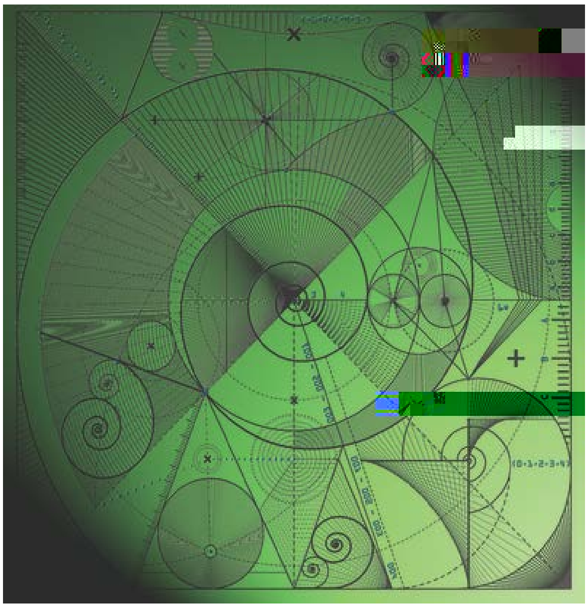

A SHAPE CALCULUS BASED METHOD FOR A TRANSMISSION PROBLEM WITH RANDOM INTERFACE

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Abstract. The present work is devoted to approximation of the statisti cal moments of the unknown solution of a class of elliptic transmission problems in $\mathbb{R}^3$ with uncertainly located transmission interfaces. Within this model, the di usion coe cient has a jump discontinuity acro ss the random transmission interface which models linear di u6.12487(n)2.80666(e)5.25645(a)633.88 -9.36016 Td TJ 219.6 0 Td [(i)-6.1264(c)5.25339(a)4.03117(l)-352.4(f)-7.23528(r)-7.5753(a)4.03117(n

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In this article we develop a deterministic method for numerical solution for a class of transmission problems with randomly perturbed interfaces. The equation to be solved is of the form

$$\nabla \quad (\nabla u) = f \quad \text{in } D \;;$$

where D is a random bounded domain in $\mathbb{R}^3$ and $D_+ = \mathbb{R}^3 \setminus \overline{D}$ is its complement. The domains share a common random surface , and the coefficient function takes (in general) distinct constant values in D and D_+, respectively. The solution u is subject to jump conditions across . A precise description of the model problem is deferred until Section 2.3, where a probabilistic perturbation model for the surface (and thus D) will be rigorously introduced. Within this model, the

with the natural inner product satisfying $\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k \rangle_{X^{(k)}} = \langle v_1, w_1 \rangle_X \cdots \langle v_k, w_k \rangle_X$.

Definition 2.1. For a random field $v \in L^k(\Omega; X)$, its k-order moment $\mathcal{M}^k[v]$ is an element of $X^{(k)}$ defined by

$$\mathcal{M}^k[v] := \int_\Omega \underbrace{v(\omega) \otimes \cdots \otimes v(\omega)}_{k\text{-times}} \, dP(\omega). \tag{2.4}$$

In the case $k = 1$, the statistical moment $\mathcal{M}^1[v]$ coincides with the mean value of v and is denoted by $\mathbb{E}[v]$. If $k \geq 2$, the statistical moment $\mathcal{M}^k[v]$ is the k

$D_+(!) := R^3 \; n \; \overline{D}$. The shape calculusin Section 3 requires a somewhat stronger smoothness assumption on , namely that the realizations of belong to $C^1(\;\, ^0)$. From (2.7) we observe that the mean random interface is represented by

$$E[\quad] = \quad x + \quad E[\;(x;\;)]n^0(x); \; x \; 2 \quad ^0 \; :$$

Without loss of generality, we may assume that the random perturbation amplitude $(x;!)$ is centered, i.e.,

$$E[\;(x;\;)] = 0 \qquad 8x \; 2 \quad ^0: \tag{2.9}$$

$s \in \mathbb{R}$, the Sobolev space $H^s_{mix}(G^k)$ is defined to be the space of all distributions $v(y_1,\dots,y_k)$ with $y_1,\dots,y_k \in G$ satisfying

