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Nonlinear error dynamics for cycled data assimilation methods

by

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1. Introduction

Combining observational data with a dynamical system presents many challenges to different areas of science. Usually, both observations and model are uncertain, which leads to formulating the problem in a statistical framework [1][2][3]. However, this problem can also considered in a deterministic setting [1][2]. In many areas of climate and forecasting science, the dimension of the system dictates the method which is used to combine observations and dynamics. Data assimilation algorithms need to be applicable within situations where the dimension of the state space, typically in numerical weather prediction, ranges between orders of $O(10^7 - 10^8)$ and observational data is of order of $O(10^6)$. Since these algorithms deal with such high dimension, it is natural to extend analysis into infinite dimension to capture the key features of large-scale systems. Using an infinite dimensional approach, we are able to work within a framework that is best suited to analyse directly, challenges that exist in high dimensional data assimilation algorithms. In this work we restrict ourselves to analysing three dimensional variational data assimilation (3DVar) type methods in a large-scale or infinite dimensional setting.

This work is primarily interested in an estimate for the ion

is equal to the true observation operator and that the given observations $y_k \in Y$ are

is called the *Tikhonov inverse***, with regularization parameter $\alpha > 0$. In other disciplines,**

such that

$$\limsup_{k\to\infty} \|e_{k+1}\| \leq \frac{\|N\|\upsilon + \|R_\alpha\|\delta}{1-\Lambda}. \tag{32}$$

Proof. We use induction as follows. For the base case we set $k = 0$, and from (29) we obtain

$$\|e_1\| \leq \Lambda\|e_0\| + \|N\|\upsilon + \|R_\alpha\|\delta, \tag{33}$$

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In this finite dimensional setting, it is clear that from Lemma 3.3 that given any global Lipschitz constant $K > 0$, i.e. any model dynamics, we can always choose an $\alpha > 0$, sufficiently small so that $\Lambda \leq K\rho < 1$. This is an interesting conclusion to draw, since the regularization parameter controls how much we trust the background term in (8). Of course reducing α means that we solve the problem in (6) more accurately which implies that solving the problem accurately keeps the data assimilation scheme stable for all time. However, from the theory of Tikhonov regularization, we know that α must be kept large enough to shift the spectrum of H^*H to combat ill-posedness in the observation operator. Representing the problem in an infinite dimension allows us to directly represent this effect of ill-posedness into the problem. We will see that for an ill-posed observation operator, significant damping must be present on higher spectral modes for us to control the behaviour of the analysis error over time. Firstly we explore the less interesting situation of well-posed observation operators H to highlight the difficulty in treating ill-posed operators in an infinite dimension.

Lemma 3.4. *For an infinite dimensional state space* X*, an injective well-posed operator* H *and a parameter* $0 < \rho < 1$*, by choosing the regularization parameter* $\alpha > 0$ *sufficiently small we can always achieve* $\Lambda \leq \rho < 1$.

Proof. As the operator H is well-posed, $G := H^*H$ has a complete orthonormal basis $\phi_{(1)}, \dots, \phi_{( )}$ of eigenvectors with eigenvalues $\lambda_{(1)}, \dots, \lambda_{( )} > 0$. If $K > 1$ we choose ρ such that $\rho < 1/K$, otherwise we choose any $\rho < 1$. Then we choose an α such that

$$\alpha \left(\frac{1}{\rho} - 1 \right) = \min_{j=1,\dots,} |\lambda_{(j)}|$$

$$+ \|N\zeta_k\| + \left\|R_\alpha \eta_{k+1}\right\| \tag{50}$$

$$\leq \left(\Lambda_k^{(1)} + \Lambda_k^{(2)}\right) \|e_k\| + \|N\| \upsilon + \|R_\alpha\| \delta, \tag{51}$$

where we have assumed Lipschitz continuity,

$$\left\| M_k^{(j)}(x_k^{(a)}) - M_k^{(j)}(x_k^{(t)}) \right\| \leq K_k^{(j)} \left\| x_k^{(a)} - x_k^{(t)} \right\| \tag{52}$$

for $j = 1, 2$, defining $\Lambda_k^{(1)} := K_k^{(1)} \cdot \|N|_{X_1}\|$ and $\Lambda_k^{(2)} := K_k^{(2)} \cdot \|N|_{X_2}\|$ with restrictions according to the singular system of H. Again, we now assume that the modelled nonlinear operator M is time invariant such

