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The Stokes Conjecture for Waves with Vorticity

by

Eugen Varvaruca and Georg S. Weiss

THE STOKES CONJECTURE FOR WAVES WITH VORTICITY

EUGEN VARVARUCA AND GEORG S. WEISS

Abstract. We study stagnation points of two-dimensional steady gravity free-surface water waves with vorticity.

We obtain for example that, in the case where the free surface is an injective curve, the asymptotics at any stagnation point is given either by the \Stokes corner ow" where the free surface has a *corner of* 120 , or the free surface ends in a *horizontal cusp*, or the free surface is *horizontally at* at the stagnation point. The cusp case is a new feature in the case with vorticity, and it is not possible in the absence of vorticity.

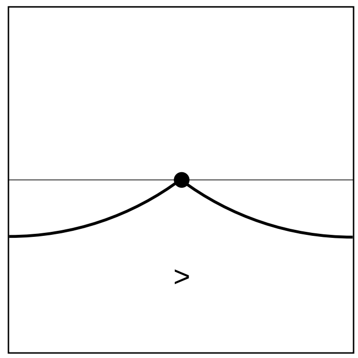
In a second main result we exclude horizontally at singularities in the case that the vorticity is 0 on the free surface. Here the vorticity may have in nitely many sign changes accumulating at the free surface, which makes this case particularly di cult and explains why it has been almost untouched by research so far.

Our results are based on calculations in the original variables and do not rely on structural assumptions needed in previous results such as isolated singularities, symmetry and monotonicity.

Contents

1. Introduction

The classical hydrodynamical problem of traveling two-dimensional gravity water waves with vorticity can be described mathematically as a free-boundary problem for a semilinear elliptic equation: given an open connected set in the $(x; y)$ plane and a function of one variable, nd a non-negative function in such that



the vorticity function , and therefore, as a a consequence of [22], the free surface of such a wave necessarily has corners of 120 at stagnation points.

The present paper is the rst study of stagnation points of steady two-dimensional gravity water waves with vorticity in the absence of structural assumptions of isolatedness of stagnation points, symmetry and monotonicity of the free boundary, which have been essential assumptions in all previous works. We obtain for example that, in the case when the free surface is an injective curve, the asymptotics at any stagnation point is given either by the \Stokes corner ow" where the free surface has a *corner of* 120 , or the free surface ends in a *horizontal cusp*,

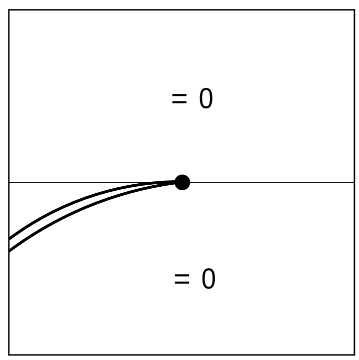


Figure 2. Cusp

or the free surface is *horizontally at* at the stagnation point.

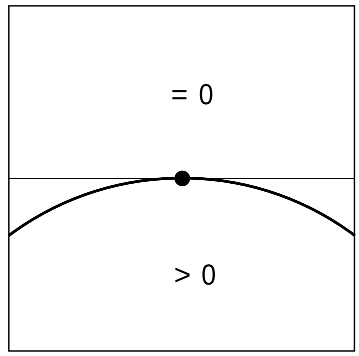


Figure 3. Horizontally flat at stagnation point

The cusp case is a new feature in the case with vorticity, and it is not possible without the presence of vorticity [23]. It is interesting to point out that Gerstner [10] constructed an explicit example of a steady wave with vorticity whose free surface has a *vertical cusp* at a stagnation point. However, this vertical cusp is due to the fact that in his example the vorticity is infinite at the free surface, while in the present paper we only consider the case of vorticities which are smooth up to the free surface. We conjecture the cusps in our paper — the existence of which is still open — to be due to the break-down of the Rayleigh-Taylor condition in the presence of vorticity.