$$\|v\|_{H^s_{mix}(G^k)} := \langle v,v\rangle^{1/2}_{H^s_{mix}(G^k)} < \infty ,$$
$$\langle v,w\rangle_{H^s_{mix}(G^k)} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\prod_{i=1}^{k} (1+\ell_i)^{2s} \right) \hat{v}_{\ell,m} \overline{\hat{w}_{\ell,m}} \tag{2.18}$$

with the Fourier coefficients

$$\hat{v}_{\ell,m} := \int_{x_1 \in S} \cdots \int_{x_k \in S} v(\ (x_1),\dots,\ (x_k)) \left(\prod_{i=1}^{k} Y_{\ell_i,m_i}(x_i) \right) d\sigma_{x_1} \cdots d\sigma_{x_k} \tag{2.19}$$

Recalling definition (2.3) we observe that $H^s_{mix}(G^k)$ is isometrically isomorphic to the tensor product space $H^s(G)^{(k)}$. These spaces will be identified in what follows. We also use the notation $H^s_{mix}(K^k)$ for the tensor product $H^s(K)^{(k)}$ where K is a compact subset of $\mathbb{R}^3$.

Sobolev spaces on bounded domains in $\mathbb{R}^3$ are defined, as usual, as spaces of all distributions whose partial derivatives are square integrable. Proper treatment of the transmission problem (2.11a)–(2.11d) in unbounded domains in $\mathbb{R}^3$ requires a special care. Following [17], for an unbounded domain $U \subset \mathbb{R}^3$ we introduce the space

$$H^1_w(U) := \left\{ v \in \mathcal{D}'(U) : \|v\|_{H^1_w(U)} = \left(\int_U |\nabla v|^2 + \frac{|v(x)|^2}{1+|x|^2}\, dx \right)^{1/2} < +\infty \right\}. \tag{2.20}$$

Specifically, for a given partition $\mathbb{R}^m$

) :

3.1. **Perturbation of deterministic interfaces.** In this subsection we collect several properties of perturbed interfaces which are important for the subsequent analysis. Assume that the perturbation function κ is a fixed deterministic function in $W^{1,\infty}(\Gamma^0)$, in particular κ is independent of ω. Then Γ_ε is defined by

$$\Gamma_\varepsilon := \{x + \varepsilon\kappa(x)n^0(x) : x \in \Gamma^0\}, \quad \varepsilon > 0. \tag{3.1}$$

As already noticed in Section 2.2, Γ_ε is a closed Lipschitz manifold in $\mathbb{R}^3$ provided $0 \le \varepsilon \le \varepsilon_0$ and ε_0 is sufficiently small. In this case Γ_ε introduces a decomposition of $\mathbb{R}^3$ into the interior and exterior subdomains D_-^ε and D_+^ε, respectively.

Following [20], we define a mapping $T_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ which transforms Γ^0 into Γ_ε and $D_\pm^0$ into $D_\pm^\varepsilon$, respectively, by

$$T_\varepsilon(x) := x + \varepsilon\tilde{\kappa}(x)\tilde{n}^0(x); \quad x \in \mathbb{R}^3; \tag{3.2}$$

where $\tilde{\kappa}$ and $\tilde{n}^0$ are any smoothness-preserving extensions of κ and n^0 into $\mathbb{R}^3$. We require in particular that $\tilde{\kappa} \in W^{1,\infty}(\mathbb{R}^3)$. Without loss of generality we assume that the extension $\tilde{\kappa}$ vanishes outside a sufficiently large ball $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ containing Γ_ε for any $0 \le \varepsilon \le \varepsilon_0$. This implies that the perturbation mapping $T_\varepsilon(x)$ is an identity in the complement B_R^c

where $_1$, $_2$, $_3$ are de ned by (3.9) and $_4$:= 0 for notational convenience later. In particular, for su ciently small > 0, there holds

$$(;x) = 1 + \ _1(x) + \ ^2\ _2(x) + \ ^3\ _3(x) \quad c > 0 \qquad 8x\ 2\ R^3: \tag{3.12}$$

Consider from now on su ciently small > 0. It follows from (3.10) and (3.12) that the ij -entry of the matrix $A(;x) \quad I$ is

$$A_{ij}\ (;\ ) \quad _{ij} = \quad (;\ )\ ^1 \sum_{n=1}^{4} \ ^n\ ^1\ h_{ijn} \quad _{ij}\ \ _n\ : \tag{3.13}$$

Hence, (3.11) yields

$$kA_{ij}\ (;\ ) \quad _{ij}\ k_{L^1(R^3)}\ ! \quad 0 \quad \text{as} \quad ! \quad 0;$$

proving (3.6).

