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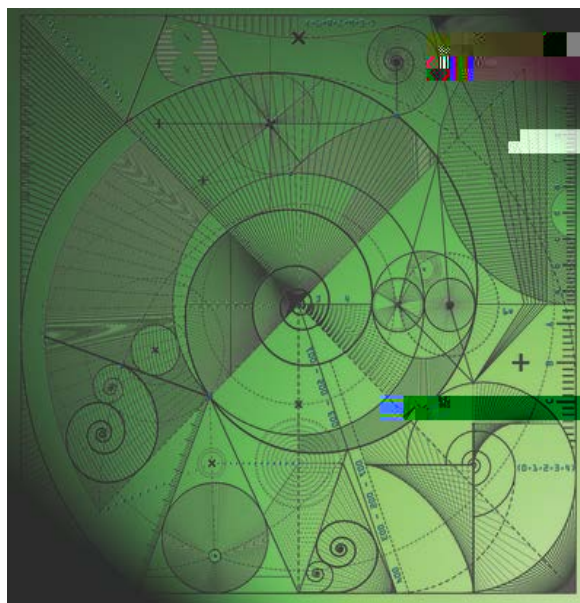
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Solutions of the Fully Compressible Semi-Geostrophic System

by

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Abstract

The fully compressible semi-geostrophic system is widely used in the modelling of large-scale atmospheric flows. In this paper, we prove rigorously the existence of weak Lagrangian solutions of this system, formulated in the original physical coordinates. In addition, we provide an alternative proof of the earlier result on the existence of weak solutions of this system expressed in the so-called geostrophic, or dual, coordinates. The proofs are based on the optimal transport formulation of the problem and on recent general results concerning transport problems posed in the Wasserstein space of probability measures.

Keywords: geophysical fluid mechanics, nonlinear PDEs, meteorology.

MSC classification: 35Q35, 35Q86

1 Introduction

The behaviour of the atmosphere is governed by the compressible Navier-Stokes equations, together with the laws of thermodynamics, equations describing phase changes, source terms and boundary fluxes. These equations are too complex to be solved accurately in a large-scale atmospheric context and therefore reductions and approximations of the Navier-Stokes equations are often used to validate and understand the solutions that have been computed.

In [6], Benamou and Brenier assumed the fluid to be incompressible, the Coriolis parameter constant and the boundaries rigid. They then used a change of variables, introduced by Hoskins in [13], to derive the so-called dual formulation. In this formulation, the equations are interpreted as a Monge-Ampere equation coupled with a transport problem, to prove the existence of stable weak solutions.

In [9], Cullen and Gangbo relaxed the assumption of rigid boundaries with a more physically appropriate free boundary condition. However, they additionally assumed constant potential temperature to obtain the 2-D system known as the semi-geostrophic shallow water system. After passing to dual variables, they showed existence of stable weak solutions for this system of equations.

In [10], Cullen and Maroo proved that stable weak solutions of the 3-D compressible semi-geostrophic system in its dual formulation exist, returning to the assumption of rigid boundaries.

The main problem posed by the existence results in [6], [9] and [10] is that they are all proved in dual space. It is difficult to relate these results directly to the Navier-Stokes equations, or indeed other reductions of them. For this reason, and in order to give the dual space results physical meaning, Cullen and Feldman [8] mapped the solutions back to the original, physical coordinates and extended the results of [6] and [9], proving existence of weak Lagrangian solutions in physical space, in the incompressible case. We mention that very recently results on the existence of Eulerian solutions have been proved in the case of a two-dimensional torus, and of a convex open subset in 3-dimensional space [2], [3].

In this paper, we make use of recent results in the analysis of ODEs in spaces of measures, in particular those of [4], [5], in order to provide an alternative proof of the dual space result in [10]. We also extend considerably the results of [10] to prove the existence of weak Lagrangian solutions of the fully compressible semi-geostrophic system in the original physical coordinates. As in the incompressible case studied in [8], the proof is based on the existence of an appropriate flow map with rather low regularity; however we also show that, if we could assume additional regularity, then the solutions derived would determine classical solutions.

The paper is organised as follows: in Section 2 we introduce various definitions, and the

- (iii) $u^g(t; x) = (u_1^g(t; x); u_2^g(t; x); 0)$ represents the geostrophic velocity;
- (iv) $p(t; x)$ represents the pressure;
- (v) $\rho(t; x)$ represents the density;
- (vi) $\theta(t; x)$ represents the potential temperature. Given its physical meaning, we assume $\theta(t; x)$ to be strictly positive and bounded;
- (vii) $\Phi(x)$ is the given geopotential representing gravitation and centrifugal forces. We assume that $\Phi \in C^2(\mathbb{R}^3)$ and that $\frac{\partial \Phi}{\partial x_i}(x) \neq 0$ for all $x \in \mathbb{R}^3$;
- (viii) f_{cor} denotes the Coriolis parameter, which we assume to be constant; indeed we will normalise this parameter to be equal to 1;
- (ix) p_{ref} is the reference value of the pressure;
- (x) c_v is a constant representing the specific heat at a constant pressure;
- (xi) c_p is a constant representing the specific heat at a constant volume;
- (xii) R represents the gas constant and satisfies $R = c_p - c_v$;
- (xiii) $\gamma = c_p/c_v$ denotes the ratio of specific heats (this is approximately 1.4 for air).

Notations and other conventions (2.2)

Throughout, we will only consider measures that are absolutely continuous with respect to Lebesgue measure. Given an open set A in $\mathbb{R}^3$, we will denote by

- $\mathcal{P}_{\text{ac}}(A)$ - the set of probability measures in $\mathbb{R}^3$ with supports contained in A ;
- χ_A - the characteristic function of A .

Unless otherwise specified, measurable means Lebesgue measurable and a.e. means Lebesgue-a.e.

D_t denotes the Lagrangian derivative, defined as $D_t = \frac{\partial}{\partial t} + u \cdot \nabla$, where u denotes the velocity of the flow.

e_3 denotes the unit vector $(0; 0; 1)$ in $\mathbb{R}^3$

For convenience, we will sometimes use the notation $F_{(t)}(\cdot) = F(t; \cdot)$ to denote the map F evaluated at fixed time t .