The second half of our paper is devoted to *excluding horizontally flat singularities* in the case that the vorticity is non-negative at the free surface. (Horizontally flat singularities are possible if the vorticity is negative at the free surface.) Of particular difficulty is the case when the vorticity is 0 at the free surface, and may have infinitely many sign changes accumulating there.

Let us briefly state our main result and give a plan of the paper:

Main Result. *Let ψ be a suitable weak solution of (1.1) (compare to Definition 3.2) satisfying*

$$|\nabla\psi(x,y)|^2 \le C\max(-y,0) \quad \textit{locally in } \Omega;$$

let the free boundary $\partial\{\psi > 0\}$ be a continuous injective curve $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma(0) = (x^0, 0)$, and assume that the vorticity function satisfies either $|\gamma(z)| \le Cz$, or $\gamma(z) \ge 0$, for all z in a right neighborhood of 0.

(i) If the Lebesgue density of the set $\{\psi > 0\}$ at $(x^0, 0)$ is positive, then the free boundary is in a neighborhood of $(x^0, 0)$ the union of two

that for every non-negative $\eta \in C_0^\infty(D)$

$$\int_D (\nabla \max(u-\epsilon;0)\cdot\nabla\eta - C_D\,\eta)\,dx$$
$$= -\int_D \eta\,(\chi_{\{u>\epsilon\}}\Delta u + C_D)\,dx - \int_{D\cap\partial\{u>\epsilon\}} \eta\,\nabla u\cdot\nu\,dH^{n-1} \le 0;$$

provided that $|f(u)| \le C_D$ in D. Letting $\epsilon \to 0$ and using that u is continuous and nonnegative in Ω, we obtain

$$\int_D (\nabla u\cdot\nabla\eta - C_D\,\eta)\,dx \le 0.$$

Thus $\Delta u + C_D$ is a nonnegative distribution in D, and the stated property follows.

Since we want to focus in the present paper on the analysis of stagnation points, we will assume that everything is smooth away from $x_n = 0$, however this assumption may be weakened considerably by using in $\{x_n > 0\}$ regularity theory for the Bernoulli free boundary problem (see [2] for regularity theory in the case $f = 0$ — which could effortlessly be perturbed to include the case of bounded f — and see [5] for another regularity approach which already includes the perturbation).

Definition 3.2 (Weak Solution). We define $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to be a *weak solution* of (3.1) if the following are satisfied:

$$M_{x^0;u}(r) = M(r) = I(r)$$

Also observe that letting $\to 0$ in

$$\int_{B_r(x^0)} \nabla u \quad \nabla \max(u \quad ; 0)^{1+} \, dx = \int_{B_r(x^0)} f(u) \max(u \quad ; 0)^{1+} \, dx + \int_{\partial B_r(x^0)} \max(u \quad ; 0)^{1+} \; \nabla u \quad dH^{n \quad 1}$$

for a.e. $r \in (0; \;)$, we obtain the integration by parts formula

$$\int_{B_r(x^0)} |\nabla u|^2 \quad u f(u) \quad dx = \int_{\partial B_r(x^0)} u \nabla u \quad dH^{n \quad 1} \tag{3.10}$$

for a.e. $r \in (0; \;)$.

Note also that

$$I'(r) = \quad (n+1) r^{\; n \; 2} \int_{B_r(x^0)} (|\nabla u|^2 \quad u f(u) + x_n \quad _{\{u>0\}}) \, dx$$

we obtain $j\nabla u j$ ($|\nabla u|$)

$$+ \, r^{\; n \; 1}$$

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Lemma 4.4. *Let u be a variational solution of (3.1) satisfying Assumption 4.1. Then:*

(i) Let $x^0 \in$ be such that $x^0_n = 0$. Then the limit $M_{x^0,u}(0+)$ exists and is finite. (Note that $u = 0$ in $\{x_n = 0\}$ by assumption.)