Lemma 3.8. *For the Hilbert space* $(X, \|\cdot\|_{B^{-1}})$, *on the subspace* X_1 *and for a parameter* $0 < \rho < 1$, *by choosing the regularization parameter* $\alpha > 0$ *sufficiently small, we can always achieve* $\|N|_{X_1}\| \leq$

Theorem 3.11. *For the Hilbert space* $(\mathbf{X}, \|\cdot\|_{\mathbf{B}^{-1}})$, *assume the system* $\mathbf{M}_\mathbf{k}$ *is Lipschitz continuous and dissipative with respect to higher spectral modes of* $\mathbf{H}$ 8.36

error. The Lorenz '63 equations are as follows,

$$\frac{dx}{dt} = -\sigma(x - y), \tag{65}$$

$$\frac{dy}{dt} = \rho x - y - xz, \tag{66}$$

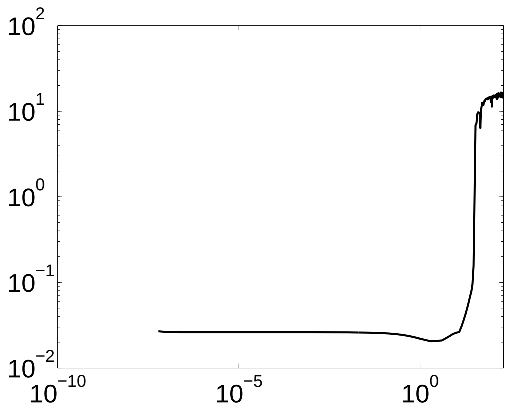
$$\frac{dz}{dt} = xy - \beta z, \tag{67}$$

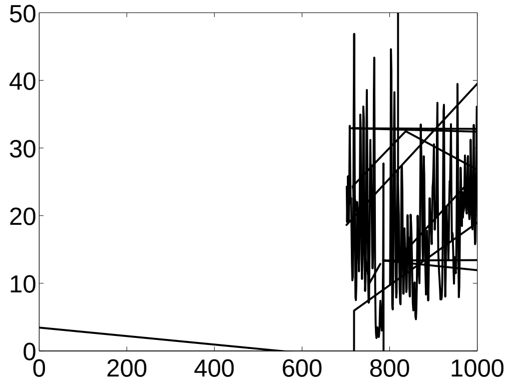
where typically σ, ρ and β are known as the Prandtl number, the Rayleigh number and a non-dimensional wave number respectively. Throughout these experiments we will use the classical parameters, σ =x 10,

We set the background weight equal to the background variance such that $w_{(b)} = \sigma^2_{(b)}$

The theory in Section 3 was for weighted norms with respect to the error covariances. As previously discussed, we consider 3DVar-type methods which involves a static covariance matrix for all time. Therefore the inverse background covariance matrix C^{-1} is acting as a scaling on the analysis error. We calculate the analysis error $\|e_k\|_{C^{-1}}$ and plot its evolution over time for regularization parameters, $\alpha = 200, 2$ and 10^{-10}. The figures are identical to Figure 2(a), Figure 3(a) and Figure 4(a) with a rescaled vertical axis. For $\alpha = 200$ the analysis error fluctuate around 0.005, then for $\alpha = 2$ the analysis error reduces to 10^{-5}. With a further inflation such that $\alpha = 10^{-10}$ the analysis error increases and fluctuates around 0.002.

We have repeated these experiments for a variety of observation operators H, different initial conditions and observation error drawn from different distributions. We obtain similar results, however we omit these experiment to keep this paper concise.

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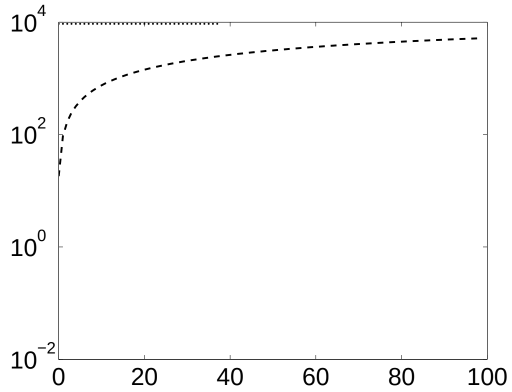
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in future work.

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