From (3.13), we have

$$\frac{A_{ij}\ (;\ ) \quad _{ij}}{} = \quad (;\ )\ ^1 \sum_{n=1}^{4} \ ^n\ ^1\ h_{ijn} \quad _{ij}\ \ _n\ : \tag{3.14}$$

Taking the limit when goes to 0, noting that (;) ! 1, we obtain

(3.15) $A^0_{ij}\ (0;\ ) = h_{ij}\ _1 \quad _{ij}$ i4 Tf 4.43984 -4.2 Td [( )-0.936115]TJ /R17 6.97385 Tf 5.16016

Using the change of variables $y = T_\varepsilon(x)$ and noting (3.11), we have

$$\|v\circ T_\varepsilon\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |v(y)|^2\, \alpha(\varepsilon;(T_\varepsilon)^{-1}(y))^{-1}\, dy \le C\|v\|_{L^2(\mathbb{R}^3)}.$$

Therefore,

$$\lim_{\varepsilon\to 0}\Big\|\sqrt{1+|\xi|^2}\,\big(\alpha(\varepsilon;x)-1\big)(v\circ T_\varepsilon)\Big\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.18}$$

Furthermore, (3.3) also gives

$$\Big\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\Big\|_{L^2(\mathbb{R}^3)} = \Big\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\Big\|_{L^2(B_R)} \le \sqrt{1+R^2}\,\|v\circ T_\varepsilon - v\|_{L^2(B_R)}.$$

Note that $\lim_{\varepsilon\to 0}\|v\circ T_\varepsilon - v\|_{L^2(B_R)} = 0$ if v is continuous. By using a density argument we deduce that $\lim_{\varepsilon\to 0}\|v\circ T_\varepsilon - v\|_{L^2(B_R)} = 0$ for $v \in L^2(B_R)$. Hence,

$$\lim_{\varepsilon\to 0}\Big\|\sqrt{1+|\xi|^2}\,\big(v\circ T_\varepsilon - v\big)\Big\|_{L^2(\mathbb{R}^3)} = 0.$$

The above identity and (3.18) together with the triangle inequality give the required result. ∎

Lemma 3.4. *For any function $v \in H^1(\mathbb{R}^3)$, there holds*

$$\lim_{\varepsilon\to 0}\Big\|\sqrt{1+|\xi|^2}\Big(\frac{\alpha(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\Big)\Big\|_{L^2(\mathbb{R}^3)} = 0.$$

Proof. Noting (3.3), Lemma 3.1, (3.17) and the triangle inequality, we obtain

$$\begin{aligned}
&\Big\|\sqrt{1+|\xi|^2}\Big(\frac{\alpha(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\Big)\Big\|_{L^2(\mathbb{R}^3)}\\
&\quad= \Big\|\sqrt{1+|\xi|^2}\Big(\frac{\alpha(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\Big)\Big\|_{L^2(B_R)}\\
&\quad\lesssim \Big\|\frac{\alpha(\varepsilon;\cdot)(v\circ T_\varepsilon) - v}{\varepsilon} - \operatorname{div}(vV)\Big\|_{L^2(B_R)}
\end{aligned}$$

$$\le \Big\|\frac{\alpha(\varepsilon;\cdot)-1}{\varepsilon}(v\circ T_\varepsilon) - v\operatorname{div}V\Big\|_{L^2(B_R)} + \Big\|\frac{v\circ T_\varepsilon - v}{\varepsilon} - V\cdot\nabla v\Big\|_{L^2(B_R)}. \tag{3.19}$$