(ii) Let $x^0 \in$ be such that $x^0_n = 0$, and let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := \frac{u(x^0 + r_m x)}{r_m^{3/2}} \tag{4.2}$$

converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ to a blow-up limit u_0. Then u_0 is a homogeneous function of degree $3/2$, i.e.

$$u_0(\lambda x) = \lambda^{3/2} u_0(x) \quad \text{for any } x \in \mathbb{R}^n \text{ and } \lambda > 0.$$

(iii) Let u_m be a converging sequence of (ii). Then u_m converges strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$.

(iv) Let $x^0 \in$ be such that $x^0_n = 0$. Then

$$M_{x^0,u}(0+) = \lim_{r}$$

(ii): For each $0 < \delta < \infty$ the sequence u_m is by assumption bounded in $C^{0,1}(B_\delta)$. For any $0 < \varrho < \sigma < \infty$, we write the identity (3.8) in integral form as

$$2 \int_\varrho^\sigma r^{-n-1} \int_{\partial B_r(x^0)} \left(\nabla u \cdot \nu - \frac{3}{2}\frac{u}{r} \right)^2 d\mathcal{H}^{n-1} dr$$
$$= M(\sigma) - M(\varrho) - \int_\varrho^\sigma r^{-n-2} K(r)\, dr. \tag{4.4}$$

It follows by rescaling in (4.4) that

$$2 \int_{B_\sigma(0)\setminus B_\varrho(0)} |x|^{-n-3} \left(\nabla u_m(x) \cdot x - \frac{3}{2} u_m(x) \right)^2 dx$$
$$\le M(r_m \sigma) - M(r_m \varrho) + \int_{r_m \varrho}^{r_m \sigma} r^{-n-2} |K(r)|\, dr \to 0 \quad \text{as } m \to \infty;$$

which yields the desired homogeneity of u_0.

(iii): In order to show strong convergence of u_m in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, it is sufficient, in view of the weak L^2-convergence of ∇u_m, to show that

$$\limsup_{m \to \infty} \int_{\mathbb{R}^n} |\nabla u_m|^2 \phi \, dx \le \int_{\mathbb{R}^n} |\nabla u_0|^2 \phi \, dx$$

for each $\phi \in C^1_0(\mathbb{R}^n)$. Let $\delta := \mathrm{dist}(x^0, \partial\Omega)/2$. Then, for each m, u_m is a variational solution of

$$\Delta u_m = -r_m^{1/2} f(r_m^{3/2} u_m) \quad \text{in } B_{\delta/r_m} \cap \{u_m > 0\}, \tag{4.5}$$
$$|\nabla u_m|^2 = x_n \quad \text{on } B_{\delta/r_m} \cap \partial\{u_m > 0\}.$$

Since u_m converges to u_0 locally uniformly, it follows from (4.5) that u_0 is harmonic in $\{u_0 > 0\}$. Also, using the uniform convergence, the continuity of u_0 and its harmonic0 Td [(0)6Td [(u0)6F11 9.962 12.6794161

; (4.5)

of u_0, we obtain that

$$\begin{aligned}\lim_{m\to\infty} M_{x^0,u}(r_m) &= \int_{B_1} |\nabla u_0|^2\,dx - \frac{3}{2}\int_{\partial B_1} u_0^2\,d\mathcal{H}^{n-1}\\ &\quad + \lim_{r\to 0+} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx\\ &= \lim_{r\to 0+} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx.\end{aligned}$$

Thus $M_{x^0,u}(0+) \geq 0$, and equality implies that for each $\delta > 0$, u_m converges to 0 in measure in the set $\{x_n > \delta\}$ as $m \to \infty$, and consequently $u_0 = 0$ in $\mathbb{R}^n$.

