Variations on the sum-product problem

by

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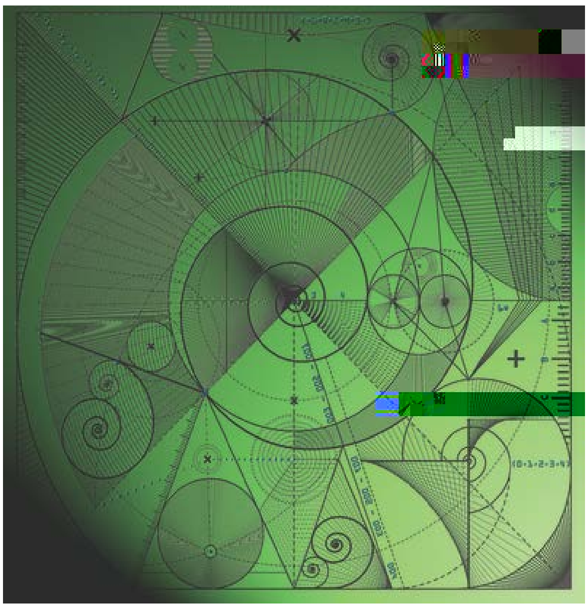

Variations on the sum-product problem

Brendan Murphy, Oliver Roche-Newton and Ilya Shkredov

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Abstract

This paper considers various formulations of the sum-product problem. It is shown that, for a finite set $A \subset \mathbb{R}$,

$$|A(A+A)| \gg |A|^{\frac{3}{2}+\frac{1}{178}},$$

giving a partial answer to a conjecture of Balog. In a similar spirit, it is established that

$$|A(A+A+A+A)| \gg \frac{|A|^2}{\log |A|},$$

a bound which is optimal up to constant and logarithmic factors. We also prove several new results concerning sum-product estimates and expanders, for example, showing that

$$|A(A+a)| \gg |A|^{3/2}$$

holds for a typical element of A.

1 Introduction

Given a finite set $A \subset \mathbb{N}$, one can define the *sum set*, and respectively the *product set*, by

$$A+A := \{a+b : a,b \in A\}$$

and

$$AA := \{ab : a,b \in A\}.$$

The Erdős-Szemerédi [7] conjecture states, for all $\epsilon > 0$,

$$\max\{|A+A|,|AA|\} \gg |A|^{2-\epsilon},$$

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and it is natural to extend this conjecture to other fields, particularly the real numbers. In this direction, the current state-of-the-art bound, due to Solymosi [23], states that for any $A \subset \mathbb{R}$

$$\max\{|A+A|, |AA|\} \gg \frac{|A|^{4/3}}{(\log|A|)^{1/3}}. \quad (1)$$

When looking to construct a set A which generates a very small sum set $A+A$, one needs to impose an additive structure on A, and an additive progression is an example of a highly additively structured set. Similarly, if A has a very small product set, it must be to some extent multiplicatively structured. Loosely speaking, the Erdős-Szemerédi conjecture reflects the intuitive observation that a set of integers, or indeed real numbers, cannot be highly structured in both a multiplicative and additive sense.

In this paper, we consider other ways to quantify this observation. In particular, one would expect that a set will grow considerably under a combination of additive and multiplicative operations. Consider the set

$$A(A+A) := \{a(b+c) : a, b, c \in A\}.$$

The same heuristic argument as the above leads us to expect that this set will always be titatively improved results. Using a straightforward application of the Szemerédi-Trotter theorem[3], one can show that

$$|A(A+A)| \gg |A|^{3/2}. \tag{3}$$

The original aim here was to improve on this lower bound, which we do by proving[4] that

$$|A(A+A)| \gtrsim |A|^{\frac{3}{2}+\frac{1}{178}}. \tag{4}$$

Although the method leads only to a small improvement for this problem, it turns out to be much more effective when more variables are involved. To this end we prove the following result:

$$|A(A+A+A+A)| \gg \frac{|A|^2}{\log|A|}. \tag{5}$$

Observe that this bound is tight, up to logarithmic factors, in the case when A is an arithmetic progression. Indeed, the aforementioned work of Ford tells us that some logarithmic factor is necessary here. The set $A(A+A+A+A)$ has similar characteristics to $A(A+A)$, and inequality (5) proves a weak version of Balog's conjecture.