Recall from (3.9) that $\alpha_1 = \operatorname{div} V$. It follows from (3.12) that

$$\frac{\alpha(\varepsilon;\cdot)-1}{\varepsilon}(v\circ T_\varepsilon) - v\operatorname{div}V = \alpha_1(v\circ T_\varepsilon - v) + (\varepsilon\alpha_2 + \varepsilon^2\alpha_3)(v\circ T_\varepsilon).$$

Employing the density argument as in proof of Lemma 3.3, we obtain

$$\lim_{\varepsilon\to 0}\|\alpha_1(v\circ T_\varepsilon - v)\|_{L^2(B_R)} = 0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\|(\varepsilon\alpha_2 + \varepsilon^2\alpha_3)(v\circ T_\varepsilon)\|_{L^2(B_R)} = 0,$$

so that

lim T

3.2. Material and shape derivatives. In this subsection, for notational convenience we use the notation D for D or D_+, and $H^1(D)$ for $H^1(D)$ or $H^1_w(D_+)$.

Definition 3.5. For any sufficiently small , let v be an element in $H^1(D)$ or $H^{1/2}()$. The material derivative of v, denoted by $\underline{v}$, is defined by

$$\underline{v} := \lim_{\to 0} \frac{v \quad T \quad v^0}{}; \tag{3.20}$$

if the limit exists in the corresponding space $H^1(D^0)$ or $H^{1/2}(^0)$. The shape derivative of v is defined by

$$v^0 = \begin{cases} \underline{v} \quad \nabla v^0 \quad V & \text{if } v \in H^1(D); \\ \underline{v} \quad \nabla_0 v^0 \quad V & \text{if } v \in H^{1/2}(); \end{cases} \tag{3.21}$$

where ∇_0 denotes the surface gradient.

Lemma 3.6. If

(i) The material and shape derivatives of the product $v\, w$ are $\underline{v}w^0 + v^0\underline{w}$ and $v^0 w^0 + v^0 w^0$, respectively.

(ii) The material and shape derivatives of the quotient $v = w$ are $(\underline{v}w^0 \quad v^0\underline{w}) = (w^0)^2$ and $(v^0 w^0 \quad v^0 w^0) = (w^0)^2$, respectively, provided that all the fractions are well-defined.

(iii) If $v = v$ for all 0, then $\underline{v} = r\ v^0 \quad V = r\ v \quad V$ and $v^0 = 0$.

(iv) If

$$J_1(D\ ) := \int_D v\ dx\,;\ J_2(D\ ) := \int v\ d\,;\quad \text{and}\ dJ_i(D\ )j_{=0} := \lim_{!\ 0} \frac{J_i(D\ ) \quad J_i(D^0)}{}\,;\ i = 1\,;2;$$

then

$$dJ_1(D\ )j_{=0} = \int_{D^0} v^0 dx + \int_0 v^0 \quad V; n^0 \quad d$$

and

$$dJ_2(D\ )j_{=0} = \int_0 v^0 d \quad + \int_0 \frac{@v^0}{@n} + \operatorname{div}_0(n^0)\, v^0 \quad V; n^0 \quad d\,:$$

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Proof. Statements (i){(iii) can be obtained by using elementary calculations. Statement (iv) is proved in [20, pages 113, 116]. □

Lemma 3.9. The material and shape derivatives of the normal field n are given by

$$n\ ) \qquad \underline{n} = n^0 = \ r \quad {}_0\,:$$

v

$) =$

Proof. nWe start by recalling that the materi4(a)-5.8887(t)-419(t)2.56084(h)2 noting from (3.8) that

$$\lim_{\varepsilon \to 0} J_{T_\varepsilon}^{\top} = \lim_{\varepsilon \to 0} J_{T_\varepsilon} = I.$$

Since $I = J_{T_\varepsilon}^{-1}(T_\varepsilon(x))\, J_{T_\varepsilon}(x)$ for all $x \in \mathbb{R}^3$, we have $0 = \frac{d}{d\varepsilon}\left(J_{T_\varepsilon}^{-1} J_{T_\varepsilon}\right)\big|_{\varepsilon=0}$, which together with the product rule and (3.8) yields

$$\frac{d}{d\varepsilon} J_{T_\varepsilon}^{-\top}(T_\varepsilon(x))\Big|_{\varepsilon=0} = -(J_{T_0})^{-\top} \frac{d}{d\varepsilon}(J_{T_\varepsilon}^{\top})\Big|_{\varepsilon=0} (J_{T_0})^{-1} = -\frac{d}{d\varepsilon}(J_{T_\varepsilon})\Big|_{\varepsilon=0} = -J_V^{\top}; \tag{3.24}$$

