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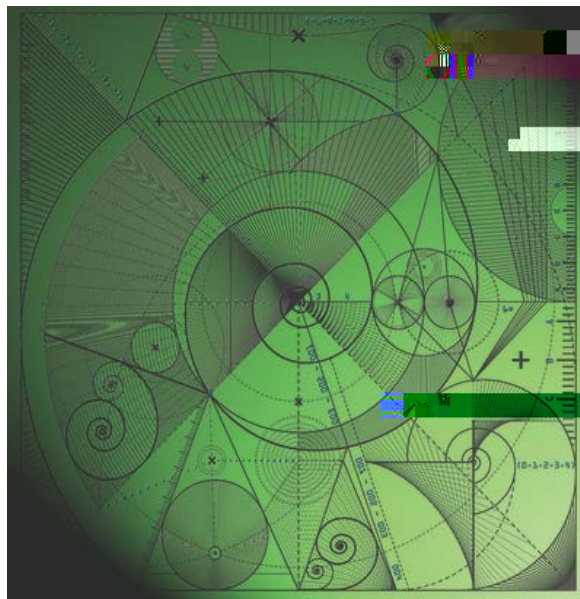
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‘Quasi’-norm of an arithmetical convolution
operator and the order of the
Riemann zeta function

by

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'Quasi'-norm of an arithmetical convolution operator and the order of the Riemann zeta function ¹

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Abstract

In this paper we study Dirichlet convolution with a given arithmetical function f as a linear mapping $'_f$ that sends a sequence (a_n) to (b_n) where (b_n) is defined by

$$b_n = \sum_{d|n} f(d) a_{n/d}.$$

For the unbounded case, we show that $'_f : M^2 \rightarrow M^2$ where M^2 is the subset of l^2 of multiplicative sequences, for many $f \in M^2$. Consequently, we study the 'quasi'-norm

for large T , which measures the 'size' of $'_f$ on M^2 . For the $f(n) = n^{i+it}$ case, we show this quasi-norm has a striking resemblance to the conjectured maximal order of $j^{3/2}(\log j)^{it}$ for $j > \frac{1}{2}$.

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Introduction

Given an arithmetical function $f(n)$, the mapping $'_f$ sends $(a_n)_{n \geq 1}$ to $(b_n)_{n \geq 1}$, where

$$b_n = \sum_{d|n} f(d) a_{n/d}. \quad (0.1)$$

Writing $a = (a_n)$, $'_f$ maps a to $f \star a$ where $\star$ is Dirichlet convolution. This is a 'matrix' mapping, where the matrix, say $M(f)$, is of 'multiplicative Toeplitz' type; that is,

$$M(f) = (a_{ij})_{i,j \geq 1}$$

where $a_{ij} = f(i=j)$ and f is supported on the natural numbers (see, for example, [6], [7]).

Toeplitz matrices (whose ij -entry is a function of $i-j$) are most usefully studied in terms of a 'symbol' (the function whose Fourier coefficients make up the matrix). Analogously, the Multiplicative Toeplitz matrix $M(f)$ has as symbol the Dirichlet series

$$\sum_{n=1}^{\infty} f(n) n^{it}.$$

Our particular interest is naturally the case $f(n) = n^{i+it}$ when the symbol is $\zeta(s+it)$. We are especially interested how and to what extent properties of the mapping relate to properties of the symbol for $0 < t < 1$.

These type of mappings were considered by various authors (for example Wintner [15]) and most notably Toeplitz [13], [14] (although somewhat indirectly, through his investigations of so-called

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(D-forms"). In essence, Toeplitz proved that $\|f\|_{l^2}^2$ is bounded if and only if $\sum_{n=1}^{\infty} f(n)n^{i s}$ is defined and bounded for all $s > 0$. In particular, if $f(n) \geq 0$ then $\|f\|_{l^2}$ is bounded iff and only if $f \in l^1$; furthermore, the operator norm is $\|f\| = \|f\|_{l^1}$. We prove this in Theorem 1.1 following Toeplitz's original idea. For example, for $f(n) = n^{-\alpha}$, $\|f\|_{l^2}$ is bounded for $\alpha > 1$ with operator norm $\zeta(2\alpha)$. In this special case, the mapping was studied in [7] for $\alpha < 1$ when it is unbounded on l^2 by estimating the behaviour of the quantity