Important definitions

Definition 2.1. We define

$$\mathcal{P}_{\text{ac}}^2(\mathbb{R}^3) := \left\{ \mu \in \mathcal{P}_{\text{ac}}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2 d\mu(x) < +\infty \right\}$$

with tangent space

$$T \mathcal{P}_{\text{ac}}^2(\mathbb{R}^3) = \overline{\left\{ \int_{\mathbb{R}^3} \cdot \nabla \phi d\mu : \phi \in C_c^\infty(\mathbb{R}^3) \right\}}^{L^2(\cdot; \mathbb{R}^3)} \quad (2.3)$$

Definition 2.2. Given two Borel probability densities $\mu_1(\cdot)$ and $\mu_2(\cdot)$ in $\mathbb{R}^3$, we define the Wasserstein-2 distance, W_2 , between μ_1 and μ_2 as follows:

$$W_2^2(\mu_1; \mu_2) := \inf_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \|x - y\|^2 d\gamma(x; y): \quad (2.4)$$

with

$\Pi(\mu_1; \mu_2) = \{ \gamma : \gamma \text{ probability measure on } \mathbb{R}^3 \times \mathbb{R}^3 \text{ with marginals } \mu_1 \text{ and } \mu_2 \}$.

We denote by $\Pi_0(\mu_1; \mu_2)$ the set of minimisers of (2.4).

The Wasserstein distance indeed defines a distance in the space of probability measures on $\mathbb{R}^3$ (or more generally, any complete separable metric space); it can be used as an optimal transport cost between these measures, see [17].

Definition 2.3. Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$!

The unknowns in the above equations are $u^g = (u_1^g; u_2^g; 0)$, $u = (u_1; u_2; u_3)$, p , θ , ρ .

Equation (3.1) is the momentum equation; (3.2) represents the adiabatic assumption; (3.3) is the continuity equation and (3.4) represents hydrostatic and geostrophic balance. The equation (3.5) is the equation of state which relates the thermodynamic quantities to each other, and (3.6) is the rigid boundary condition, where n is the outward normal to $\partial\Omega$.

The semi-geostrophic equations are a valid approximation to the compressible Euler equations when $UL \ll 1$; and are accurate when $\frac{H}{L} < \frac{1}{N}$; where U is a typical scale for horizontal speed; L is a typical horizontal scale; H is a typical vertical scale; N is the buoyancy frequency.

The energy associated with the flow, known as the *geostrophic energy*, is defined as

$$E(t) = \int_{\Omega} \frac{1}{2} |u^g|^2(t; x) + \frac{1}{2} |u|^2(t; x) + c_v \theta(t; x) - \frac{p(t; x)}{\rho_{ref}}(t; x) dx: \quad (3.7)$$

In what follows, we set $f_{cor} = 1$.

3.1 Dual formulation

In [10], solutions were obtained using a transformation into the so-called *dual (geostrophic) coordinates* $y = (y_1; y_2; y_3)$. The coordinate transformation is given by:

$$T : \Omega \rightarrow \mathbb{R}^3; \quad T(t; x) = (T_1(t; x); T_2(t; x); T_3(t; x)) = (y_1; y_2; y_3);$$

with

$$y_1 = x_1 + u_2^g(t; x); \quad y_2 = x_2 - u_1^g(t; x); \quad y_3 = x_3: \quad (3.8)$$

Note that, by (2.1)(vi), $\Omega \subset \mathbb{R}^2 \times [0, 1]$ for some $0 < \epsilon < 1$.

Using this transformation, as well as (3.5), it was shown in [10] that we can write the energy in (3.7) as

$$E(t) = E(t; \cdot; T) = \int_{\Omega} \frac{1}{2} f_j x_1 - y_1 j^2 + j x_2 - y_2 j^2 g + \theta(x)$$

where

$$E(\cdot) = \inf_{\bar{T} \in \mathcal{T}} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_i} \left(\frac{1}{\bar{T}} \right) \right|^2 dx$$

- (ii) S is optimal in the transport of μ_0 to μ_1 with cost $e(y; x) = c(x; y)$,
- (iii) S and T are inverses, i.e. $S \circ T(x) = x$ for $x \in \mathbb{R}^3$ and $T \circ S(y) = y$ for $y \in \mathbb{R}^3$,
- (iv) $\mu_0 = (\text{id}; T)^\#$ is a minimiser of the relaxed optimal transport problem (3.22),
- (v) For $J_{(\cdot; \cdot)}(f; g)$ defined by (3.23), the following equality holds:

$$\sup_{(f; g) \in 2\text{Lip}_c} J_{(\cdot; \cdot)}(f; g) = \inf_{\mu \in \mathcal{M}(\mathbb{R}^3)} \bar{I}[\mu] = \inf_{T^\# = \mu_0} I[\bar{T}]:$$

4 The main existence result in dual space - an alternative proof

4.1 Statement of the theorem

Theorem 4.1. Let $1 < r < \infty$ and $\mu_0 \in L^r(\mathbb{R}^3)$ be an initial potential density with support in Ω_0 , where Ω_0 is a bounded open set in $\mathbb{R}^3$ with $\overline{\Omega_0} \subset \mathbb{R}^2 \times [\gamma, 1-\gamma]$ for some $0 < \gamma < 1$. Let Ω be an open bounded convex set in $\mathbb{R}^3$. Assume that $c(\cdot; \cdot)$ is given by (3.13) and that μ_0 satisfies (2.1)(ix). Then the system of semi-geostrophic equations in dual variables (3.14)-(3.19) has a stable weak solution $(\mu; T)$ such that, with $(\mu; T) = T(\mu; \cdot)^\#(\mu; \cdot)$ and w as in (3.15),

- (i) $(\mu; T) \in L^r([0; \infty); \mathbb{R}^3)$; $\|\mu(t; \cdot)\|_{L^r(\Omega)} \leq \|\mu_0\|_{L^r(\Omega)}; \quad \forall t \in [0; \infty);$
- (ii) $(\mu; T) \in W^{1,1}(\mathbb{R}^3)$; $\|\mu(t; \cdot)\|_{W^{1,1}(\Omega)} \leq C = C(\mu_0; c(\cdot; \cdot); K_1); \quad \forall t \in [0; \infty);$
- (iii) $\|\mu(t; \cdot)\|_{L^1(\Omega)} \leq C = C(\mu_0; \cdot) \quad \forall t \in [0; \infty);$

where Ω is a bounded open domain in $\mathbb{R}^3$ containing $\text{supp}(\mu_0)$, such that $\overline{\Omega} \subset \mathbb{R}^2 \times [\gamma, 1-\gamma]$ for some $0 < \gamma < 1$.