We also have, using the fact that $|J_{T_0}^{-\top} n^0| = 1$,

$$\frac{d}{d\varepsilon} \left|J_{T_\varepsilon}^{-\top} n^0\right|\Big|_{\varepsilon=0} = \left|J_{T_0}^{-\top} n^0\right| \frac{d}{d\varepsilon}\left|J_{T_\varepsilon}^{-\top} n^0\right|\Big|_{\varepsilon=0} = \frac{1}{2}\frac{d}{d\varepsilon}\left|J_{T_\varepsilon}^{-\top} n^0\right|^2\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\left\langle \frac{d}{d\varepsilon}\left(J_{T_\varepsilon}^{-1} J_{T_\varepsilon}^{-\top}\right) n^0; n^0\right\rangle = -\frac{1}{2}\left\langle (J_V^{\top} + J_V)\, n^0; n^0\right\rangle. \tag{3.25}$$

Simple calculation reveals that

$$J_V^{\top} = \nabla \kappa\, (n^0)^{\top} \quad \text{and} \quad (J_V^{\top} + J_V)\, n^0 = \nabla \kappa + \langle \nabla \kappa; n^0\rangle\, n^0. \tag{3.26}$$

Inserting (3.24)–(3.26) into (3.23), we obtain

$$\dot{n} = -J_V^{\top} n^0 + \frac{1}{2}\left\langle (J_V^{\top} + J_V)\, n^0; n^0\right\rangle n^0 = -\nabla \kappa + \langle \nabla \kappa; n^0\rangle\, n^0 = -\nabla_{\Gamma^0} \kappa;$$

finishing the proof of the lemma. □

3.3. **Shape derivative of solutions of transmission problem.** In this subsection, we shall discuss the existence of material and shape derivatives of the solutions of transmission problems on perturbed interfaces. Consider a deterministic problem with respect to the reference interface Γ^0:

$$-\Delta u^0 = f \quad \text{in } D_-^0 \cup D_+^0; \tag{3.27a}$$

$$[u^0] = 0 \quad \text{on } \Gamma^0; \tag{3.27b}$$

$$\left[\frac{\partial u^0}{\partial n}\right] = 0 \quad \text{on } \Gamma^0; \tag{3.27c}$$

$$u^0(x) = O(|x|^{-1}) \quad \text{when } |x| \to \infty. \tag{3.27d}$$

The perturbed problem corresponding to the perturbed interface Γ_ε is given by

$$-\Delta u_\varepsilon = f \quad \text{in } D_-^\varepsilon \cup D_+^\varepsilon; \tag{3.28a}$$

$$[u_\varepsilon] = 0 \quad \text{on } \Gamma_\varepsilon; \tag{3.28b}$$

$$\left[\frac{\partial u_\varepsilon}{\partial n}\right] = 0 \quad \text{on } \Gamma_\varepsilon; \tag{3.28c}$$

$$u_\varepsilon(x) = O(|x|^{-1}) \quad \text{when } |x| \to \infty; \tag{3.28d}$$

where (cf. (2.10))

$$(x) = \begin{cases} \quad ; & x \in D \\ {}_{+} ; & x \in D_{+} : \end{cases}$$

Lemma 3.10. Suppose $f \in L^2(\mathbb{R}^3) \cap W_0$ and $\in C^1(\ ^0)$, then

$$\lim_{\to 0} \left\| u \quad T \quad u^0 \right\|_{W_0} = 0 : \tag{3.29}$$

Here, W_0 denotes the dual space of W_0 with respect to the L^2-inner product.