$$\mathcal{C}_f(N) = \sup_{\|k\|_2=1} \left\| \sum_{n=1}^N k_n j^n \right\|_{l^2}^2$$

for large N . Approximate formulas for $\mathcal{C}_f(N)$ were obtained and it was shown that, for $\frac{1}{2} < \alpha < 1$, $\mathcal{C}_f(N)$ is a lower bound for $\max_{1 \leq t \leq T} |j^{3(\alpha+it)}|$ with $N = T^\alpha$ (some $\epsilon > 0$ depending on α only). In this way, it was proven that the measure of the set

$$\{t \in [1; T] : |j^{3(1+it)}| \geq e^{\epsilon \log \log T}\} \gg A^{-1}$$

is at least $T \exp\left(-\frac{a \log T}{\log \log T}\right)$ (some $a > 0$) for A sufficiently large, while for $\frac{1}{2} < \alpha < 1$ one has

$$\max_{1 \leq t \leq T} |j^{3(\alpha+it)}| \ll \exp \frac{c(\log T)^{1+\alpha/4}}{\log \log T}$$

for some $c > 0$ depending on α only, as well providing an estimate for how often $|j^{3(\alpha+it)}|$ is as large as the right-hand side above. The method is akin to Soundararajan's 'resonance' method and incidentally shows the limitation of this approach for $\alpha > \frac{1}{2}$ since $|j^{3(\alpha+it)}|$

Similarly one can study the quantity

$$m_f(T) = \inf_{\substack{k \leq T \\ a \leq M}} \frac{k' f(ak)}{kak}.$$

With $f(n) = n^{i\theta}$ this is shown to behave like the known and conjectured minimal order of $j^3(\theta + iT)$ for $\theta > \frac{1}{2}$. It should be stressed here that, unlike the case of $\mathcal{O}(N)$ which was shown to be a lower bound for $Z_\theta(T)$ in [7], we have not proved any connection between $\mathfrak{f}(\theta + iT)$ and $M_f(T)$. Even to show $M_f(T)$ is a lower bound would be very interesting.

Our results, though motivated by the special case $f(n) = n^{i\theta}$, extend naturally to completely multiplicative f for which f_{j_P} is regularly varying (see section 2 for the definition).

Addendum. I would like to thank the anonymous referee for some useful comments and for pointing out a recent paper by Aistleitner and Seip [1]. They deal with an optimization problem which is different yet curiously similar. The function $\exp(c_\theta(\log T)^{1-i\theta}(\log \log T)^{i\theta}g)$ appears in the same way, although their c_θ is expected to remain bounded as $\theta \rightarrow \frac{1}{2}$. It would be interesting to investigate any links further.

1. Bounded operators

Notation: Let l^1 and l^2

$\| \cdot \|_f : l^2 \rightarrow l^2$ is bounded if and only if $\sum_{n=1}^{\infty} \|k_n\|_f^2 < \infty$, in which case $\| \cdot \|_f = \sqrt{\sum_{n=1}^{\infty} \|k_n\|_f^2} \| \cdot \|_1$.

Proof. After (1.1), and since $\phi_f(N)$ increases with N , we need only provide a lower bound for an infinite sequence of N 's. Let $a_n = d(n)^{\frac{1}{2}}$ for $n \leq N$ and zero otherwise (N to be chosen later), where $d(n)$ is the divisor function. Thus $a_1^2 + \dots + a_N^2 = N$ and

$$\phi_f(N) \leq \sum_{n \leq N} a_n b_n = \frac{1}{d(N)} \sum_{n \leq N} \sum_{d|n} f(d) = \frac{1}{d(N)} \sum_{d \leq N} f(d) d \frac{N}{d} ; \quad (1.2)$$

say. We choose N

(see [2] for a detailed treatise on the subject). For example $x^{\frac{1}{2}}(\log x)^{\epsilon}$ is regularly-varying of index $\frac{1}{2}$ for any ϵ . The Uniform Convergence Theorem says that the above asymptotic formula is automatically uniform for λ in compact subsets of $(0, 1)$. Note that every regularly varying function of non-zero index is asymptotic to one which is strictly monotonic and continuous. We shall make use of Karamata's Theorem: for λ regularly varying of index $\frac{1}{2}$

