set problems respectively for a special case when $= 1/2$. Wang et al. proved that both these problems are NP-hard. Thus, it is important to study approximability of the problems. In their work Dinh et al. [4] generalise PIDS and TPIDS by allowing any $0 < \; < 1$ and show that both problems can be approximated within a factor ln + O(1) and present linear time exact algorithm for trees.

3 Randomised rounding algorithm

of an integer program IP:

min

Showing our goal (4) is equivalent to showing

$$\frac{1}{2} \le \sum_{l=k}^{d_v} \sum_{A \in F_l} \Big(\prod_{i \in A} x_i\Big)\Big(\prod_{j \in A^c} (1 - x_j)\Big) = Pr(k \le X). \qquad (5)$$

So we are looking for a minimum of the right hand side of (5) subject to $\sum_{i=1}^{d_v} x_i \ge k$ (this minimum must exist by continuity and compactness). Clearly the minimum will be found when $\sum_{i=1}^{d_v} x_i = k$; increasing one of the x_is without changing the others will clearly only increase the RHS. So we may assume that

$$\sum_{i=1}^{d_v} x_i = k. \qquad (6)$$

Now we can use the result from [10], Theorem 5, that shows that tail distribution function of Poisson's binomial distribution attains its minimum in binomial distribution, i.e. when all probabilities are equal. The theorem states that for two integers b, and c such that $0 \le b \le np \le c \le n$, the probability $P(b \le X \le c)$ reaches its minimum where all the probabilities $p_1 = \dots = p_n = p$, unless $b = 0$ and $c = n$. Here p_is are probabilities (or parameters) of Poisson's binomial distribution, and n and p are parameters of related binomial distribution. We apply that theorem taking two integers b and c to be our k and d_v respectively. We have that p, the equal probability is $\frac{k}{d_v}$ from (6), whence np equals our k. The theorem gives us

$$\sum_{l=k}^{d_v} \binom{d_v}{l} p^l (1-p)^{d_v - l} \le Pr(k \le X).$$

Thus, we will be done if we can show that

$$\sum_{l=k}^{d_v} \binom{d_v}{l} p^l (1-p)^{d_v - l} \qquad (7)$$

is at least $\frac{1}{2}$. Let Y be a random variable of binomial distribution with d_v trials each of probability p. Then observe that in fact $Pr(Y \ge k)$ is equal to (7) above. The median of Y is bounded by $\lfloor d_v p \rfloor$ and $\lceil d_v p \rceil$ [13], but $d_v p$ is exactly the integer k, so k is the unique median of Y. It follows from the defining property of medians that $Pr(Y \ge k) \ge \frac{1}{2}$, and thus $Pr(Y < k) < \frac{1}{2}$ and the proof is complete.

Hence, the probability is lower bounded by $\frac{1}{2}$, and the feasibility follows. Let A_i denote the event that vertex v_i is -rate dominated and let $B = \cap_{i=1}^{n} A_i$ be the event that all vertices are dominated. Using a technique identical to one carried out in [3], with amplification approach (repeating randomised rounding $t = O(\log_2 \;)$ times) which results in $Pr([x_i = 1]) = 1 - (1 - \hat{x}_i)^t$. We obtain that the expected value of the solution resulted from randomised rounding, given that event B happens, (i.e. that the solution is feasible) is

$$
\begin{aligned}
E\left[\sum_{i=1}^{n} w_i x_i | B\right] &= \sum_{i=1}^{n} w_i \Pr([x_i = 1] | B) \\
&= \sum_{i=1}^{n} w_i \frac{\Pr(B | [x_i = 1])}{\Pr[B]} \Pr(x_i = 1) \\
&\leq \sum_{i=1}^{n} w_i \prod_{j \in N(v_i)} \frac{1}{\Pr(A_j)} (1 - (1 - \hat{x}_i))^t \\
&\leq \frac{t}{\frac{1}{2}} \sum_{i=1}^{n} w_i \hat{x}_i \\
&\leq O(\log_2 \Delta \cdot OPT).
\end{aligned}
$$

Hence, there exists a particular solution that it is within $O(\log_2 \Delta)$ ratio to the optimal solution. □

A simple randomised rounding algorithm AlgRR follows immediately, by first solving LP and then rounding the solutions to zero or one. All vertices with ones then create an ρ-rate domination set with the sum of the weights within $O(\log_2 \Delta)$ factor of the optimal solution. We implemented AlgRR in Python.