The proof of this theorem is given in [10, Theorem 5.5], using a time-approximation argument, similar in spirit to the original argument of the proof given by Benamou and [0

Definition 4.1. Let $H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-1, +1]$ be a proper, upper semicontinuous function and let $\gamma \in D(\mathbb{R}^3)$

Definition 4.3. Let $H : P$

Let μ, ν be as in (3.20). For $\gamma \in \mathcal{P}_{ac}(\mathbb{R}^d)$, define the Hamiltonian H by

$$H(\gamma) = \inf_{\gamma \in \mathcal{P}_{ac}(\mathbb{R}^d)} \int_{\mathbb{R}^d} E(\gamma; \cdot) + K_1(\cdot(x)) dx; \quad (4.7)$$

where

$$E(\gamma; \cdot) = \inf_{\bar{S} \in \mathcal{S}(\gamma)} \int_{\mathbb{R}^d} e(y; \bar{S}(y)) \gamma(y) dy \quad (4.8)$$

with $e(y; x) = c(x; y)$ defined by (3.13). We begin with the following:

Proposition 4.6. *Let μ and ν satisfy (3.20). Let the Hamiltonian $H(\gamma)$ on $\mathcal{P}_{ac}(\mathbb{R}^d)$ be defined by (4.7). Then H is superdifferentiable, upper semicontinuous and (2) concave.*

Proof. Given $\gamma \in \mathcal{P}_{ac}(\mathbb{R}^d)$, denote by γ_h the minimiser in (4.7). The existence and uniqueness of this minimiser follows from Theorem 3.3. For any $\gamma_h \in \mathcal{P}_{ac}(\mathbb{R}^d)$ we have

$$H(\gamma_h) = \inf_{\gamma \in \mathcal{P}_{ac}(\mathbb{R}^d)} \int_{\mathbb{R}^d} E(\gamma_h; \cdot) + K_1(\cdot(x)) dx \\ = \int_{\mathbb{R}^d} E(\gamma_h; \cdot) + K_1(\cdot(x)) dx:$$

First, recall that we can guarantee that there exists a unique optimal transport map R_h from μ to γ_h with respect to the Wasserstein cost function $d(y; y_h) = \frac{1}{2}|y - y_h|^2$; see Remark 2.1.

We consider now transport with respect to the cost function $e(y; x) = c(x; y)$ given by (3.13). Let S be the optimal map in the transport of μ to γ and let S_h be the optimal map in the transport of μ_h to γ_h . Therefore, we have

$$\int_{\mathbb{R}^d} e(y; S(y)) \gamma(y) dy = \int_{\mathbb{R}^d} e(y; S_h(y)) \gamma_h(y) dy$$

and

$$\int_{\mathbb{R}^d} e(y; S(y)) \gamma_h(y) dy = \int_{\mathbb{R}^d} e(y; S_h(y)) \gamma_h(y) dy:$$

The existence of S and S_h follows from Theorem 3.4. Note that, since (S, S_h)

(4.9)

where we have used (2.5).

Hence, using Definition 4.1, we conclude that $r \in (y; S(y)) \subset \partial H(y)$. Thus, $\partial H(y)$ is non-empty, H is superdifferentiable and we can use [17, Proposition 10.12] to conclude that H is semi-concave, i.e.

H is $(\frac{1}{2})$ -concave: (4.10)

Also, from the narrow continuity of $E(\cdot; \cdot)$ (see [10, Theorem 3.4]) and the uniform convergence of ϕ_n as the minimiser of (4.7) (see [10, Lemma 4.3]), we have that

H is upper semicontinuous. (4.11)

From (4.10) and (4.11), we have that (H3) holds. □

The following proposition yields a proof of Theorem 4.1, alternative to the proof given in [10].

Proposition 4.7. *Let $1 < r < \infty$ and $\phi_0 \in L^r(\mathbb{R}^d)$*

and

$$\int_Z (y) (R^{-s}(y) - y) (y) dy = \int_Z (y) (g_s(y) - y) (y) dy = s \int_Z (y) r'(y) (y) dy:$$

Combining this with (4.12), we therefore obtain

$$\begin{aligned} & \int_Z (y) r'(y) (y) dy + s^2 \int_Z |r'(y)|^2 (y) dy \leq H(\cdot) - H(s) \\ & \leq \frac{6}{Z} E(\cdot; s) - E(s; s) = \frac{6}{Z} \int_Z \epsilon(y; S^{-s}(y)) (y) dy - \int_Z \epsilon(y; S_s^{-s}(y)) s(y) dy \\ & \leq \frac{6}{Z} \int_Z \epsilon(y; S_s^{-s} \circ g_s(y)) (y) dy - \int_Z \epsilon(y; S_s^{-s}(y)) s(y) dy \\ & = \int_Z \epsilon(g_s^{-1}(y); S_s^{-s}(y)) s(y) dy - \int_Z \epsilon(y; S_s^{-s}(y)) s(y) dy; \end{aligned} \quad (4.13)$$

since $g_s \# \mu = \mu_s$. Here S^{-s} denotes the optimal transport map from μ to μ_s and S_s^{-s} denotes the optimal transport map from μ_s to μ_s with respect to the cost function $\epsilon(\cdot; \cdot)$. The existence of S^{-s} and S_s^{-s} follows from Theorem 3.4.