Proof. By multiplying both sides of (3.28a) with an arbitrary function $v \in C_0^1(\mathbb{R}^3)$ and integrating over $D \cup D_+$, we obtain

$$\int_{\mathbb{R}^3} f v \, dx = \int_{D} 4 u (x) v(x) \, dx \quad + \int_{D_+} 4 u (x) v(x) \, dx : \tag{3.30}$$

Applying Green's identity and noting (3.28c), we obtain

$$\int_{D_+ \cup D} (x) \nabla u (x) \nabla v(x) = \langle f, v \rangle_{L^2(\mathbb{R}^3)} \quad \forall v \in C_0^1(\mathbb{R}^3): \tag{3.31}$$

Since the space $C_0^1(\mathbb{R}^3)$ is dense in W (see [17, Remark 2.9.3]), there holds

$$\int_{D_+ \cup D} (x) \nabla u (x) \nabla v (x) = \langle f, v \rangle_{L^2(\mathbb{R}^3)} \quad \forall v \in W : \tag{3.32}$$

Choosing $v = u$ gives

$$|u|_W^2 \simeq \langle f, u \rangle_{L^2(\mathbb{R}^3)} \quad \|f\|_W \|u\|_W :$$

It follows from Lemma 2.2 that

$$\|u\|_W \lesssim \|f\|_W \simeq \|f\|_{W_0} : \tag{3.33}$$

On the other hand, using the change of variables $x = T(y)$ in (3.32), we have (noting that $(T(y)) = (y)$)

$$\int_{D_+^0 \cup D^0} (y) (\nabla w(y))^\top A(\,;y) \nabla (u \quad T)(y) \, dy = \int_{D_+^0 \cup D^0} f(T(y)) w(y) (\,;y) \, dy; \tag{3.34}$$

for any $w \in W_0$. We also obtain from problem (3.27a)–(3.27d)

$$\int_{D_+^0 \cup D^0} (y) (\nabla w(y))^\top \nabla u^0(y) \, dy = \int_{D_+^0 \cup D^0} f(y) w(y) \, dy; \tag{3.35}$$

for any $w \in W_0$. Subtracting (3.35) from (3.34) we deduce

$$\int_{D_+^0 \cup D^0} (y) \nabla w(y)^\top \nabla \quad (u \quad T)(y) \quad u^0$$

Subtracting (3.37) from (3.38) yields

$$
\begin{aligned}
&\int_{D_+^0 [D^0} (y)\, r \;(z\;(y) \quad z(y)) \;\; r\; w(y)\, dy \\
&= \int_{D_+^0 [D^0} (y) \;\; r \;(u \quad T\;)(y)\, \frac{A(\;;y) \quad I}{} \;\; r \quad u^0(y)A^0(0;y) \quad r\; w(y)\, dy \\
&\quad + \int_{D_+^0 [D^0} \frac{(\;;y)f\,(T\;(y)) \quad f\,(y)}{} \;\; \mathrm{div} \;\; V(y)f\,(y) \quad w(y)\, dy
\end{aligned}
$$

(3.39) $$=: I_1(w) + I_2(w):$$

The rst integral in the right hand side of (3.39) can be written as

$$
\begin{aligned}
I_1(w) = &\int_{D_+^0 [D^0} (y)\, r \;(u \quad T\;)(y)\, \frac{A(\;;y) \quad I}{} \;\; A^0(0;y) \quad r\; w(y)\, dy \\
+ &\int_{D_+^0 [D^0} (y)\, r \quad y
\end{aligned}
$$

and

$$dJ_2(D\ )j_{=0} = \int_{@\ _0} \left(u^0 \frac{@v}{@n} \quad u^0\ r\ _0 v\ \ r\ _0\ V;n^0 \right) d\ + \int_{\ _0} \frac{@}{@n}\left(u^0 \frac{@v}{@n} \right) V;n^0\ d$$

$$= \ \ + \int_{\ _0} \mathrm{div}\ _0(n^0)u^0 \frac{@v}{@n}\ V;n^0\ d\ ;$$

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since u^0 = u^0 on the interface ^{0} by (3.27b). Therefore, di erentiating by both sides of (3.46), using Green's formula, the jump condition (3.27c)9