(v): For each $\delta > 0$ we obtain from the Monotonicity Formula (Theorem 3.4), Remark 4.2 as well as the fact that $\lim_{x\to x^0} M_{x,u}(r) = M_{x^0,u}(r$

Proof. Consider a blow-up sequence u_m

$\partial\{u > 0\}$ is in a neighborhood of x^0 a continuous injective curve $\sigma : (-t_0, t_0) \to \mathbb{R}^2$ such that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma(0) = x^0$. Then the following hold:

(i) If $M(0+) = \int_{B_1} x_2^+ \chi_{\{x: \pi/6 < \theta < 5\pi/6\}} \, dx$, then (cf. Figure 4) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$ and, depending on the parametrization, either

$$\lim_{t \to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}} \text{ and } \lim_{t \to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}},$$

or

$$\lim_{t \to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}} \text{ and } \lim_{t \to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}}.$$

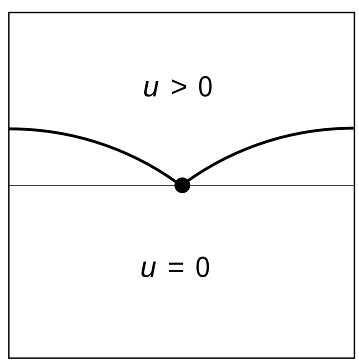

Figure 4. Stokes corner

(ii) If $M(0+) = \int_{B_1} x_2^+ \, dx$, then (cf. Figure 5) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ changes sign at $t = 0$ and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

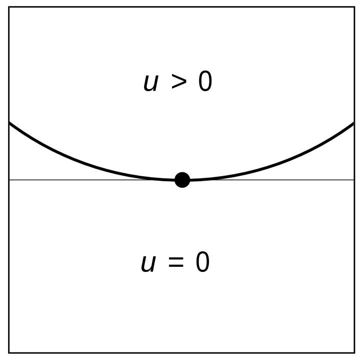

Figure 5. Full density singularity

(iii) If $M(0+) = 0$*, then (cf. Figure 6 and Figure 7)* $_1(t) \neq x_1^0$ *in* $(\ t_1, t_1)\setminus\{0\}$*,* $_1 \ \ x_1^0$ *does not change its sign at* $t = 0$*, and*

$$\lim_{t \to 0} \frac{_2(t)}{_1(t) \ \ x_1^0} = 0.$$

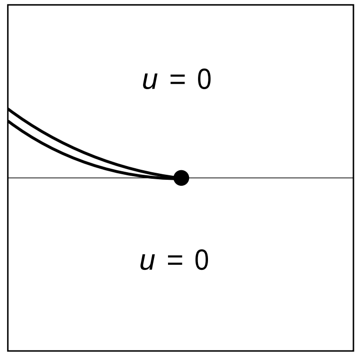

Figure 6. Left cusp

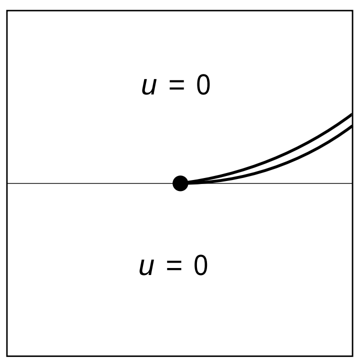

Figure 7. Right cusp

Proof. We may assume that $x_1^0 = 0$. Moreover, for each y 2 R^2 we de ne arg y as the complex argument of y, and we de ne the sets

L := f $_0$ 2 [0;

Remark 4.7. In [23] we used a strong version of the Rayleigh-Taylor condition (which is always valid in the case of zero vorticity) in order to prove that the cusps of case (iii) are not possible. Unfortunately we do not have the Rayleigh-Taylor condition (4.1) in the case with nonzero vorticity, and the method of [23] breaks down here. Still we *conjecture* that the cusps in case (iii) are *not* possible when assuming the Rayleigh-Taylor condition.

5. Partial regularity at non-degenerate points

Definition 5.1 (Stagnation Points). Let u be a variational solution of (3.1). We call $S^u := \{x \in \Omega : x_n = 0 \text{ and } x \in \partial\{u > 0\}\}$ the set of *stagnation points*.

Throughout the rest of this section we assume that $n = 2$.

Definition 5.2 (Non-degeneracy). Let u be a variational solution of (3.1).