In a slight generalisation of the earlier definition, the multiplicative energy of A and B, denoted $E(A,B) = E_2(A,B)$, is defined to be the number of solutions to the equation

$$a_1 b_1 = a_2 b_2,$$

such that $a_i \in A$ and $b_i \in B$. This quantity is also the number of solutions to

$$\frac{a_1}{a_2} = \frac{b_2}{b_1}$$

and

$$\frac{a_1}{b_2} = \frac{a_2}{b_1}.$$

Observe that $E(A,B)$ can also be defined in terms of the representation function r as follows:

$$\begin{aligned} E(A,B) &= \sum_x r^2_{A:B}(x) \\ &= \sum_x r_{A:A}(x) r_{B:B}(x) \\ &= \sum_x r^2_{AB}(x). \end{aligned}$$

We use $E(A)$ as a shorthand for $E(A,A)$.

One of the fundamental basic properties of the multiplicative energy is the following well-known lower bound:

$$E(A,B) \geq \frac{|A|^2|B|^2}{|AB|}. \qquad (8)$$

The proof is short and straightforward, arising from a single application of the Cauchy-Schwarz inequality. The full details can be seen in Chapter 2 of [27].

The above definitions can all be extended in the obvious way to define the *additive energy* of A and B, denoted $E^+(A,B)$. So,

$$E^+(A,B) := \sum_x r^2_{A-B}(x).$$

The *third moment multiplicative energy* is the quantity

$$E_3(A) := \sum_x r^3_{A:A}(x); A) := \sum_x r^2_{A} \quad (x).$$

We will use the Katz–Koester trick [14], which is the observation that

$$|(A+A)\cap(A+A-s)| \geq |A+A_s|,$$

and

$$|(A-A)\cap(A-A-s)| \geq |A-(A\cap(A+s))|,$$

where $A_s = A\cap(A-s)$. We also need the following identity (see [22], Corollary 2.5)

$$\sum_s |A - A_s| = |A^2 - \Delta(A)|, \tag{9}$$

where

$$\Delta(A) = \{(a,a) : a \in A\}.$$

2 Statement of results

2.1 Preliminary Results - Applications of the Szemerédi-Trotter Theorem

The most important ingredient for the sum-product type results in this paper is the Szemerédi-Trotter Theorem [26]:

Theorem 2.1. Let $P \subseteq \mathbb{R}^2$ be a finite set of points and let L be a collection of lines in the real plane. Then

$$I(P;L) := |\{(p,l) \in P \times L : p \in l\}| \ll |P|^{2/3}|L|^{2/3} + |L| + |P|.$$

Here by $I(P;L)$ we denote the number of incidences between a set of points P and a set of lines L. Given a set of lines L, we call a point that is incident to at least t lines of L a t-rich point, and we let P_t denote the set of all t-rich points of L. The Szemerédi-Trotter theorem implies a bound on the number of t-rich points:

Corollary 2.2. Let L be a collection of lines in $\mathbb{R}^2$, let $t \geq 2$ be a parameter and let P_t be the set of all t-rich points of L. Then

$$|P_t| \ll \frac{|L|^2}{t^3} + \frac{|L|}{t}.$$

Further, if no point of P_t is incident to more than $|L|^{1/2}$ lines, then

$$|P_t| \ll \frac{|L|^2}{t^3}.$$

This result is used to prove the main preliminary results in this paper, which give us information about various kinds of energies.

Lemma 2.3. Let A, B and X be finite subsets of $\mathbb{R}$ such that $|X| \leq |A||B|$. Then

$$\sum_{x\in X} E^+(A, xB) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2}.$$

Note that $E^{+}(A; xB) \quad |A||B|$ for all x, so the condition $|X| \quad |A||B|$ is necessary. Bourgain formulated a similar theorem ("Theorem C" of [2]) for subsets of fields with prime cardinality. Bourgain's theorem is closely related to the Szemerédi-Trotter theorem for finite fields [5, 11].

Theorem 2.9. Let $A \subseteq \mathbb{R}$ be a finite set. Then there exists a subset $A' \subseteq A$, such that $|A'| \geq \frac{|A|}{2}$, and for all $a \in A'$,

$$|A(A+a)| \gg |A|^{3/2}.$$

Adding more variables to our set leads to better lower bounds:

Theorem 2.10. Let $A \subseteq \mathbb{R}$ be a finite set. Then there exists a subset $A' \subseteq A$ with cardinality $|A'| \geq \frac{|A|}{2}$, such that for all $a \in A'$,

$$|(A+A)(A+a)| \gg \frac{|A|^{5/3}}{(\log |A|)^{1/3}}.$$

Theorem 2.10 is similar to the result of Theorem 2.6, especially if we think of the set $A(A+A)$ in the terms $(A+0)(A+A)$. This result tells us that we can usually do better than Theorem 2.6 if 0 is replaced by an element of A.

The next theorem is quantitatively worse than Theorem 2.10, but is more general, since it applies not only for most $a \in A$, but to all real numbers except for a single problematic value.