4 Twitter UK mentions network

The Twitter data-set was collected on our behalf by Datasift, a certified Twitter partner, which allowed us to access the full Twitter firehose rather than being rate-limited. The data-set consists of all UK based[3] Twitter users that sent tweets with at least one mention between 8 Dec 2011 and 4 Jan 2012 (28 days in total). Mentions are messages that include an @ followed by a username and are used to address people. Thus, if person A posts a tweet containing "@B" that means A is addressing the tweet to B specifically. Mentions are not private messages and can be read by anyone who searches for them. A tweet can be addressed to several users simultaneously using @ repetitively.

4.1 Data

We preprocessed the data, removing empty mentions and self-addressing which left us with $3{,}614{,}705$ time-stamped arcs (individual mentions) from a total of $819{,}081$ distinct usernames, or nodes. We then removed all users who didn't tweeted but just received messages, as we did not have a weight measure for them. There were approximately 50k nodes that appeared both as tweeters and receivers. We aggregated data on weekly basis and kept only two-directional arcs (thus if person A mentioned B and person B mentioned A at least once during

[3] All Twitter users appearing in our data-set had selected the UK as their location.

a week there is a bi-directional edge between A and B in a weekly graph). For simplicity, we treated those bi-directional edges as undirected. This left us with 4 undirected weekly graphs with around 5k nodes in each and around 5.3k edges in average. For each vertex we retrieved its Klout score and used it as a weight. The Klout score measures an individual's influence based on her/his social media activity [4] It is a single number that represents the aggregation of multiple pieces of data about individuals' social media activity, based on a score model which is not publicly available [14]. The descriptive statistics of the Twitter mentions weekly graphs are given in Table 1 below. As the 4 mentions graphs are quite sparse, we

Table 1. Twitter mentions network statistics, ME denotes number of multi-edges in

4.2 Results

In this section we investigate how our randomised rounding algorithm AlgRR performs on some real and created networks. We also compare it with the existing -rate domination algorithm for simple (non-weighted) graphs from [5] (denoted here as AlgA). We have run both algorithms on 4 weekly Twitter random and preferential graphs. As algorithms are randomised, we have run both algorithms 100 times taking averages. Results are presented in Tables 3 and 4 below. The results show that for dense networks (networks denoted with pref-d and rnd-d) the algorithm AlgRR outperforms algorithm A signi cantly and not only on minimum weights (which would be expected, as algorithm A optimises the size of -rate dominating set, while algorithm AlgRR optimises the weights) but also on the sizes of -rate dominating sets. According to the theoretical bounds of algorithm A the probability with which each candidate vertex for -rate dominating set is selected gets close to 1 for dense network pref-d, thus resulting in selecting all the nodes of the network. However on sparse networks such as twitt1-4 Algorithm A slightly outperforms algorithm AlgRR.

Table 3. Alpha-rate domination sets' sizes (#), weights(W) and running times(T) for AlgA, for di erent graphs and = 0 :25; 0:5; 0:75 respectively.

Graph	Avg#	AvgW	Min#	Max#	MaxW	MinW	AvgT(ms)
pref-d0.25	5000	193419	5000	5000	193419	193419	12.71
pref-d0.5	5000	194267	5000	5000	194267	194267	12.87
pref-d0.75	5000	188938	5000	5000	188938	188938	13.09
rnd-d0.25	4730	182675	4689	4768	184149	180909	12.11
rnd-d0.5	4991	191934	4985	4998	192222	191573	12.35
rnd-d0.75	4998	194278	4994	5000	194343	194082	12.58
twitt10.25	4328	146960	4259	4408	149577	144171	3.21
twitt10.5	5291	179701	5222	5334	181377	176986	4.21
twitt10.75	5056	171733	4993	5113	173600	169609	3.80
twitt20.25	4612	157207	4539	4670	159495	154516	3.22
twitt20.5	5453	185872	5436	5475	186780	185343	3.77
twitt20.75	5259	179254	5217	5297	180618	177888	3.58
twitt30.25	3960	135557	3879	4031	138291	132624	2.70
twitt30.5	4897	167738	4854	4946	169551	165730	3.60
twitt30.75	4753	162770	4697	4794	164233	160732	3.24
twitt40.25	4195	142143	4119	4289	145122	139323	3.24
twitt40.5	5127	173809	5069	5183	175550	171776	3.64
twitt40.75	5036	170727	4979	5092	172826	168874	3.53