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Differentiating by both sides, applying Lemma 3.8 we have

$$0 = d\int [u\,]\,v\,d = d\left(\int u\,v\,d \quad \int_{+} u_+\,v\,d\right)$$

$$= \int_0 (u^0 v)^0 + \int_0 \frac{\partial(u^0 v)}{\partial n}$$

and (3.55) follows. An analogous estimate holds for

$$M^k[u \quad E[u]] \quad {}^k M^k[u^0] = \quad {}^k \; M^k[u^0 + (h \quad E[h$$

These equalities together with (4.7) imply

$$u^0(x) = \begin{cases} (\nabla S_- - W_-)(u^0_-)(x) =: E_-(u^0_-)(x), & x \in D^0_-\\ (W_+ - \nabla S_+)(u^0_+)(x) =: E_+(u^0_+)(x), & x \in D^0_+. \end{cases} \tag{4.10}$$

The randomness of the interface $\Gamma(\omega)$ which is given via the randomness of the vector field $V(\cdot, x, \omega)$ implies the randomness in the solution u. From (4.10), we have

$$u^0(x,\omega) = \begin{cases} E_-(u^0_-(\omega)|_{\Gamma^0})(x), & x \in D^0_-,\\ E_+(u^0_+(\omega)|_{\Gamma^0})(x), & x \in D^0_+. \end{cases}$$

Tensorizing and integrating both sides of the above equation, we deduce

$$\mathrm{Cov}[u^0](x_1,x_2) = \begin{cases} (E_{-,x_1} \otimes E_{-,x_2})\mathrm{Cor}[u^0_-|_{\Gamma^0}](x_1,x_2), & x_1,x_2 \in D^0_-,\\ (E_{+,x_1} \otimes E_{+,x_2})\mathrm{Cor}[u^0_+|_{\Gamma^0}](x_1,x_2), & x_1,x_2 \in D^0_+, \end{cases} \tag{4.11}$$

and in general

$$M^k[u^0](x_1,\ldots,x_k) = \begin{cases} (E_{-,x_1} \otimes \cdots \otimes E_{-,x_k})M^k[u^0_-|_{\Gamma^0}](x_1,\ldots,x_k), & x_1,\ldots,x_k \in D^0_-,\\ (E_{+,x_1} \otimes \cdots \otimes E_{+,x_k})M^k[u^0_+|_{\Gamma^0}](x_1,\ldots,x_k), & x_1,\ldots,x_k \in D^0_+. \end{cases} \tag{4.12}$$

Equation (4.11) suggests that the covariance of the solution u^0 in D^0 can be computed from the correlation function of the Dirichlet data $u^0|_{\Gamma^0}$ on the transmission interface.

The jump conditions in (4.1) gives

$$u^0_-(\omega) = u^0_+(\omega) + g_D(\omega) \quad \text{on } \Gamma^0, \tag{4.13}$$

and

$$\underbrace{(\mathcal{S}_- + \mathcal{S}_+)}_{=:[\mathcal{S}]} u^0_+(\omega) = g_N(\omega) - (\;\mathcal{S}_-)g_D(\omega) \quad \text{on } \Gamma^0. \tag{4.14}$$

We note that for a fixed $\omega \in \Omega$, the right hand side $g_N(\omega) - (\;\mathcal{S}_-)g_D(\omega) \in H^{1/2}(\Gamma^0)$. The solution $u^0_+(\omega)$ of (4.14) belongs to $H^{1/2}(\Gamma^0)$

and $H^{1/2}(\Gamma)$-elliptic, i.e.

$$B(v,v) \geq C_2 \|v\|^2_{H^{1/2}(\Gamma')} \quad \forall v \in H^{1/2}(\Gamma'); \tag{4.18}$$

where the positive constants C_1 and C_2 are independent of v.