Note that

$$g_s^{-1}(y) = y - sr'(y) + s^2 \ell(y) = y - sy + \mathfrak{g}.$$

existence of S

Dividing both sides first by $s > 0$, then by $s < 0$ and letting $|s| \rightarrow 0$ we obtain

$$\int_Z (y - r'(y)) dy = \int_Z \frac{S(y) - y}{y^3} (y - r'(y)) dy$$

(iii) There exists a Borel map

$$F : [0; \infty) \rightarrow \mathbb{R}^3$$

such that, for every $t \in (0; \infty)$, the map

$$F_{(t)} = F(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

satisfies

$$F_{(t)}^{-1} \circ F_{(t)}(x) = x \text{ and } F_{(t)} \circ F_{(t)}^{-1}(x) = x;$$

for $a.e. x \in \mathbb{R}^3$, and

$$F_{(t)} \# (t, \cdot) = \mu_0(\cdot):$$

(iv) The function

$$Z(t; x) := T(t; F_{(t)}(x)) \quad (5.12)$$

is a weak solution of

$$\begin{aligned} \partial_t Z(t; x) &= e_3 \cdot [Z(t; x) - F_{(t)}(x)] \quad \text{in } [0; \infty) \times \mathbb{R}^3; \\ Z(0; x) &= T_0(x) \quad \text{in } \mathbb{R}^3; \end{aligned} \quad (5.13)$$

in the following sense:

for any $\varphi \in C_c^1([0; \infty) \times \mathbb{R}^3)$,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} \left(\partial_t Z(t; x) \varphi(t; x) + e_3 \cdot [Z(t; x) - F_{(t)}(x)] \varphi'(t; x) g_0(x) \right) dt dx \\ & + \int_{\mathbb{R}^3} Z(0; x) \varphi(0; x) dx = 0. \end{aligned}$$

In order to justify Definition 5.2, we must show that a weak Lagrangian solution corresponds to a weak (Eulerian) solution of (5.1)-(5.5), as defined by Definition 5.1. Indeed, we prove that, with additional regularity property $\Phi \in L^1([0; \infty); \mathbb{R}^3)$, a weak Lagrangian solution $(F; T; \cdot)$ as defined in Definition 5.2 determines a weak (Eulerian) solution of (5.1)-(5.5) and, furthermore, that a smooth Lagrangian solution determines a classical solution of (5.1)-(5.5). This is the content of the following result:

Proposition 5.3. *Let Ω be an open bounded convex set in $\mathbb{R}^3$ and let $\tau > 0$. Let $(F; T; \cdot)$ be a weak Lagrangian solution of (5.1)-(5.5) in $[0; \tau]$.*

(i) If $\Phi \in L^1([0; \tau]; \mathbb{R}^3)$, then the function

$$u(t; x) := (\Phi)(t; F_{(t)}(x)) \quad (5.16)$$

satisfies $u \in L^1([0; \tau]; \mathbb{R}^3)$ and $(u; T; \cdot)$ is a weak Eulerian solution of (5.1)-(5.5) in $[0; \tau]$ in the sense of Definition 5.1.

(ii) If $(F; F'; T) \in C^2([0; \tau]; \mathbb{R}^3)$, then the function (5.16) satisfies $u \in C^1([0; \tau]; \mathbb{R}^3)$, and $(u; T; \cdot)$ is a classical solution of (5.1)-(5.5) in $[0; \tau]$.

Proof. Let us first prove (i). Since F is a Borel map and, by our additional regularity

If we make the change of variables $X = F_{(t)}(x)$ in the second integral above, then, by (ii), (iii) in Definition 5.2 and by (2.5), we have that

$$\begin{aligned} & \int_{[0; \infty)} (0; x) \phi_0(x) dx - \int_{[0; \infty)} (\phi(F_{(t)}))(F_{(t)}(X)) r_{\phi}(t; X) (t; X) dt dX \\ &= \int_{[0; \infty)} \phi(t; X) (t; X) dt dX : \end{aligned}$$

Then, rearranging and using the definition of u in (5.16), we obtain

$$\int_{[0; \infty)} \phi(t; X) + u(t; X) r_{\phi}(t; X) g(t; X) dt dX + \int_{[0; \infty)} (0; x) \phi_0(x) dx = 0;$$

Changing notations X to x gives us (5.9).

We now prove that (5.8) also holds. By the properties of F and T in Definition 5.2, we have that $Z(t; x)$ as defined in (5.12) satisfies $Z \in L^1([0; \infty))$. Also, applying the definition of $Z(t; x)$ in (5.12) to equation (5.14) gives

$$\begin{aligned} & \int_{[0; \infty)} \phi(T(t; F_{(t)}(x))) \phi'(t; x) + e_3 [T(t; F_{(t)}(x)) - F_{(t)}(x)]' (t; x) g_0(x) dt dx \\ &+ \int_{[0; \infty)} [T_0(x) - (0; x)] \phi_0(x) dx = 0; \end{aligned} \quad (5.17)$$

for any $\phi \in C_c^1([0; \infty))$. Now, since $\mathcal{S}$ is a bounded set and $F_{(t)} \# \phi_0(\cdot) = (t; \cdot)$ for all $t \in [0; \infty)$, equation (2.5) allows us to make the change of variables $X = F_{(t)}(x)$ in the first integral of (5.17). Thus, by (iii) of Definition 5.2, we have that $x = F_{(t)}(X)$ for a.e. $x \in \mathcal{S}$ for every $t \in [0; \infty)$ and then, from (5.17), we obtain

$$\begin{aligned} & \int_{[0; \infty)} \phi(T(t; X)) \phi'(t; F_{(t)}(X)) + e_3 [T(t; X) - X]' (t; F_{(t)}(X)) g(t; X) dt dX \\ &+ \int_{[0; \infty)} [T_0(x) - (0; x)] \phi_0(x) dx = 0; \end{aligned} \quad (5.18)$$

for any $\phi \in C_c^1([0; \infty))$. We now show that (5.18) also holds for all ϕ such that

$$\begin{aligned} & \phi \in L^1([0; \infty)); \\ & \phi' \in L^1([0; \infty)); \\ & \text{supp}(\phi) \subset [0; \infty) \text{ for some } \epsilon > 0 \end{aligned} \quad (5.19)$$