$$\lambda^{-\frac{1}{2}} \gg \frac{x^{\frac{1}{2}}(x)}{\frac{1}{2}+1} \quad \text{if } \frac{1}{2} > \lambda, \quad \lambda^{-\frac{1}{2}} \gg \frac{x^{\frac{1}{2}}(x)}{\frac{1}{2}+1} \quad \text{if } \frac{1}{2} < \lambda,$$

while if $\frac{1}{2} = \lambda$, $\lambda^{-\frac{1}{2}}$ is slowly varying (regularly varying with index 0) and $\lambda^{-\frac{1}{2}} \sim x^{\frac{1}{2}}(x)$.

Notation. Let M^2

On the other hand, the RHS of (2.2) is greater than

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} f(p^r) f(p^s) g(p^{k_i r+1}) g(p^{k_i s+1}):$$

Hence $2M^2$ if and only if

$$\sum_{p \equiv m; n, 1}^{\infty} \sum_{k=0}^{\infty} f(p^m) g(p^n) f(p^k)$$

Thus, in particular, $M_{\mathbb{C}}^2 \approx M_0^2$. For $f \in M_{\mathbb{C}}^2$ if and only if $|f(p)| < 1$ for all primes p and $\sum_p |f(p)|^2 < \infty$. Thus

$$\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \leq A;$$

independent of p (since $f(p) \neq 0$).

The "quasi-norm" $M_f(T)$

Let $f \in M_{\mathbb{C}}^2$. From above we see that $\|f\|_f(M^2) \approx M^2$ but, typically, $\|f\|_f$ is not "bounded" on M^2 (if $f \neq 1$) in the sense that $\|f\|_f(a) = \|f\|_f(a)$ is not bounded by a constant for all $a \in M^2$. It therefore makes sense to define, for $T \geq 1$,

$$M_f(T) = \sup_{\substack{a \in M^2 \\ \|a\|_f = T}} \|f\|_f(a).$$

We aim to find the behaviour of $M_f(T)$ for large T .

We shall consider f completely multiplicative and such that $f|_P$ is regularly varying of index α with $\alpha > -1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index α) with $\tilde{f}(p) = f(p)$ for every prime p .

Our main result here is the following:

Theorem 2.3

Let $f \in M_{\mathbb{C}}^2$, such that $f \neq 0$ and $f|_P$ is regularly varying of index α where $\alpha \in (-1/2; 1)$. Then

$$\log M_f(T) \sim c(\alpha) \tilde{f}(\log T \log \log T) \log T$$

where $\tilde{f}$ is any regularly varying extension of $f|_P$ and

$$c(\alpha) = B\left(\frac{1}{\alpha}\right)$$

Collecting those terms for which $(c; d) = k$, writing $c = km$, $d = kn$

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subject to $0 < \alpha_p < 1$ and $\sum_p \frac{1}{1-\alpha_p^2} = T^2$. The maximum clearly occurs for α_p decreasing (if $\alpha_{p^0} > \alpha_p$ for primes $p < p^0$, then the sum increases in value if we swap α_p and α_{p^0}). Thus we may assume that α_p is decreasing.

By interpolation we may write $\alpha_p = g(\frac{p}{P})$ where $g : (0; 1) \rightarrow (0; 1)$ is continuously differentiable and decreasing. Of course g will depend on P . Let $h = \log \frac{1}{1-g^2}$, which is also decreasing. Note that

$$2 \log T = \sum_p h \frac{p}{P} = \sum_{p \leq aP} h \frac{p}{P} = h(a) \frac{1}{2} (aP) + o(a \log T);$$

for P sufficiently large, for some constant $c > 0$. Thus $h(a) \sim C_a$ (independent of T).