Proof. The boundedness of the bilinear form B is derived directly from the boundedness of V^{-1} and K. To prove ellipticity we first note that the hypersingular operator D is $H^{1/2}(\Gamma')$-semi-elliptic for all closed interface Γ', i.e.,

$$\langle Dv, v\rangle_{L^2(\Gamma')} \geq C |v|_{H^{1/2}(\Gamma')} \quad \forall v \in H^{1/2}(\Gamma'); \tag{4.19}$$

see e.g. [21, Corollary 6.25]. The Cauchy data $(u, \frac{\partial u}{\partial n})$ on Γ' satisfy

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}: \tag{4.20}$$

Substituting (4.8) into the second equation of (4.20) gives

$$\frac{\partial u}{\partial n} = D u + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) u \quad \text{on } \Gamma':$$

This equation and (4.8) yield

$$S_- = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K):$$

Noting that K' is the adjoint operator of K, we have

$$\langle S_- v, v\rangle = \langle Dv, v\rangle + \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}I + K)v\rangle \quad \forall v \in H^{1/2}(\Gamma'): \tag{4.21}$$

Similarly, the exterior Dirichlet-to-Neumann operator S_+ satisfies

$$S_+ = -D - (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)$$

and

$$\langle S_+ v, v\rangle = -\langle D v, v\rangle - \langle V^{-1}(\frac{1}{2}I - K)v, (\frac{1}{2}I - K)v\rangle \quad \forall v \in H^{1/2}(\Gamma'): \tag{4.22}$$

From (4.21), (4.22), (4.19) and noting the $H^{1/2}$-ellipticity of the inverse operator of V, we derive

$$\langle [S]v, v\rangle = ({}_- + {}_+)\langle Dv, v\rangle + \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}$$

hS

implies

Lemma 4.2. The bilinear form $B(\cdot\,;\cdot) : H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0) \times H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0) \to \mathbb{R}$ is bounded and $H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0)$-elliptic, i.e.,

$$B(v;w) \le C_1 \|v\|_{H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0)} \|w\|_{H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0)}; \tag{4.28}$$

and

$$C_2 \|v\|^2_{H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0)} \le B(v;v) \tag{4.29}$$

for all $v;w \in H^{1/2}_{mix}(\Gamma_0 \times \Gamma_0)$. By Lemma 4.2 there exists a unique solution of (4.27).

5. Examples. In this section, we consider the transmission problem (2.11a)–(2.11d) where the random interface $\Gamma(\omega)$ is given by

$$\Gamma(\omega) = \{x + \varepsilon(x;\omega)n(x) : x \in \mathbb{S}\}.$$

Here, the reference interface Γ^0 is the unit sphere $\mathbb{S}$. The perturbation parameter $\varepsilon(x;\omega) = \varepsilon a(\omega)$, where $a(\omega)$ is uniformly distributed in $[-1;1]$. The mean value $\mathbb{E}[\varepsilon] = 0$ and the covariance $\mathrm{Cov}[\varepsilon](x;y) = \mathrm{Cor}[\varepsilon](x;y) = 1/3$. The interface $\Gamma(\omega)$ is a sphere of radius $R(\omega) = 1 + \varepsilon a(\omega)$.

5.1. Analytic example. Firstly, we choose the right hand side f to be

$$f(x) = \begin{cases} (4r_x^2 - 1)^2 & \text{if } 0 \le r_x \le 1/2; \\ 0 & \text{if } 1/2 \le r_x; \end{cases}$$

where $r_x = |x|$. Then solution of the transmission problem with respect to the random interface $\Gamma(\omega)$ can be analytically computed as follows:

$$u(x;\omega) = \begin{cases} \frac{1}{\ }\left(\frac{8}{21}r_x^6 - \frac{2}{5}r_x^4 + \frac{r_x^2}{6}\right) - \frac{3}{105}r_x - \frac{23}{840} + \frac{+}{105\ \ +R(\omega)} & \text{if } 0 \le r_x \le \frac{1}{2}; \\ \frac{1}{105\ \ r_x} + \frac{+}{105\ \ +R(\omega)} & \text{if } \frac{1}{2} \le r_x \le R(\omega); \\ \frac{1}{105\ +r_x} & \text{if } R(\omega) \le r_x: \end{cases} \tag{5.1}$$

In particular, the exact solution u^0 of the transmission problem on the reference interface Γ^0 ... $[0; 1] =$

0 6 rr, tx 1 [( )-5.92546TJ ET Q 1 q 40.8 0 0 -4.8 2459.88 3711.88 cm28f +

We then compute the covariance of the solution u by elementary calculations, noting (5.1), to obtain

(5.4) $$\mathrm{Cov}_u(x; y) = \Big(\frac{1}{3} \frac{[\quad]^2}{(105}$$

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