In order to do this, we construct an approximating sequence for such ϕ . Let us extend ϕ to $(-\infty; 1)$ by defining, for $x \in \mathcal{S}$, $\phi(t; x) = \phi(t; x)$ for $t < 0$ and $\phi(t; x) = 0$ for $t > 1$. Then, we let $h > 0$ and define $\phi_h = \phi \chi_{\mathcal{S}_h}$ where $\chi_{\mathcal{S}_h}$ denotes the characteristic function of the set $\mathcal{S}_h$. Thus, ϕ_h is now defined on $\mathbb{R}^1 \times \mathbb{R}^3$, where $\chi_{\mathcal{S}_h}$ denotes the characteristic function of the set $\mathcal{S}_h$. Next, let $j_h(t; x) = \frac{1}{h^4} j(\frac{t(x)}{h})$, where $j(\cdot)$ is a standard mollifier, and let $k > \frac{1}{4}$ be an integer. We then have that functions $\phi_k = (\phi_h)_{4h}$, with $h = \frac{1}{k} < 1$, satisfy

$$\phi_k \in C_c^1([0; \infty)) \text{ with } \|\phi_k\|_{L^1([0; \infty))} + \|\phi_k'\|_{L^1([0; \infty))} \leq C;$$

where C does not depend on k , and

$$(\phi_k; \phi_k') \rightarrow (\phi; \phi') \text{ a.e. on } [0; \infty) \text{ as } k \rightarrow \infty;$$

Thus, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_{[0; \infty)} ('_k; @'_k)(t; x) \phi_0(x) dt dx = \int_{[0; \infty)} ('; @'_t)(t; x) \phi_0(x) dt dx :$$

Also, since $' \in L^1([0; \infty))$ by (5.19) and $F_{(t)} \# (t; \cdot) = \phi_0(\cdot)$, we have from (2.5) that

$$\int_{[0; \infty)} ('_k; @'_k)(t; x) \phi_0(x) dt dx = \int_{[0; \infty)} ('_k; @'_k)(t; F_{(t)}(X)) (t; X) dt dX$$

and
$$\int_{[0; \infty)} ('; @'_t)(t; x) \phi_0(x) dt dx = \int_{[0; \infty)} ('; @'_t)(t; F_{(t)}(X)) (t; X) dt dX :$$

Hence,

$$\lim_{k \rightarrow \infty} \int_{[0; \infty)} ('_k; @'_k)(t; F_{(t)}(X)) (t; X) dt dX = \int_{[0; \infty)} ('; @'_t)(t; F_{(t)}(X)) (t; X) dt dX :$$

We now state the main result, which we will prove in Sections 6 and 7:

Theorem 5.4. *Let* $\mathbb{R}^3$

by Theorem 4.1 we have that $w \in L^1([0; \infty) \times \mathbb{R}^3)$,

by (6.1), (6.2) and (6.3) we have that $\|Dw(t; \cdot)\| \in L^1_{loc}([0; \infty) \times \mathbb{R}^3)$.

Since χ has compact support in $[0; \infty) \times \mathbb{R}^3$, we can modify w away from χ so that the modified function $\tilde{w}$ satisfies

$$\tilde{w} \in L^1([0; \infty) \times \mathbb{R}^3); \quad \tilde{w}(t; \cdot) \in BV_{loc}(\mathbb{R}^3) \text{ for a.e. } t \in [0; \infty); \quad (6.4)$$

$$\|D\tilde{w}(t; \cdot)\| \in L^1_{loc}([0; \infty) \times \mathbb{R}^3); \quad \int_{\mathbb{R}^3} \tilde{w}(t; \cdot) = 0 \text{ in } \mathbb{R}^3 \text{ for every } t \in [0; \infty) \quad (6.5)$$

and

$$w = \tilde{w} \quad \text{in } \chi. \quad (6.6)$$

We construct such a modification as follows. Following [10, Section 5], define $\chi := B(0; R)$ $(\cdot; \frac{1}{2})$, where $0 < \epsilon < 1$ and $B(0; R)$ represents the open ball of radius R centered at the origin in $\mathbb{R}^2$. Define χ, R as in [10, (55), (56)] ensures that $\text{supp}(\chi)$ is contained in χ . Define $\eta \in C^1(\mathbb{R})$ as $\eta = 1$ on $[-R, R]$, $\eta = 0$ on $[-R, R]$ and $0 \leq \eta \leq 1$ on $\mathbb{R}$. Define $\tilde{\eta} \in C^1(\mathbb{R})$ as $\tilde{\eta} = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\tilde{\eta} = 0$ when $s \leq -\frac{1}{2}$ or when $s \geq \frac{1}{2}$ and $0 \leq \tilde{\eta} \leq 1$ on $\mathbb{R}$. Then, define for $y \in \mathbb{R}^3$

$$M(y) = (M_1(y); M_2(y); M_3(y)) = (\eta(y_1)y_1; \eta(y_2)y_2; \eta(y_3)y_3) \quad (6.7)$$

and define the modified velocity as

$$\tilde{w} = e_3 \cdot (M(y) \cdot S(t; M(y))) = (y_3)y_3e_3 + \eta_0(M(y)). \quad (6.8)$$

Then $\tilde{w}$ satisfies (6.4)-(6.6). These conditions enable us to apply the theory of [1] to the transport equation (3.14) with w replaced by our modified velocity $\tilde{w}$:

Lemma 6.1. *There exists a unique locally bounded Borel measurable map $\tilde{w} : [0; \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying*

- (i) $\tilde{w}(t; y) \in W^{1,1}([0; \infty) \times \mathbb{R}^3)$ for $0 \leq t < \infty$ and $y \in \mathbb{R}^3$;
- (ii) $\tilde{w}(0; y) = y$ for $0 \leq t < \infty$ and $y \in \mathbb{R}^3$;
- (iii) for $0 \leq t < \infty$ and $y \in \mathbb{R}^3$,
$$\partial_t \tilde{w}(t; y) = \tilde{w}(t; \tilde{w}(t; y)); \quad (6.9)$$
- (iv) there exists a Borel map $\tilde{w} : [0; \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that, for every $t \in [0; \infty)$, the map $\tilde{w}(t; \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is Lebesgue-measure preserving, and such that $\tilde{w}(t; \tilde{w}(t; y)) = y$ for $0 \leq t < \infty$ and $y \in \mathbb{R}^3$;
- (v) $\tilde{w}(t; \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a Lebesgue-measure preserving map for every $t \in [0; \infty)$.