We say that a point $x^0 \in \Omega \cap \partial\{u > 0\} \cap \{x_2 = 0\}$ is *degenerate* if

$$\frac{u(x^0 + rx)}{r^{3/2}}$$

By the proof of Theorem 4.5(ii), the sequence u_m converges strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ to the homogeneous solution

$$u_0(\ ;\ ) = \frac{\sqrt{2}}{3}\ ^{3/2}\cos(\frac{3}{2}(\min(\max(\ ;$$

Lemma 6.6. *Let $r_0 > 0$ and $\alpha > 1$. Let*

$$G := \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < r_0, |\theta| < \pi/(2\alpha)\}.$$

Let $w \in C^2(G) \cap C(\overline{G})$ be a superharmonic function in G, such that $w(0,0) = 0$ and $w > 0$ in $\overline{G} \setminus \{(0,0)\}$. Then there exists $\delta > 0$ such that

$$w(\rho\cos\theta, \rho\sin\theta) \geq \delta\rho^\alpha \cos(\alpha\theta) \quad \text{in } \overline{G},$$

and in particular

$$w(\rho, 0) \geq \delta\rho^\alpha \quad \text{for all } \rho \in (0, r_0).$$

Suppose for a contradiction that $M(0+) = \int_{B_1} x_2^+ \, dx$. Then, the assumption on f and Theorem 4.6 yield the existence of $r_0 > 0$ and $\delta \in (0, \pi/6)$, such that u is superharmonic in $\{u > 0\} \cap B_{r_0}$ and $\overline{G} \setminus \{(0,0)\} \subset \{u > 0\} \cap B_{r_0}$, where $G := \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < r_0, \delta < \theta < \pi - \delta\}$. After a suitable rotation, we may apply Lemma 6.6, obtaining the existence $\delta > 0$ such that

$$u(0, x_2) \geq \delta x_2^\alpha \quad \text{for all } x_2 \in (0, r_0),$$

where $\alpha := \pi/(\pi - 2\delta)$, so that $\alpha < 3/2$. But this contradicts the estimate

$$u(0, x_2) \leq C x_2^{3/2},$$

which is a consequence of the Bernstein estimate assumption 4.1.

Motivated by Remark 6.4, we will focus in the present paper on the case $f(0) = 0$.

Theorem 6.7 (Frequency Formula). *Let u be a variational solution of (3.1) satisfying Assumption 4.1, let x^0 be a stagnation point, and let $\delta := \operatorname{dist}(x^0, \partial\Omega)/2$. Let*

$$D_{x^0,u}(r) = D(r) = \frac{r \int_{B_r(x^0)} (|\nabla u|^2 - u f(u)) \, dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}$$

and

$$V_{x^0,u}(r) = V(r) = \frac{r \int_{B_r(x^0)} x_n^+ (1 - \chi_{\{u>0\}}) \, dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}.$$

Then the "frequency"

$$\begin{aligned} H_{x^0,u}(r) = H(r) &= D(r) - V(r) \\ &= \frac{r \int_{B_r(x^0)} \left(|\nabla u|^2 - u f(u) + x_n^+ (\chi_{\{u>0\}} - 1) \right) dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \end{aligned}$$

satisfies for a.e. $r \in (0,\infty)$ *the identities*

$$H'(r) = \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)} u^2\, dH^{n-1}\right)^{1/2}} - D(r)\frac{u}{\left(\int_{\partial B_r(x^0)} u^2\, dH^{n-1}\right)^{1/2}} \right]^2 dH^{n-1} + \frac{2}{r}V^2(r) + \frac{2}{r}V(r)\left(H(r) - \frac{3}{2}\right) + \frac{K(r)}{\int_{\partial B_r(x^0)} u^2\, dH^{n-1}} \tag{6.1}$$

and

$$H'(r) = \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)} u^2\, dH^{n-1}\right)^{1/2}} - H(r)\frac{u}{\quad} \right.$$