Since the algorithm AlgRR is based on LP -relaxation technique it runs in polynomial time. Our resetwOurpolynomialin Tab.no6138291LP0

Table 4. Alpha-rate domination sets' sizes, weights and running times for AlgRR, for di erent graphs and = 0 :25; 0:5; 0:75 respectively.

Graph	Avg#	AvgW	Min#	Max#	MaxW	MinW	AvgT(ms)
pref-d0.25	979	16647	578	1512	30845	7124	65.70
pref-d0.5	2158	51541	1498	2777	75523	28143	139.24
pref-d0.75	3489	109794	2661	4279	149581	71461	256.96
rnd-d0.25	1604	28367	960	2420	56342	10869	341.29
rnd-d0.5	2688	68953	1843	3664	115395	34074	870.87
rnd-d0.75	3901	130250	2747	4770	180502	70105	1180.02
twitt10.25	4628	153367	4607	4655	154355	152560	8.79
twitt10.5	4817	159901	4792	4837	160658	158939	9.11
twitt10.75	5665	191648	5665	5665	191648	191648	9.29
twitt20.25	4443	148065	4413	4469	148980	147007	7.77
twitt20.5	4595	153249	4566	4625	154311	152196	8.06
twitt20.75	5431	184481	5431	5431	184481	184481	8.13
twitt30.25	4191	140094	4166	4220	141094	139172	6.99
twitt30.5	4360	145765	4321	4391	146869	144362	7.13
twitt30.75	5159	175854	5159	5159	175854	175854	7.26
twitt40.25	4459	147854	4447	4471	148289	147384	7.80
twitt40.5	4637	153552	4624	4651	154067	153058	8.12
twitt40.75	5468	184486	5468	5468	184486	184486	8.16

The analysis of AlgA has shown similar spread of solutions for di erent runs, and were relatively stable. For algorithm AlgRR the results in Table 4 show signi cant di erence between minimum and maximum cardinalities of -rate dominating sets for dense networks pref-d and rnd-d. This indicates that the values of variables in the solutions obtained by LP relaxation are spread out over (0; 1) interval (i.e. are fractional). We have veri ed the spread and consistency by performing additional 200 runs for algorithm AlgRR where = 0:25 for pref-d and rnd-d networks and recordeF52 9.9ueetn

-rate dominating sets does not change signi cantly compared with the results obtained from 100 runs. Thus it can be concluded that more runs are unlikely to achieve better results.

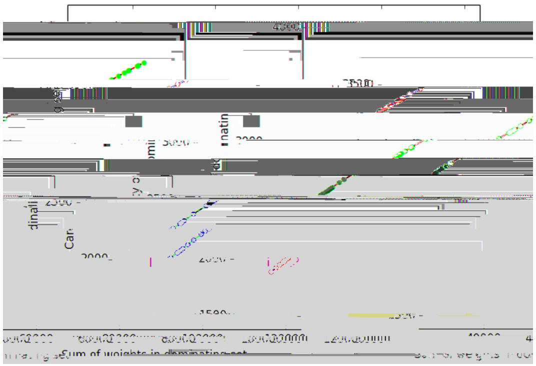

Fig. 1. Variation in size and weight for 100 runs of AlgRR, for rnd-d, $= 0.5$

On Figure 1 given are sizes and weights for 100 runs for AlgRR on rnd-d networks where $= 0.5$. Similar plots were obtained for all graphs and for both algorithms.

5 Conclusion

We have explored how to pick optimal sets of individuals for interventions in social networks. If each person in network has assigned a cost, the aim was to nd a group of people having minimum sum of costs so that each individual in network has at least 100 percent of its neighbourhood in this designated group. We presented a randomised algorithm for nding approximation of minimum weight rate domination set in graphs. We proved that this algorithm's output is within $O(\log_2$

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