Proof. The proof is essentially identical to that of [8, Lemma 2.8]. □

We now show that the image of the flow map $\tilde{w}$ is contained in χ , and therefore corresponds to the velocity field w .

Lemma 6.2. *Let χ be as in Theorem 4.1. Let $\tilde{w}$ be the map defined in Lemma 6.1 and let w be defined as in (3.15). Then*

$$\tilde{w}(t; y) \in \chi \quad \text{for } 0 \leq t < \infty \text{ and } y \in T_0(\chi) \text{ and every } t \in [0; \infty); \quad (6.10)$$

In particular,

$$\partial_t \tilde{w}(t; y) = w(t; \tilde{w}(t; y)) \quad \text{for } 0 \leq t < \infty \text{ and } y \in T_0(\chi) \text{ and every } t \in [0; \infty); \quad (6.11)$$

Proof. For h, k, j_h, g_h^k as in [10, Section 5] define

$$\mathbf{w}_h^k(y) := (y_3)y_3\mathbf{e}_3 - r \cdot (j_h \cdot g_h^k)(M(y)); \quad (6.19)$$

where $M, \cdot$ are defined in (6.7). Define functions $\mathbf{w}_h$ on $[0; \infty) \times \mathbb{R}^3$ by setting them equal to $\mathbf{w}_h^k$ on the time-interval $t \in [kh; (k+1)h)$. Following [10, Lemma 5.3], the corresponding potential density ϕ_h is a weak solution of

$$\begin{aligned} \partial_t \phi_h + r \cdot (\nabla_h \mathbf{w}_h) &= 0 \quad \text{in } (0; \infty) \times \mathbb{R}^3; \\ \phi_h(0; y) &= \phi_h^0(y); \end{aligned} \quad (6.20)$$

The construction of $\mathbf{w}_h$ implies that $\mathbf{w}_h$ is a divergence-free vector field satisfying (6.4)-(6.6) and

$$\begin{aligned} (i) \quad & \int_0^Z \int_{\mathbb{R}^3} |\mathbf{w}_h|_{-h}^2 dt dy < +\infty; \\ (ii) \quad & \end{aligned}$$

Passing to the limit $j \rightarrow 1$ in the last equality, using (6.24), the fact that $\rho_{h_j}^0 \rightarrow \rho_0$ as $j \rightarrow 1$ in $L^1(\mathbb{R}^3)$ and the dominated convergence theorem in the left-hand side, and using (6.22) in the right-hand side, we obtain

$$\int_{\mathbb{R}^3} \rho(t; y) \rho_0(y) dy = \int_{\mathbb{R}^3} \rho(Y)(t; Y) dY \quad (6.26)$$

for any $\rho \in C_c(\mathbb{R}^3)$. This implies (6.17).

Since $\gamma(t)$ is a measure preserving map, we use Lemma 6.1 (iv) to conclude that the left-hand side of (6.26) is equal to

$$\int_{\mathbb{R}^3} \rho(Y) \rho(\gamma(t)(Y)) dY;$$

and now (6.26) implies (6.18). □

Remark 6.5. It would be desirable to be able to avoid using the approximating solutions in dual space when showing that ρ is a weak Lagrangian solution of the transport equation. However, we have not been able to approximate in L^1 the velocity $\mathbf{w}$ directly. We appeal instead to the sequence of solutions of the approximating equations in dual space constructed in [10], as was done for the proof of the analogous result for the incompressible case given in [8].

7 Lagrangian flow in physical space

Throughout this section we will assume that $\rho, r, T_0, T, w, \mathbf{w}$ are as in Proposition 6.4. Note that, by Theorem 4.1, we can apply (2.5) to ρ_0, ρ, ρ_0 throughout this section.

We now perform the last step of the analysis and prove the existence of a Lagrangian flow $F : [0; \infty) \rightarrow \mathbb{R}^3$ in the physical space. Indeed, we define $F(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for $t \in [0; \infty)$ as

$$F(t) := S(t) \circ \gamma(t) \circ T_0; \quad (7.1)$$

where T_0 is as in (5.5), $S(t)$ is the inverse of $T(t)$ (see Theorem 3.4) and $\gamma(t)$ is the Lagrangian flow in dual space constructed in Lemma 6.1. To justify this definition, we prove the following lemma:

Lemma 7.1. *For any $t \in [0; \infty)$, the right hand side of (7.1) is defined ρ_0 a.e. in $\mathbb{R}^3$. The map $F : [0; \infty) \rightarrow \mathbb{R}^3$ defined by (7.1) is Borel.*

Proof. Since T_0 exists and is unique ρ_0 a.e. in $\mathbb{R}^3$, we have that T_0 exists and is unique on $\mathbb{R}^3 \setminus N_0^1$ where N_0^1 is a Borel subset of $\mathbb{R}^3$ with $\rho_0[N_0^1] = 0$. Also, since S exists and is unique ρ a.e. in $\mathbb{R}^3$ for every $t \in [0; \infty)$, we have that S exists and is unique on $\mathbb{R}^3 \setminus N^2$ for every $t \in [0; \infty)$, where N^2 is a Borel subset of $\mathbb{R}^3$ with $\rho[N^2] = 0$. Then, the right-hand side

Fix $t \in [0; \infty)$. Then, using that $T_0 \# \phi_0 = \phi_0$ and thus $T_0 \# \phi_0 = \phi_0$ for all $x \in \mathbb{R}^n \setminus N_0^1$, and using (6.17) as well as Lemma 6.1 (iv), we can apply (2.5) and compute

$$\begin{aligned} \int_{\mathbb{R}^n} M_{(t)} &= \int_{\mathbb{R}^n} \phi_0(x) \mathbb{1}_{\mathbb{R}^n \setminus N_0^1} \circ T_0(x) \mathbb{1}_{\mathbb{R}^n \setminus N_0^1} \circ T_0^{-1}(x) dx \\ &= \int_{T_0^{-1}(\mathbb{R}^n \setminus N_0^1)} \phi_0(x) dx = \int_{\mathbb{R}^n \setminus N_0^1} \phi_0(y) dy \\ &= \int_{\mathbb{R}^n} \phi_0(y) dy = 0. \end{aligned}$$

Thus, we can define $F : [0; \infty) \rightarrow \mathbb{R}$ by (7.1). Then, by Lemma 6.1, F is a Borel mapping. $\square$

It remains to prove that, if F is defined by (7.1), then $(F; T; \cdot)$ is a weak Lagrangian solution of (5.1)-(5.5) in the sense of Definition 5.2. We begin by showing that the initial condition for the flow in Definition 5.2 (i) is satisfied.

Proposition 7.2. *Let F be defined as in (7.1). Then, $F(0; x) = x$ for ϕ_0 -a.e. $x \in \mathbb{R}^n$.*

Proof. By (7.1) we have that $F_{(0)}(x) = S_{(0)} \circ T_0(x)$ for all $x \in \mathbb{R}^n \setminus N_0$ where N_0 is a Borel set with $\phi_0[N_0] = 0$.

By Lemma 3.4, there exist Borel sets N_1, N_2 with $\phi_0[N_1] = \phi_0[N_2] = 0$ such that T_0 exists and is unique in $\mathbb{R}^n \setminus N_1$ and $S_{(0)}$ exists and is unique in $\mathbb{R}^n \setminus N_2$. Moreover, if $x \in \mathbb{R}^n \setminus [N_1 \cup T_0^{-1}(N_2)]$, then $S_{(0)} \circ T_0(x) = x$. Also, by Lemma 6.1 (ii), we have that $\phi_{(0)}(y) = y$ in $\mathbb{R}^n \setminus N_3$, where N_3 is a Borel set with $\phi_0[N_3] = 0$.

Therefore, $F_{(0)}(x) = x$ for all $x \in \mathbb{R}^n \setminus [N_0 \cup N_1 \cup T_0^{-1}(N_2 \cup N_3)]$. We must now show that $\phi_0 \setminus T_0^{-1}(N_2 \cup N_3) = 0$.

We have that $\phi_0[N_2 \cup N_3] = 0$. Then, since $T_0 \# \phi_0 = \phi_0$, we obtain

$$\phi_0 \setminus T_0^{-1}(N_2 \cup N_3) = \int_{T_0^{-1}(N_2 \cup N_3)} \phi_0(x) dx = \int_{N_2 \cup N_3} \phi_0(y) dy = \phi_0[N_2 \cup N_3] = 0. \quad \square$$

since $\|S_{(t)} - T_{(t)}\|_2 \in L^1(\mathbb{R})$. Then, since $\|S_{(t)}\|_2 \in L^1(\mathbb{R})$ we can use (6.17) and apply (2.5) to get

$$\int_{\mathbb{R}} S_{(t)}(y) \phi(y) dy = \int_{\mathbb{R}} S_{(t)}(Y) \phi(Y) dY :$$

Finally, since $S_{(t)}$ satisfies $S_{(t)}^\# = 0$, we have that

$$\int_{\mathbb{R}} S_{(t)}(Y) \phi(Y) dY = \int_{\mathbb{R}} (X) \phi(X) dX :$$

Thus, we have shown that

$$\begin{aligned} \int_{\mathbb{R}} (F_{(t)}(x) - T_{(t)}(x)) \phi(x) dx &= \int_{\mathbb{R}} S_{(t)}(x) - T_{(t)}(x) \phi(x) dx \\ &= \int_{\mathbb{R}} S_{(t)}(y) \phi(y) dy \\ &= \int_{\mathbb{R}} S_{(t)}(Y) \phi(Y) dY \\ &= \int_{\mathbb{R}} (X) \phi(X) dX ; \end{aligned}$$

as required. □

We now prove that (5.10) holds for all $q \in [1; 1]$.

Proposition 7.4. *For any $t_0 \in [0; \infty)$ and any $q \in [1; 1]$,*

$$\lim_{t \downarrow t_0} \int_{\mathbb{R}} |F_{(t)}(x) - F_{(t_0)}(x)|^q \phi(x) dx = 0 :$$

Proof. By Lemma 7.1 we have that, for any $t \in [0; \infty)$, (7.1) holds ϕ -a.e. in $\mathbb{R}$. Thus, since $T_0^\# \phi = \phi$, we see that, for any $t, t_0 \in [0; \infty)$,

$$\int_{\mathbb{R}} F_{(t)}(x) - F_{(t_0)}(x) \phi(x) dx = 0$$

(6.17) and Holder's inequality to estimate

$$\begin{aligned}
 I_1 &= \int_{\mathbb{R}^3} |S_{(t)}(y) - S_{(t_0)}(y)|^q \rho(y) dy \\
 &= \int_{\mathbb{R}^3} |S_{(t)}(y) - S_{(t_0)}(y)|^q \rho(y) dy \\
 &\leq \left(\int_{\mathbb{R}^3} |S_{(t)}(y) - S_{(t_0)}(y)|^{qr} dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^3} \rho(y)^r dy \right)^{\frac{1}{r'}} \\
 &= \|S_{(t)} - S_{(t_0)}\|_{q, \rho}^q \|\rho\|_r^{\frac{q}{r'}} \\
 &= \|S_{(t)} - S_{(t_0)}\|_{q, \rho}^q k_0 k_r \rightarrow 0 \text{ as } t \rightarrow t_0.
 \end{aligned}$$

Next, we show that $I_2 \rightarrow 0$ as $t \rightarrow t_0$. Since $S_{(t)} \in \mathcal{M}^+$ for each t and for $a \in \mathcal{E}$, then, by the dominated convergence theorem, it remains to prove that for every t_0 ,

$$S_{(t_0)}(y) - S_{(t)}(y) \rightarrow 0 \text{ as } t \rightarrow t_0 \quad (7.2)$$

for $a \in \mathcal{E}$. First we note that, since $\rho_{(t)}$ is measure preserving, then it follows from Lemma 6.1 (i), and the fact that $\rho \in L^1([0, \infty) \times \mathbb{R}^3)$ by (6.4), that

$$\rho_{(t)}(y) \rightarrow \rho_{(t_0)}(y) \text{ as } t \rightarrow t_0$$

in $[0, \infty) \times \mathbb{R}^3$ for $a \in \mathcal{E}$. If y is such a point and if, in addition, $\rho_{(t_0)}(y)$ is a point of continuity for $S_{(t_0)}$, then convergence in (7.2) holds at y . Since $\rho_{(t_0)}$ is measure preserving, it follows that $\rho_{(t_0)}(y)$ is a point of continuity for $S_{(t_0)}$ for $a \in \mathcal{E}$. Thus, (7.2) holds for $a \in \mathcal{E}$. $\square$

Lemma 7.5. *Let Z be defined as in (5.12) with F defined as in (7.1). Then, for all $t \in [0, \infty)$,*

$$Z_{(t)}(x) = T_{(t)} T_0(x)$$

for $0 \leq a \in \mathcal{E}$.

Proof. Using (5.12), we have that $Z_{(t)} = T_{(t)} F_{(t)}$. Therefore, we need to justify the following formal computation:

$$T_{(t)} F_{(t)} = T_{(t)}$$

Let $M = \{y \in \mathbb{R}^n : T(t; S(t; y)) \in y\}$. Then M is a Borel set. Now, the proof of the lemma will be completed if we show that

$$\int_0^h \int_{\mathbb{R}^n} f(x) \chi_M(x) dx dt = 0 \quad (7.3)$$

From Theorem 3.4 (iii) we have that, for any $t \in [0, \infty)$,

$$T_{(t)}^{-1}(S_{(t)}(y)) = y \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Thus, we have

$$\int_M f(y) dy = 0$$

for any $t \in [0, \infty)$. Therefore, using that $\int_0^h \int_{\mathbb{R}^n} f(x) dx dt = 0$, which implies that

thus $T_{(t)}^\# = \text{id}$ for all $x \in \mathbb{R}^n$, and using (6.17), we can apply (2.5) and compute

$$\begin{aligned} [M] &= \int_{\mathbb{R}^n} x^2 \cdot T_{(t)}^\#(x) \, dx = \int_{\mathbb{R}^n} x^2 \, dx \\ &= \int_{\mathbb{R}^n} x^2 \, dx = \int_{\mathbb{R}^n} y^2 \, dy \\ &= \int_{\mathbb{R}^n} y^2 \, dy = 0. \end{aligned}$$

Thus, we can define $F_{(t)} = S_{(0)} \circ T_{(t)}$ and F is a Borel mapping.

We can now prove that property (iii) of Definition 5.2 holds. Since $F_{(t)}^\# = \text{id}$, we have that $F_{(t)}(x) = S_{(0)}(T_{(t)}(x))$ for a.e. $x \in \mathbb{R}^n$. Then, using Lemma 7.5, we get $F_{(t)}(x) = S_{(0)}(T_{(t)}(x))$ for a.e. $x \in \mathbb{R}^n$. Since, by Lemma 6.1 (iv), $T_{(t)}(y) = y$ for a.e. $y \in \mathbb{R}^n$ and thus for a.e. $x \in \mathbb{R}^n$, and since $T_{(0)}^\# = \text{id}$, we have $T_{(t)}(x) = T_{(0)}(x)$ for a.e. $x \in \mathbb{R}^n$. Thus, $F_{(t)}(x) = S_{(0)}(T_{(0)}(x)) = x$ for a.e. $x \in \mathbb{R}^n$ by Lemma 3.4 (iii).

By a similar argument, we have that $F_{(t)}(x) = x$ for a.e. $x \in \mathbb{R}^n$. $\square$

Finally, we show that property (iv) of Definition 5.2 holds for F defined in (7.1).

Proposition 7.7. *Let F be defined as in (7.1). Then, equality (5.14) holds for any $\gamma \in C_c^1([0, \infty); \mathbb{R}^3)$. Moreover, we have that $Z(\gamma; x) \in W^{1,1}([0, \infty))$ for a.e. $x \in \mathbb{R}^n$, and (5.21) holds.*

Proof. From the definition of the Lagrangian flow in Lemma 6.1, we have that

$$F_{(t)}(y) = y + \int_0^t \mathbf{w}(s; F_{(s)}(y)) \, ds$$

for a.e. $y \in \mathbb{R}^n$ and every $t \in [0, \infty)$. Thus, this equality holds for all $y \in \mathbb{R}^n$ where $\mathbf{w}[N] = 0$. Since $T_{(0)}^\# = \text{id}$ it follows that

$$\mathbb{R}^n \setminus T_{(0)}^{-1}(N) = \mathbb{R}^n \setminus T_{(0)}^{-1}(N) \quad \text{a.e.}$$

$$\int_{[0; \infty)} \left(\frac{\partial}{\partial t} (t; x) - (t; T_0(x)) \right) \phi_0(x) dt dx = \int_{[0; \infty)} \left(\frac{\partial}{\partial t} (0; x) T_0(x) - \phi_0(x) \right) dt dx + \int_{[0; \infty)} \left(\frac{\partial}{\partial t} (t; x) w(t; (T_0(x))) - \phi_0(x) \right) dt dx; \quad (7.5)$$

References

- [1] Ambrosio, L. Transport equation and cauchy problem for BV vector fields. *Invent. Math.* 158 (2004), 227{260.
- [2] Ambrosio, L., Colombo, M., De Philippis, G., and Figalli, A. Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case. *arXiv:1111.7202v1 math.AP* (2011).
- [3] Ambrosio, L., Colombo, M., De Philippis, G., and Figalli, A. A global existence result for the semigeostrophic equations in three dimensional convex domains. *arXiv:1205.5435 math.AP* (2012).
- [4] Ambrosio, L., and Gangbo, W. Hamiltonian ODE's in the wasserstein space of probability measures. *Comm. Pure. Appl. Math.* 61 (2008), 18{53.
- [5] Ambrosio, L., Gigli, N., and Savaré, G. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, vol. 58 of *Lectures in Mathematics ETH Zürich*. Birkhauser Verlag, Basel, 2005.
- [6] Benamou, J.-D., and Brenier, Y. Weak existence for the semigeostrophic equations formulated as a coupled Monge-Ampere/transport problem. *SIAM J. Appl. Math.* 58 (1998), 1450{1461626 Tf 163.25lpplp450{14012).