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Using the identities (3.6) and (3.7), we therefore obtain that, for a.e. $r \in (0, \delta)$,

$$
\begin{aligned}
H'(r) = {} & \frac{2r \int_{\partial B_r(x^0)} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1} - 3 \int_{\partial B_r(x^0)} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1} + K(r)}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \\
& - (D(r) - V(r)) \frac{1}{r} \frac{2r \int_{\partial B_r(x^0)} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1} - 3 \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \\
= {} & \frac{2}{r} \left(\frac{r^2 \int_{\partial B_r(x^0)} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1}}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} - \frac{3}{2} D(r) \right) \\
& - \frac{2}{r} (D(r) - V(r)) \left(D(r) - \frac{3}{2} \right) + \frac{K(r)}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}, \qquad (6.3)
\end{aligned}
$$

Note that when f is a C^1 function, the above is a consequence of $f(0) = 0$. Assumption 6.10 also implies that

$$|F(z)| \le Cz^2/2 \quad \text{for all } z \in (0, z_0).$$

As a corollary of Lemma 6.9 we obtain thus:

Corollary 6.11. *Let u be a variational solution of (3.1) such that Assumption 4.1 and Assumption 6.10 hold. Then there exists $r_0 > 0$ such that*

$$r \int_{\partial B_r(x^0)} u^2 \, dH^{n-1} \le \int_{B_r(x^0)} u^2 \quad \text{for all } r \in (0, r_0)$$

and

$$|K(r)| \le C_0 r \int_{\partial B_r(x^0)} u^2 \quad \text{for all } r \in (0, r_0).$$

Theorem 6.12. *Let u be a variational solution of (3.1) such that Assum and Assumption 6.10 hold, let $x^0 \in$*

Let $Y : (0, r_0) \to \mathbb{R}$ be given by

$$Y(r) = \int_0^r t^{-n-1} \int_{\partial B_t(x^0)} u^2 \, d\mathcal{H}^n$$

Since, by part (ii), $r \mapsto H(r)$ is bounded below as $r \to 0$, we obtain (iii). We also deduce from (6.16) and part (i) that $H(r)$ has a limit as $r \to 0+$, and that $H(0+) \ge 3/2$, thus proving (iv).

We now consider (6.2), and deduce from part (i) using (6.15) that, for a.e. $r \in (0, r_0)$,

$$H'(r) \ge \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u \cdot \nu)}{\left(\int_{\partial B_r(x^0)} u^2\, dH^{n-1}\right)^{1/2}} - H(r)\frac{u}{\left(\int_{\partial B_r(x^0)} u^2\, dH^{n-1}\right)^{1/2}} \right]^2 dH^{n-1} \tag{6.17}$$

$$- 2C_0 r V(r) - C_2 r$$

$$\ge -\frac{1}{r}V^2(r) - C_1^2 r^3 - C_0 r, \tag{6.18}$$

which, together with part (iii), proves (v).

7. Blow-up limits

The Frequency Formula allows passing to blow-up limits.

Proposition 7.1. *Let u be a variational solution of* (3.1), *and let $x^0 \in \Sigma^u$. Then:*

(i) There exist $\lim_{r \to 0+} V(r) = 0$ and $\lim_{r \to 0+} D(r) = H_{x^0,u}(0+)$.

(ii) For any sequence $r_m \to 0+$ as $m \to \infty$, the sequence

$$v_m(x) := \frac{u(x^0 + r_m x)}{\sqrt{r_m^{1-n}\int_{\partial B_{r_m}(x^0)} u^2\, dH^{n-1}}} \tag{7.1}$$

is bounded in $W^{1,2}(B_1)$.