Theorem 2.11. Let $A \subseteq \mathbb{R}$ be a finite set. Then, for all but at most one value $x \in \mathbb{R}$,

$$|(A+A)(A+x)| \gg \frac{|A|^{11/7}}{(\log |A|)^{3/7}}. \tag{12}$$

Unfortunately, this does not lead to an improvement to Theorem 2.6, since the single bad x that violates (12) may be equal to zero.

2.4 Further results

Finally, we formulate a theorem of a slightly different nature.

Theorem 2.12. Let $A, B \subseteq \mathbb{R}$ be finite sets.

Then

$$|A+B|^3 \gg \frac{|B| E^{*}(A)}{\log |A|} \geq \frac{|A|^4 |B|}{|AA^{-1}| \log |A|}, \tag{13}$$

and

$$|B+AA|^3 \gg \frac{|B||A|^{12}}{(E_3(A))^2 |AA^{-1}| \log |A|}. \tag{14}$$

Let us say a little about the meaning of these two bounds. If we fix $A = B$, then (13) tells us that $|AA|$ is very large if $|A+A|$ is very small. Similar results are already known; for example, a quantitatively improved version of this statement is a consequence of Solymosi's sum-product estimate in [23]. The benefit of (13) is that it also works for a mixed sum set $A+B$.

One of the main objectives of this paper is to study the set $A(A+A)$, and inequality (14) considers the dual problem of the set $A+AA$. As stated earlier, it is easy to show that $|A+AA| \gg |A|^{3/2}$. If we fix $A = B$ in (14), then this bound gives an improvement in the case when $E_3(A)$ is small. We hope to carry out (cl

3 Proofs of Preliminary Results

Proof of Lemma 2.3

Recall that Lemma 2.3 states that for $|X| \leq |A||B|$,

$$\sum_{x \in X} E^+(A; xB) \lesssim |A|^{3/2}|B|^{3/2}|X|^{1/2}.$$

Note that

$$\sum_{x \in X} E^+(A; xB) = \sum_{x \in X} \sum_{y} r^2_{A+xB}(y). \tag{15}$$

We will interpret $r_{A+xB}(y)$ geometrically and use corollary 2.2 to show that there are not too many pairs $(x; y)$ for which the quantity $r_{A+xB}(y)$ is large.

Claim. Let $R_t = \{(x; y) : r_{A+xB}(y) \geq t\}$. Then for any integer $t \geq 2$,

$$|R_t| \lesssim |A|^2$$

To bound the first term in (18), observe that

$$\sum_{x\in X}\sum_{y: r_{A+xB}(y)\le\Delta} r^2_{A+xB}(y) \le \Delta\sum_{x\in X}\sum_{y} r_{A+xB}(y) \tag{19}$$

$$= \Delta|A||B|\sum_{x\in X}1 \tag{20}$$

$$= |A||B|\Delta|X|. \tag{21}$$

To bound the second term in (18), we decompose dyadically and then apply (16) to bound the size of the dyadic sets we are summing over:

$$\sum_{(x,y): r_{A+xB}(y)>\Delta} r^2_{A+xB}(y) = \sum_{j\ge 1}\ \sum_{(x,y): \Delta 2^{j-1}<r_{A+xB}(y)\le\Delta 2^j} r^2_{A+xB}(y) \tag{22}$$

$$\ll \sum_{j\ge 1}\frac{|A|^2|B|^2}{(\Delta 2^j)^3}(\Delta 2^j)^2 \tag{23}$$

$$= \frac{|A|^2|B|^2}{\Delta}\sum_{j\ge 1}\frac{1}{2^j} \tag{24}$$

$$= \frac{|A|^2|B|^2}{\Delta}. \tag{25}$$

For an optimal choice, set the parameter $\Delta = \left\lceil\frac{|A|^{1/2}|B|^{1/2}}{|X|^{1/2}}\right\rceil \approx \frac{|A|^{1/2}|B|^{1/2}}{|X|^{1/2}} > 1$. The approximate equality here is a consequence of the assumption $\frac{|A|^{1/2}|B|^{1/2}}{|X|^{1/2}} > 1$.

Combining the bounds from (21) and (25) with (18), it follows that

$$\sum_{x\in X} E^{+}(A;xB) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2},$$

as required.

This completes the proof of Lemma 2.3.

□

The proof of Lemma 2.4 is essentially the same, with the roles of addition and multiplication reversed. For completeness, a full proof is provided.

Proof of Lemma 2.4

Recall that Lemma 2.4 states that for $|X|\le|A||B|$,

$$\sum_{x\in X} E^{\times}(A;B+x) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2}.$$

Define a set of lines $L := \{l_{a,b} : (a,b)\in A\times B\}$, where $l_{a,b}$ now represents the line with equation $y = a(b+x)$. These lines are all distinct and so $|L| = |A||B|$. Since $r_{A(B+x)}(y)$ is the number of such lines incident to a point $(x;y)$, we can apply Corollary 2.2 and argue as before to show that

$$|\{(x;y) : r_{A(B+x)}(y) \quad t\}| \quad \frac{|A|^2|B|^2}{t^3}; \qquad (26)$$

for any integer $t \quad 1$.

Next, we use the bound (26) in the following calculation, which holds for any integer $4 > 1$:

$$\sum_{x \in X} E(A;B+x) = \sum_{x \in X}\sum_{y} r^2_{A(B+x)}(y)$$

$$\sum_{x \in X}\ \sum_{y : r_{A(}}$$

respectively, provided that $|X| \quad |A||B|$. Putting $B = A$ and $X = (A \quad A)/(A \quad A)$ into (30) proves (27). Similarly, putting $B = A$ and

$$X = \frac{a_2 b_2 \quad a_1 b_1}{a_1 \quad a_2} : a_1; a_2 \in A;\ b_1; b_2 \in B$$

into (31), we obtain (28).

Let $D = A \quad A$ Taking $A = B = D$, $X = D/D$, summing just over $x; y \in D$ in (30), and using Katz–Koester trick as well as identity (9), we get

$$|A \quad A|^3 \quad \frac{A \quad A}{A \quad A}^{1/2} \quad \sum_{x \in D} r_{D \quad D}(x) \Bigg)^2 \quad \sum_{x \in D} |A \quad A_x| \Bigg)^2 = |A|^2 \quad (A)|^2$$

which coincides with (29). □

Inequality (27) can also be deduced from Beck's Theorem, which states that a set of N points in the plane which does not have a single very rich line, will determine (N^2) distinct lines. See Exercise 8.3.2 in [27]. A geometric result of Ungar [29], concerning the number of different directions determined by a set of points in the plane, also yields (27) as a corollary. Although the result here is not new, it has been stated in order to illustrate the sharpness of Lemma 2.3. Similar results to (28) were established in [12]; it seems likely that (28) is suboptimal.

Proof of Lemma 2.5

Recall that Lemma 2.5 states that

$$E\ (A)|A(B + C)|^2 \quad \frac{|A|^4|B||C|}{\log|A|}.$$

Let S

Now we apply the Cauchy-Schwarz inequality:

$$\frac{1}{4}(|A||B||C|)^2 \le \left(\sum_{x \in A(B+C)\setminus\{0\}} r_{A(B+C)}(x) \right)^2 \tag{33}$$

$$\le |A(B+C)| \sum_{x \neq 0} r^2_{A(B+C)}(x) \tag{34}$$

$$= |A(B+C)| S^*. \tag{35}$$

The rest of the proof is concerned with finding a satisfactory upper bound for the quantity S^*. We will eventually conclude that

$$S^* \ll E(A)^{1/2}|B|^{3/2}|C|^{3/2}(\log|A|)^{1/2}. \tag{36}$$

If this is proven to be true, one can combine the upper and lower bounds on S^* from (36) and (35) respectively, and then a simple rearrangement completes the proof of the lemma.

It remains to prove (36). To do this, first observe that (32) can be rewritten in the form

$$\frac{a_1}{a_2} = z = \frac{b_2 + c_2}{b_1 + c_1}.$$

Note that we can divide by $b_1 + c_1$

□

Now we will prove the claimed estimate for the distribution of $r_Q(z)$.

Proof of Claim. First we will get an easy estimate for $|Z_t|$ from Markov's inequality. Since[7]

$$t|Z_t| \le \sum_{z\in Z_t} r_Q(z) \le \sum_{z\in Q} r_Q(z) = |B|^2|C|^2;$$

we have

$$|Z_t| \le \frac{|B|^2|C|^2}{t}. \tag{38}$$

Note that if $|Z_t| \ge |B||C|$, then it follows from (38) that $t \le |B||C|$. But then

$$\frac{|B|^2|C|^2}{t} \le \frac{|B|^3|C|^3}{t^2};$$

so we have proved the claim in the case $|Z_t| \ge |B||C|$.

Now we will prove the claim when $|Z_t| \le |B||C|$ using Lemma 2.3. To do this we make a key observation, which is inspired by the Elekes-Sharir set-up from [17]: every solution of the equation

$$z = \frac{b_2 + c_2}{b_1 + c_1}$$

is a solution to the equation

$$b_2 - zc_1 = zb_1 - c_2 = y$$

for some y. Thus

$$r_Q(z) \le \sum_y r_{zB-C}(y)r_{B-zC}(y).$$

By the arithmetic-geometric mean inequality

$$r_{zB-C}(y)r_{B-zC}(y) \le \frac{r^2_{zB-C}(y) + r^2_{B-zC}(y)}{2};$$

so

$$r_Q(z) \le \frac{E^+(zB;-C) + E^+(B;-zC)}{2}.$$

Now if $|Z_t| \le |B||C|$, we can sum over Z_t and apply Lemma 2.3:

$$t|Z_t| \le \sum_{z\in Z_t} r_Q(z) \le \frac{1}{2}\sum_{z\in Z_t} E^+(zB;-C) + \frac{1}{2}\sum_{z\in Z_t} E^+(B;-zC) \lesssim |B|^{3/2}|C|^{3/2}|Z_t|^{1/2}.$$

Rearranging yields the estimate

$$|Z_t| \lesssim \frac{|B|^3|C|^3}{t^2};$$

as claimed. □

[7] $r_Q(z)$ is supported on Q, so if $t \ge 1$ we have $Z_t \subseteq Q$.

We remark here that this is not the only proof we have found of Lemma 2.5 during the process of writing this paper. In particular, it is possible to write a "shorter" proof which is a relatively straightforward application of an upper bound from [17] on the number of solutions to the equation

$$(a_1 - b_1)(c_1 - d_1) = (a_2 - b_2)(c_2 - d_2);$$

such that $a_i \in A, \dots, d_i \in D$.

Although this proof may appear to be shorter, it relies on the bounds from [17], which in turn rely on the deeper concepts used by Guth and Katz [10] in their work on the Erdős distinct distance problem. For this reason, we believe that this proof is the more straightforward option. In addition, this approach leads to better logarithmic factors and works over the complex number (see the discussion at the end of the paper).

The following corollary gives an analogous result for third moment multiplicative energy, however, unlike Lemma 2.5, this result does not appear to be optimal.

Corollary 3.2. For any finite sets $A, B, C \subseteq \mathbb{R}$, we have

$$E_3(A)|A(B+C)|^4 \gg \frac{|A|^6|B|^2|C|^2}{(\log|A|)^2}.$$

Proof. By the Cauchy-Schwarz inequality,

$$\sum_x r^2_{A:A}(x) = \sum_x r^{3/2}_{A:A}(x) r^{1/2}_{A:A}(x) \le \left(\sum_x r^3_{A:A}(x)\right)^{1/2} \sum_x r$$

and

$$|A' - A'| \lesssim K^4 \frac{|A'|^3}{|A|^2}.$$

We remark that the first preprint of this paper used a different version of the Balog-Szemerédi-Gowers Theorem, due to Schoen [18]. Shortly after uploading this, we were informed by M. Z. Garaev of a quantitatively improved version of the Balog-Szemerédi-Gowers Theorem, in the form of Theorem 4.1. This leads to a small improvement in the statement of Theorem 2.6, since our earlier result had an exponent of $\frac{3}{2} + \frac{1}{234}$. The proof of Theorem 4.1 result is short, arising from an application of Lemmas 2.2 and 2.4 in [3]. It is possible that further small improvements can be made to Theorem 2.6 by combining more suitable versions of the Balog-Szemerédi-Gowers Theorem with our approach.

We will also need a sum-product estimate which is effective in the case when the product set or ratio set is relatively small. The best bound for our purposes is the following (see [16], Theorem 1.2):

Theorem 4.2. Let $A \subset \mathbb{R}$. Then

$$|A : A|^{10}|A + A|^9 \gtrsim |A|^{24}.$$

Proof of Theorem 2.6

Recall that Theorem 2.6 states that

$$|A(A + A)| \gtrsim |A|^{\frac{3}{2} + \frac{1}{178}}.$$

Write $E(A) = \frac{|A|^3}{K}$. Applying Lemma 2.5 with $A = B = C$, it follows that

$$\frac{|A|^3}{K}|A(A + A)|^2 \gtrsim |A|^6,$$

and so

$$|A(A + A)| \gtrsim |A|^{3/2}K^{1/2}. \qquad (39)$$

On the other hand, by Lemma 4.1, there exists a subset $A' \subset A$ such that

$$|A'| \quad |A|$$

so that after rearranging, and applying the crude bound $|A'| \leq |A|$, we obtain

$$K^{40}|A'+A'|^{9} \gtrsim \frac{|A|^{20}}{|A'|^{6}} \geq |A|^{14}$$

Using another crude bound,

$$|A(A+A)| \geq |A+A| \geq |A'+A'|; \qquad (42)$$

yields

$$|A(A+A)| \gtrsim \frac{|A|^{14/9}}{K^{40/9}}. \qquad (43)$$

Finally, we note that the worst case occurs when $K \approx |A|^{\frac{1}{89}}$. If $K \leq |A|^{\frac{1}{89}}$, then (39) implies that

$$|A(A+A)| \gtrsim |A|^{3/2}K^{1/2} \geq |A|^{\frac{3}{}}$$

Proof of inequality (44). To get "$_0$ we need to improve (42), that is to show jA(A +

Proof of Theorem 2.8

Recall that Theorem 2.8 states that

$$|A(A+A+A)| \gtrsim |A|^{\frac{7}{4}+\frac{1}{284}}.$$

For the ease of the reader, we begin by writing down a short proof of the fact that

$$|A(A+A+A)| \gtrsim \frac{|A|^{7/4}}{(\log|A|)^{3/4}}. \tag{51}$$

First note that, since $r_{A:A}(x) \leq |A|$ for any x,

$$E_3(A) = \sum_{x \in A:A} r^3_{A:A}(x) \leq |A| \sum_{x \in A:A} r^2_{A:A}(x) = |A|E(A); \tag{52}$$

so that (50) yields

$$E_3(A) \ll |A||A+A|^2 \log|A|. \tag{53}$$

Now, apply Corollary 3.2, with $B = A$ and $C = A + A$. We obtain

$$E_3(A)|A(A+A+A)|^4 \gg \frac{|A|^8|A+A|^2}{(\log|A|)^2}.$$

Combining this with the upper bound on $E_3(A)$ from (53), it follows that

$$|A(A+A+A)| \gg \frac{|A|^{7/4}}{(\log|A|)^{3/4}};$$

which proves (51).

Now, we will show how a slightly more subtle argument can lead to a small improvement in this exponent. Apply (50) and Lemma 2.5, with $B = A$ and $C = A + A$, so that

$$|A|^5|A+A| \lesssim E(A)|A(A+A+A)|^2 \lesssim |A+A|^2|A(A+A+A)|^2; \tag{54}$$

and thus

$$|A+A||A(A+A+A)|^2 \gtrsim |A|^5. \tag{55}$$

Write $E(A) = \frac{|A|^3}{K}$, for some value $K \geq 1$. By the first inequality from (54), it follows that

$$|A(A+A+A)| \gtrsim |A|K^{1/2}|A+A|^{1/2}. \tag{56}$$

Applying Solymosi's bound for the multiplicative energy then yields

$$|A(A+A+A)| \gtrsim |A|^{7/4}K^{1/4}. \tag{57}$$

Now, by Theorem 4.1 there exists a subset $A' \subseteq A$ such that

$$|A'| \gtrsim \frac{|A|}{K} \tag{58}$$

and

$$|A' : A'| \lesssim K^4 \frac{|A'|^3}{|A|^2}. \tag{59}$$

By Theorem 4.2 and (59),

$$|A'|^{24} \lesssim |A' + A'|^9 |A' : A'|^{10}$$
$$\lesssim |A + A|^9 K^{40} \frac{|A'|^{30}}{|A|^{20}};$$

and then

$|A + A|$ [redacted]

From the latter inequality we now have $|A + A|$ [redacted] $K^{40/9}$. Comparing this with (56) leads to the following bound:

$$|A(A + A + A)| \gtrsim \frac{|A|^{16/9}}{K^{31/18}}. \qquad (60)$$

The worst case occurs when $K \approx |A|^{1/71}$. It can be verified that if $K < |A|^{1/71}$, then

$$|A(A + A + A)| \gtrsim |A|^{\frac{7}{4}+\frac{1}{284}};$$

by inequality (60). On the other hand, if $K \geq |A|^{1/71}$, then it follows from inequality (57) that

$|A(A +$ [redacted]

Therefore, we have proved that (10) [redacted] which concludes the proof.

□

5 Proofs of Results on Products of Translates

We record a short lemma which will be used in the proofs of Theorem 2.10 and 2.11

Lemma 5.1. Let $A \subset \mathbb{R}$, $|\ldots|^4 \ldots |A|$ ...

Proof of Theorem 2.9

Recall that Theorem 2.9 states that

$$|A(A+a)| \gg |A|^{3/2}$$

holds for at least half of the elements a belonging to A. Lemma 2.4 tells us that, for some fixed constant C

$$\sum_{a \in A} E(A; a+A) \leq C|A|^{7/2}.$$

Let $A' \subseteq A$ be the set

$$A' := \{a \in A : E(A; a+A) \leq 2C|A|^{5/2}\},$$

and observe that

$$2C|A|^{5/2}|A \setminus A'| \leq \sum_{a \in A \setminus A'} E(A; a+A) \leq C|A|^{7/2},$$

which implies that

$$|A \setminus A'| \leq \frac{|A|}{2}.$$

This implies that $|A'| \geq \frac{|A|}{2}$. To complete the proof, we will show that for every $a \in A'$ we have $|A(A+a)| \gg |A|^{3/2}$. To see this, simply observe that, for any $a \in A'$,

$$\frac{|A|^4}{|A(A+a)|} \leq E(A; A+a) \ll |A|^{5/2}.$$

The lower bound here comes from (8), whilst the upper bound comes from the definition of A'. Rearranging this inequality gives

$$|A(A+a)| \gg |A|^{3/2},$$

as required.

□

We remark that it is straightforward to adapt this argument slightly—switching the roles of addition and multiplication and using Lemma 2.3 in place of Lemma 2.4—in order to show that there exists a subset $A' \subseteq A$, such that $|A'| \geq \frac{|A|}{2}$, with the property that

$$|A + aA| \gg |A|^{3/2},$$

for any $a \in A'$.

It is also easy to adapt the proof of Theorem 2.9 in order to show that, for any $0 < \epsilon < 1$ and any $A \subseteq \mathbb{R}$, there exists a subset $A' \subseteq A$ such that $|A'| \geq (1-\epsilon)|A|$, and for all $a \in A'$,

$$|A(A+a)| \gg_\epsilon |A|^{3/2}.$$

In other words, the set $A(A+a)$ is large for all but a small positive proportion of elements $a \in A$. The analogous statement for $A + aA$ is also true.

Proof of Theorem 2.10

Recall that Theorem 2.10 states that

$$|(A+a)(A+A)| \gg \frac{|A|^{5/3}}{(\log|A|)^{1/3}}$$

holds for at least half of the elements a belonging to A. This proof is similar to the proof of Theorem 2.9. Again, Lemma 2.4 tells us that for a fixed constant C, we have

$$\sum_{a\in A} E(A+A, a+A) \leq C|A|^2|A+A|^{3/2}.$$

Define $A' \subseteq A$ to be the set

$$A' := \{a \in A : E(A+A, a+A) \leq 2C|A||A+A|^{3/2}\},$$

and observe that

$$2C|A||A+A|^{3/2}|A\setminus A'| \leq \sum_{a\in A\setminus A'} E(A+A, a+A) \leq C|A|^2|A+A|^{3/2}.$$

This implies that $|A\setminus A'| \leq \frac{|A|}{2}$, and so

$$|A'| \geq \frac{|A|}{2}.$$

Next, observe that, for any $a \in A'$,

$$\frac{|A|^2|A+A|^2}{|(A+a)(A+A)|} \leq E(A+A, A+a) \ll |A||A+A|^{3/2}.$$

The lower bound here comes from (8), whilst the upper bound comes from the definition of A'. After rearranging, we have

$$|(A+a)(A+A)| \gg |A||A+A|^{1/2}, \qquad (64)$$

for any $a \in A'$. To complete the proof we need a useful lower bound on $|A+A|$. This comes from Lemma 5.1, which tells us that for any $a \in \mathbb{R}$, and so certainly any $a \in A$,

$$|A+A|^{1/2} \gg \frac{|A|^{3/2}}{(\log|A|)^{1/2}|(A+a)(A+A)|^{1/2}}.$$

Finally, this bound can be combined with (64), to conclude that

$$|(A+a)(A+A)| \gg \frac{|A|^{5/3}}{(\log|A|)^{1/3}},$$

as required.

□

Another upper bound on the multiplicative energy

Before proceeding to the proof of Theorem 2.11, it is necessary to establish another upper bound on the multiplicative energy. This is essentially a calculation, based on earlier work from [9] and [13]. We will need the following lemma:

Lemma 5.2. *Suppose that A, B and C are finite subsets of $\mathbb{R}$ such that $0 \notin A, B$, and $\lambda \in \mathbb{R} \setminus \{0\}$. Then, for any integer $t \geq 1$,*

$$|\{s : r_{AB}(s) \geq t\}| \ll \frac{|(A+\lambda)\cdot C|^2|B|^2}{|C|t^3}.$$

This statement is a slight generalisation of Lemma 3.2 in [13]. We give the proof here for completeness.

Proof. For some values p and b, define the line $l_{p,b}$ to be the set $\{(x,y) : y = (px - \lambda)b\}$. Let L be the family of lines

$$L := \{l_{p,b} : p \in (A+\lambda)C, b \in B\}.$$

Observe that, since λ is non-zero, $|L| = |(A+\lambda)C||B|$.[10] Let P_t denote the set of all t-rich points in the plane. By Corollary 2.2, for $t \geq 2$,

$$|P_t| \ll \frac{|B|^2|(A+\lambda)C|^2}{t^3} + \frac{|B||(A+\lambda)C|}{t}, \tag{65}$$

and it can once again be simply assumed that

$$|P_t| \ll \frac{|B|^2|(A+\lambda)C|^2}{t^3}. \tag{66}$$

This is because, if the second term from (65) is dominant, it must be the case

$$t > |(A+\lambda)C|^{1/2}|B|^{1/2} \geq \min\{|A|, |B|\}.$$

However, in such a large range $|\{s : r_{AB}(s) \geq t\}| = 0$, and so the statement of the lemma is trivially true.

Next, it will be shown that for every $s \in \{s : r_{AB}(s) \geq t\}$, and for every element $c \in C$,

$$\left(\frac{1}{c}, s\right) \in P_t. \tag{67}$$

Once, (67) has been established, it follows that $|P_t| \geq |C||\{s : r_{AB}(s) \geq t\}|$. Combining this with (66), it follows that

$$|\{s : r_{AB}(s) \geq t\}| \ll \frac{|B|^2|(A+\lambda)\cdot C|^2}{|C|t^3}, \tag{68}$$

[10] Note that it is not true in general that $|L| = |(A+\lambda)C||B|$. Indeed, if $0 \in B$, then $l_{p,0} = l_{p',0}$ for $p \neq p'$, and so the lines may not all be distinct. However, we may assume again that zero does not cause us any problems. To be more precise, we assume that $0 \notin B$, as otherwise 0 can be deleted, and this will only slightly change the implied constants in the statement of the lemma. If $0 \notin B$, then the statement that $|L| = |(A+\lambda)C||B|$ is true.

for all t 2. We can then check that (68) is also true in the case when = 1, since

$$\frac{jBj^2j(A + \quad)Cj^2}{1^3jCj} \quad j \; Bj^2j(A + \quad)Cj \quad j \; ABj = jf s : r_{AB}(s) \quad 1gj:$$

It remains to establish (67). To do so, x s with $r_{AB}(s)$ t and c 2 C. The element s can be written in the form s = a_1

Proof of Theorem 2.11

Let a and b be distinct real numbers. We will show that

$$|(A+a)(A+A)|^5 |(A+b)(A+A)|^2 \gg \frac{|A|^{11}}{(\log |A|)^3}. \qquad (69)$$

Once we have established (69), the theorem follows, since this implies that for any $a, b \in \mathbb{R}$ with $a \neq b$, we have

$$\max\{|(A+a)(A+A)|, |(A+b)(A+A)|\}$$

as required. Here we have used the fact

$$|A\times A+\Delta(A)| = \sum_{s\in D} |A + A_s| = \sum_{x \in S} |A + (A \cap (x - A))|;$$

which follows from the consideration of the projections of the set $A\times A+\Delta(A)$. More precisely, one has $A\times A+\Delta(A) = \{(a_1 + a, a_2 + a) : a, a_1, a_2 \in A\}$. Whence, writing $s = (a_1 + a) - (a_2 + a) = a_1 - a_2 \in D$, we get $a_2 \in A_s$, $a + a_2 \in A + A_s$ and viceversa. Similarly, put $x = a_1 + a_2 \in S$, one get $a_2 \in A \cap (x - A)$, $a + a_2 \in A + (A \cap (x - A))$ and viceversa.

Further, by Lemma 5.5

$$|A|^6 \lesssim E_3(A) \sum_x D(x) r_{S-S}(x):$$

Applying the Cauchy–Schwarz inequality, we get

$$|A|^{12} \lesssim E_3^2(A)E(S)|D|$$

and formula (78) follows. The result for the set D is similar. □

Finally, we can prove Theorem 2.12:

Proof of Theorem 2.12. We begin with the first formula of the result.

Take $C = A - B$ in Corollary 5.4. Note that $r_{(A-B)+B}(a) \geq |B|$ for all $a \in A$, which implies that $r_{A(B+C)}(x) \geq r_{AA}(x)|B|$. Thus by Corollary 5.4 we have[11]

$$|B|^2 E_2(A) \leq \sum_{x\neq 0} r^2_{A(B+C)}(x) \lesssim E_2(A)^{1/2}|B|^{3/2}|A-B|^{3/2}(\log|A|)^{1/2}:$$

Rearranging and applying the Cauchy-Schwarz lower bound for $E_2(A)$ yields

$$\frac{|A|^4|B|}{|AA^{-1}|} \leq |B|E_2(A) \lesssim |A-B|^3 \log|A|;$$

as required.

Combining (13) with Corollary 5.6, we obtain (14). This completes the proof. □

Concluding remarks - the complex case

We conclude by pointing out that almost all of the results in this paper also hold in the more general case whereby A is a finite set of complex numbers, since the tools we have made use of can all be extended in this direction. Indeed, the Szemerédi-Trotter was extended to points and lines in $\mathbb{C}^2$

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