Now, for $F : (0; 1) \rightarrow [0; 1]$ decreasing,

$$\sum_{ax < p \leq bx} F \frac{p}{x} = \frac{x}{\log x} \int_a^b F + O \left(\frac{x F(a)}{(\log x)^2} \right); \quad (2.9)$$

where the implied constant is independent of F (and x). For, on writing $\psi(x) = \text{li}(x) + e(x)$, the LHS is

$$\begin{aligned} \int_{ax}^{bx} F \frac{t}{x} d\psi(t) &= x \int_a^b \frac{F(t)}{\log xt} dt + \int_a^b F(t) d\psi(xt) \\ &= \frac{x}{\log \mu x} \int_a^b F + \int_a^b F(t) e(xt) dt + \int_a^b e(xt) dF(t) \quad (\text{some } \mu \in [a; b]) \\ &= \frac{x}{\log x} \int_a^b F \end{aligned}$$

me $\mu \in [a; b]$)

approximate arbitrarily closely with such functions. On writing $g = s \pm h$ where $s(x) = \frac{1}{1 + e^{-x}}$, we have

$$\begin{aligned} \int_0^1 \frac{g(u)}{u^\alpha} du &= \int_0^1 \frac{h(u)u^{1-\alpha}}{1 + e^{-u}} du + \int_0^1 \frac{g^0(u)u^{1-\alpha}}{1 + e^{-u}} du \\ &= \int_0^1 \frac{1}{1 + e^{-u}} s^0(h(u))h^0(u)u^{1-\alpha} du = \frac{1}{1 + e^{-u}} \int_0^{h(0^+)} s^0(x)l(x)^{1-\alpha} dx; \end{aligned}$$

where $l = h^{i-1}$, since $\frac{1}{u^\alpha} \leq 1$ as $u \leq 1$. The final integral is, by Hölder's inequality at most

$$\left(\int_0^{h(0^+)} s^{0(1-\alpha)} \right)^{\frac{1}{1-\alpha}} \left(\int_0^{h(0^+)} l^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} \quad (2.10)$$

But $\int_0^{h(0^+)} l^{1-\alpha} = \int_0^1 u h^0(u) du = \frac{R_1}{\alpha} h^{\alpha-2}$, so

$$\int_0^1 \frac{g(u)}{u^\alpha} du \leq \frac{2^{1-\alpha}}{1 + e^{-u}} \left(\int_0^1 s^{0(1-\alpha)} \right)^{\frac{1}{1-\alpha}}.$$

A direct calculation shows that $\int_0^1 s^{0(1-\alpha)} = 2^{1-\alpha} B\left(\frac{1}{\alpha}; 1, \frac{1}{2\alpha}\right)$. This gives the upper bound.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (2.10) if $l = (s^0)^{1-\alpha}$ is constant; i.e. $l(x) = c(s^0(x))^{1-\alpha}$ for some constant $c > 0$ chosen so that $\int_0^1 l = 2$. This means we take

$$h(x) = (s^0)^{\frac{1}{1-\alpha}} \frac{x^{\frac{1}{\alpha}}}{c} = \log \frac{1}{2} + \frac{1}{2} \frac{1}{1 + \frac{c}{x}^{2\alpha}};$$

from which we can calculate g . In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use (2.3), and define a_p by:

$$a_p = g_0 \frac{p^{\frac{1}{\alpha}}}{P};$$

where $P = \log T \log \log T$ and g_0 is the function

$$g_0(x) = \frac{1}{1 + \frac{2}{1 + \left(\frac{c}{x}\right)^{2\alpha}}};$$

with $c = 2^{1+\alpha} B\left(\frac{1}{\alpha}; 1, \frac{1}{2\alpha}\right)$. As such, by the same methods as before, we have $kak = T^{1+o(1)}$ and

$$\log \frac{k'_{\alpha} ak}{kak} = \sum_p f(p) g_0 \frac{p^{\frac{1}{\alpha}}}{P} + O(1) \gg \frac{Pf(P)}{\log P} \int_0^1 \frac{g_0(u)}{u^\alpha} du.$$

By the choice of g_0 , the integral on the right is $\frac{B\left(\frac{1}{\alpha}; 1, \frac{1}{2\alpha}\right)}{(1 + e^{-u})^{2\alpha}}$, as required. $\square$

Remark. From the above proof, we see that the supremum $(\frac{k'_{\alpha} ak}{kak})$ over $M_{\frac{2}{\alpha}}$ is roughly the same size as the supremum over M^2 ; i.e. they are log-asymptotic to each other. Is it true that these respective suprema are closer still; eg. are they asymptotic to each other for $\frac{1}{2} < \alpha < 1$?

3. The special case $f(n) = n^{i-\alpha}$.

In this case we can take $f(x) = x^{i-\alpha}$ which is regularly varying of index $i-\alpha$. Here we shall write $'_{\alpha}$ for $'_f$ and M_{α} for M_f .

³The integral is $2^{1-\alpha} \int_0^1 e^{i-x} (1 + e^{-x})^{i-2\alpha} dx = 2^{1-\alpha} \int_0^1 t^{1-\alpha} (1 + t)^{i-1-2\alpha} dt$.

Theorem 3.1

We have

$$M_1(T) = e^{\circ} (\log \log T + \log \log \log T + 2 \log 2 + 1 + o(1)); \quad (3.1)$$

while for $\frac{1}{2} < \beta < 1$,

$$\log M_{\beta}(T) = \frac{\mu_{\beta} \left(\frac{1}{\beta}, 1 + \frac{1}{2\beta} \right)^{\beta}}{(1 + \beta) 2^{\beta}} + o(1) \frac{(\log T)^{1+\beta}}{(\log \log T)^{\beta}}; \quad (3.2)$$

Remark. As noted in the introduction, these asymptotic formulae bear a strong resemblance to the (conjectured) maximal order of $j^3(\beta + it)j$. It is interesting to note that the bounds found here are just larger than what is known about the lower bounds for $Z_{\beta}(T) = \max_{1 \leq t \leq T} j^3(\beta + it)j$. In a recent paper (see [8]), Lamzouri suggests $\log Z_{\beta}(T) \gg C(\beta)(\log T)^{1+\beta}(\log \log T)^{\beta}$ with some specific function $C(\beta)$ (for $\frac{1}{2} < \beta < 1$). We note that the constant appearing in (3.2) is not $C(\beta)$ since, for β near $\frac{1}{2}$, the former is roughly $p^{\frac{1}{\beta-1/2}}$, while $C(\beta) \gg p^{\frac{1}{2\beta-1}}$. For $\beta = 1$, see the comment in the introduction.

It would be very interesting to be able to extend these ideas (and results) to the $\beta = \frac{1}{2}$ case. As we show in the appendix, we cannot do this by restricting $\frac{1}{2}$ to smaller domains in $\mathbb{N}^2$. Somehow the analogy | if such exists | between M_{β} and Z_{β} breaks down just here.

Proof of Theorem 3.1. For $\frac{1}{2} < \beta < 1$ the result follows from Theorem 2.3, so we only concern ourselves with $\beta = 1$.

For an upper bound we use (2.5) with $f(p) = 1/p$ (and $A = 1$). Thus

$$\frac{k'_{1ak}}{kak} \leq (2) \sum_p \mu \left(1 + \frac{p}{p} \right) :$$

by (2.9). Thus

$$\frac{k' - 1}{k} \log_2 T + \log_3 T + \int_a^A \frac{g(u)}{u} du + \int_a^1 \frac{1}{u} du + \frac{2}{A} + o(1)$$

for all $A > 1 > a > 0$. We need to minimise the constant term. Since $g(u) < 1$, the minimum occurs for a arbitrarily small. On the other hand $\int_A^1 \frac{g(u)}{u} du \sim \left(\frac{1}{A}\right) \int_A^1 g^2)^{1/2} = o(1)$.

after some calculation.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (3.3) if $l = s^0$ is constant; i.e. $l(x) = cs^0(x)$ for some constant $c > 0$ chosen so that $\sum_{n=1}^{\infty} l = 2$ (i.e. we take $c = 2$). Thus, actually $\gamma = 2 \log 2$ and the supremum is achieved for the function g_0 , where

$$g_0(x) = \prod_{p \leq x} \frac{1}{1 + \frac{2}{1 + (\frac{2}{x})^2}}.$$

In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use Corollary 2.5, and define a_p by:

$$a_p = g_0\left(\frac{p}{P}\right);$$

where $P = \log T \log \log T$. As such, by the same methods as before, we have $k = T^{1+o(1)}$. Let $a > 0$ and $P = \log T \log \log T$. By Corollary 2.5

$$\frac{k^{1/2} a k}{k a k} \leq \prod_{p \leq aP} \frac{1}{1 + \frac{a_p}{p}} = \prod_{p \leq aP} \frac{1}{1 + \frac{1}{p}} \prod_{p \leq aP} \frac{1}{1 + \frac{1/p - 1}{p}} \prod_{p > aP} \frac{1}{1 + \frac{a_p}{p}}. \quad (3.4)$$

Using Merten's Theorem, the first product on the right is $e^{o(\log aP + o(1))}$, while the second product is greater than

$$\exp \left(\sum_{p \leq aP} \frac{1}{p} \left(\frac{a_p}{p} \right)^{3/4} \right) \geq \exp \left(\sum_{p \leq aP} \frac{1}{p} g_0(p/P) \right).$$

The sum is asymptotic to $\frac{a}{\log P} \int_0^a \frac{1 - g_0(u)}{u} du < \frac{a}{\log P}$, for any given $\epsilon > 0$, for sufficiently small a . The third product in (3.4) is greater than

$$\exp \left(\sum_{p > aP} \frac{1}{p} \left(\frac{a_p}{p} \right)^{3/4} \right) = \exp \left(\frac{(1 + o(1))}{\log P} \int_a^\infty \frac{g_0(u)}{u} du \right)$$

by (2.9). Thus

$$\frac{k^{1/2} a k}{k a k} \geq e^{\mu \log P + \int_a^\infty \frac{g_0(u)}{u} du + \log a - \epsilon}, \quad e^{\mu \log P + L(g_0) - \epsilon}$$

for a sufficiently small a . As $L(g_0) = 2 \log 2$ and ϵ arbitrary, this gives the required lower bound. $\square$

Lower bounds for γ $\circledast$

We see that $m_{\mathbb{R}}(T)$ corresponds closely to the conjectured minimal order of $j^3(\mathbb{R} + iT)$ (see [3] and [9]). We omit the proofs, but just point out that for an upper bound (for $1 = m_{\mathbb{R}}(T)$) we use

$$\frac{kak}{k'_{\mathbb{R}}ak} \leq \prod_p \left(1 + \frac{p^{\frac{1}{2}}}{p^{\mathbb{R}}} \right);$$

which can be obtained in much the same way as (2.5). For the lower bound, we choose $\mathbb{R}$ as i 1 times the choice in Theorem 3.1 and use Corollary 2.5.

The above formulae suggest that the supremum (respectively infimum) of $k'_{\mathbb{R}}ak=kak$ with $a \leq M^{-2}$ and $kak = T$ are close to the supremum (resp. infimum) of $j^3_{\mathbb{R}}j$ on $[1; T]$. One could therefore speculate further that there is a close connection between $k'_{\mathbb{R}}ak=kak$ (for such a) and $j^3(\mathbb{R} + iT)$, and hence between $Z_{\mathbb{R}}(T)$ and $M_{\mathbb{R}}(T)$. Recent papers by Gonek [4] and Gonek and Keating [5] suggest this may be possible, or at least that $M_{\mathbb{R}}$ is a lower bound for $Z_{\mathbb{R}}$. On the Riemann Hypothesis, it was shown in [4] (Theorem 3.5) that $j^3(s)$ may be approximated for $\frac{3}{4} > \frac{1}{2}$ up to height T by the truncated Euler product

$$\prod_{p \leq P} \frac{1}{1 + p^{i-s}} \quad \text{for } P \leq T.$$

Thus one might expect that, with $a \leq M_c^{-2}$ maximizing $\frac{k'_{\mathbb{R}}ak}{kak}$ subject to $kak = T$, and $A(s) = \prod_{p \leq P} \frac{1}{1 + a_p p^{i-s}}$ (with $P \leq T$),

$$\int_{-T}^T j^3(\mathbb{R} + it) j^2 j A(it) j^2 dt \gg \int_{-T}^T \prod_{p \leq P} \frac{1}{(1 + \frac{p^{it}}{p^{\mathbb{R}}})(1 + a_p p^{it})} dt = \int_{-T}^T \prod_{p \leq P} j B_p(it) j^2 dt, \quad \text{with } \frac{1}{2} < \mathbb{R} = T J/J/F20 \ 9.96 \ T f \ 10.52$$