(iii) For any sequence $r_m \to 0+$ as $m \to \infty$ such that the sequence v_m in (7.1) *converges weakly in $W^{1,2}(B_1)$ to a blow-up limit v_0, the function v_0 is homogeneous of degree $H_{x^0,u}(0+)$ in B_1, and satisfies*

$$v_0 \ge 0 \text{ in } B_1,\ v_0 \equiv 0 \text{ in } B_1 \cap \{x_n \le 0\} \text{ and } \int_{\partial B_1} v_0^2\, dH^{n-1} = 1.$$

Proof. We first prove that, for any sequence $r_m \to 0+$, the sequence

Indeed, for any such ϱ [redacted]ows by scaling from (6.18) that, for every m such that $r_m <$,

$$\int_{\varrho} \frac{2}{r} \int_{\partial B_r} \left[\frac{r(\text{[redacted]}}{\int_{\partial B_r} v\,\text{[redacted]}} \quad H(r_m r) \frac{v_m}{\left(\int_{\partial B_r} v_m^2 \, dH^{n-1}\right)^{1/2}} \right]^2 dH^{n-1} dr$$

$$H(r_m \;) \quad \text{[redacted]} \quad \frac{1}{r} V^2(r) + C_1^2 r^3 + C_0 r \, dr \to 0 \quad \text{as } m \to \infty;$$

as a consequence of T[redacted])-(v). The above implies that

$$\int_{\varrho} \frac{2}{r} \int_{\partial B_r} \left[\frac{r(\text{[redacted]}}{\int_{\partial B_r} v_m^2 \, dH^{n-1}\,\text{[redacted]}} \quad H(0+) \frac{v_m}{\left(\int_{\partial B_r} v_m^2 \, dH^{n-1}\right)^{1/2}} \right]^2 dH^{n-1} dr$$

$$\to 0 \quad \text{as } m \to \infty. \tag{7.3}$$

Now note that, for every $r \in (\varrho, \;) \quad (0, 1)$ and all m as before, it follows by using Theorem 6.12 (ii), that

$$\int_{\partial B}$$

and

$$(\partial_1 \tilde v_m)^2 - (\partial_2 \tilde v_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B as $m \to \infty$. It follows that

$$\partial_1 v_m \partial_2 v_m \to \partial_1 v_0 \partial_2 v_0 \tag{8.4}$$

and

$$(\partial_1 v_m)^2 - (\partial_2 v_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B as $m \to \infty$. Let us remark that this alone would allow us to pass to the limit in the domain variation formula for v_m in the set $\{x_2 > 0\}$.

z 2 f(1=2; 0); (1=2; 0)g. Consider the blow-up sequence v_m given by (7.1), and also the sequence

$$u_m(x) = \frac{u(x^0 + r_m x)}{r_m^{3=2}}:$$

Note that each u_m is a variational solution of (4.5), and v_m is a scalar multiple of u_m. Since x_m 2 ^{u}, it it it 9226 Tfra73.y7itu

$N(x^0) \quad 2$ such that

$$v_r(x) := \frac{u(x^0 + rx)}{\sqrt{r^{\;\;1} \int_{\partial B_r(x^0)} u^2 \, dH^1}} \tag{10.1}$$

$$\to \quad {}^{N(x^0)} j \sin(N(x^0) \min(\max(\;\; ; 0)$$

We are going to use a atness-implies-regularity result of [5]. Note that although not stated in [5], [5, Lemma 4.1] yields as in the proof of [5, Theorem 1.1] that for each $2\ (0;\ _0)$

$$\max(x \quad ;0) \quad w \quad \max(x \quad + \; ;0) \text{ in } B_1 \qquad (10.3)$$

implies that the outward unit normal ^{w} on the free boundary $@fw > 0g$ satis es

$$j\ ^w(0) \quad j \quad C\ ^2:$$

Note that $^w(0) = \ (\ )$. Since (10.3) is by (10.2) satis ed for $= (1{=}2;\ ^p\overline{3}{=}2)$, $r = r(\ )$ and every su ciently small > 0, we obtain that the outward unit normal (x) on $@fu > 0g$ converges to as $x\ !\ 0; x_1 > 0$. It follows that the present curve component is the graph of a C^1-function (up to $x_1 = x_1^0$) in the x_2-direction.

The remaining statements of the Theorem follow from Theorem 4.6.

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Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, U.K.

E-mail address: