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by

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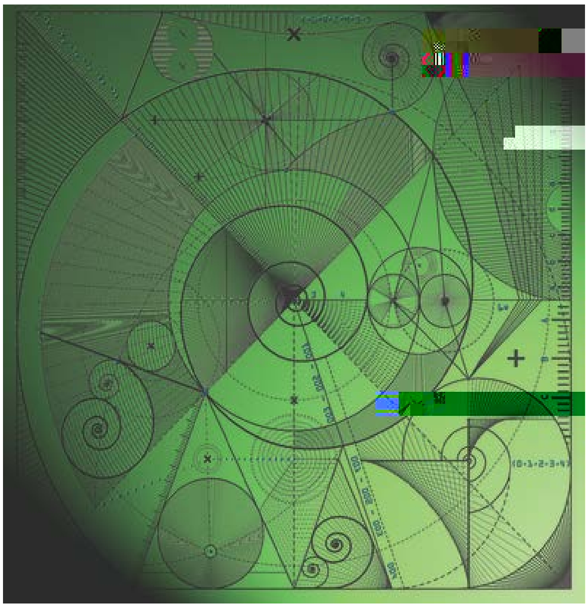

Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens

S. N. Chandler-Wilde and D. P. Hewett

Abstract. We study the classical first-kind boundary integral equation reformulations of time-harmonic acoustic scattering by planar sound-soft (Dirichlet) and sound-hard (Neumann) screens. We prove continuity and coercivity of the relevant boundary integral operators (the acoustic single-layer and hypersingular operators respectively) in appropriate fractional Sobolev spaces, with wavenumber-explicit bounds on the continuity and coercivity constants. Our analysis is based on spectral representations for the boundary integral operators, and builds on results of Ha-Duong (Jpn J Ind Appl Math 7:489–513 (1990) and Integr Equat Oper Th 15:427–453 (1992)).

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1. Introduction

This paper concerns the mathematical analysis of a class of time-harmonic acoustic scattering problems modelled by the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1}$$

where u is a complex scalar function and $k > 0$ is the wavenumber. We study the reformulation of such scattering problems in terms of boundary integral equations (BIEs), proving continuity and coercivity estimates for the associated boundary integral operators (BIOs) which are explicit in their k-dependence.

Our focus is on scattering by a thin planar screen occupying some bounded and relatively open set $\Gamma \subset \Gamma_\infty := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$ (we assume throughout that $n = 2$ or 3), with (1) assumed to hold in the domain $D := \mathbb{R}^n \setminus \overline{\Gamma}$. We shall assume throughout that Γ is a Lipschitz relatively open subset of Γ_∞ (in the sense of [26], viewing Γ and Γ_∞ as subsets of $\mathbb{R}^{n-1}$). But in fact our continuity and coercivity estimates can be used to prove analogous estimates for the corresponding BIOs on arbitrary relatively open or relatively closed $\Gamma \subset \Gamma_\infty$ (in suitable Sobolev spaces), as is discussed in [7, 6, 9].

We consider both the Dirichlet (sound-soft) and Neumann (sound-hard) boundary value problems (BVPs), which we now state. The function space notation in the following definitions, and the precise sense in which the boundary conditions are to be understood, will be explained in §2.

Definition 1.1 (Problem D). Given $g_D \in H^{1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0, \quad \text{in } D, \tag{2}$$

$$u = g_D, \quad \text{on } \Gamma, \tag{3}$$

and u satisfies the Sommerfeld radiation condition at infinity.

Definition 1.2 (Problem N). Given $g_N \in H^{-1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0, \quad \text{in } D, \tag{4}$$

$$\frac{\partial u}{\partial n} = g_N, \quad \text{on } \Gamma, \tag{5}$$

and u satisfies the Sommerfeld radiation condition at infinity.

Example 1.3. Consider the problem of scattering by Γ of an incident plane wave

$$u^i(x) := e^{ikx\cdot d}, \qquad x \in \mathbb{R}^n, \tag{6}$$

where $d \in \mathbb{R}^n$ is a unit direction vector. A 'sound-soft' and a 'sound-hard' screen are modelled respectively by problem D (with $g_D = -u^i|_\Gamma$) and problem N (with $g_N = -\partial u^i/\partial n|_\Gamma$). In both cases u represents the scattered field, the total field being given by $u^i + u$.

Such scattering problems have been well-studied, both theoretically [30, 29, 32, 17, 18, 23] and in applications [13, 12]. For $\Gamma \subset \Gamma_\infty$ sufficiently smooth, it is well known (see, e.g., [30, 29, 32, 23]) that problems D and N are uniquely solvable for all $g_D \in H^{1/2}(\Gamma)$ and $g_N \in H^{-1/2}(\Gamma)$. (Many of the references cited above assume $\Gamma \subset \Gamma_\infty$ is C^∞ smooth, or do not explicitly specify the regularity of Γ, but in fact unique solvability extends to the more general Lipschitz case we consider in this paper, as clarified e.g. in [7].) Their solutions can be represented respectively in terms of the single and double layer potentials (for notation and definitions see §2)

$$\mathcal{S}_k : \tilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^1_{\mathrm{loc}}(D), \qquad \mathcal{D}_k : \tilde{H}$$

where $\Phi(x,y)$ denotes the fundamental solution of (1),

$$\Phi(x,y) := \begin{cases} \dfrac{e^{ik|x-y|}}{4\pi|x-y|}, & n = 3, \\[2mm] \dfrac{i}{4}H_0^{(1)}(k|x-y|), & n = 2, \end{cases} \qquad x, y \in \mathbb{R}^n. \tag{8}$$

The densities of the potentials are the unique solutions of certain first-kind BIEs involving the single-layer and hypersingular BIOs (again, for definitions see §2)

$$S_k : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) = (\widetilde{H}^{-1/2}(\Gamma))^*,$$

$$T_k : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) = (\widetilde{H}^{1/2}(\Gamma))^*,$$

which for $\phi \in D(\Gamma)$ and $x \in \Gamma$ have the following integral representations:

$$S_k\phi(x) = \int_\Gamma \Phi(x,y)\phi(y)\,ds(y), \qquad T_k\phi(x) = \frac{\partial}{\partial n(x)}\int_\Gamma \frac{\partial\Phi(x,y)}{\partial n(y)}\phi(y)\,ds(y). \tag{9}$$

These standard statements are summarised in the following two theorems. Here $[u]$ and $[\partial u/\partial n]$ represent the jump across Γ of u and of its normal derivative respectively.

Theorem 1.4. Problem D has a unique solution u satisfying

$$u(x) = -\mathcal{S}_k[\partial u/\partial n](x), \qquad x \in D, \tag{10}$$

where $[\partial u/\partial n] \in \widetilde{H}^{-1/2}(\Gamma)$ is the unique solution of the BIE

$$S_k[\partial u/\partial n] = -g_D. \tag{11}$$

Theorem 1.5. Problem N has a unique solution satisfying

$$u(x) = \mathcal{D}_k[u](x), \qquad x \in D, \tag{12}$$

where $[u] \in \widetilde{H}^{1/2}(\Gamma)$ is the unique solution of the BIE

$$T_k[u] = g_N. \tag{13}$$

1.1. Main results and outline of the paper

In this paper we present new k-explicit continuity and coercivity estimates for the operators S_k and T_k appearing in (11) and (13). Our main results are contained in the following four theorems.

Theorem 1.6. For any $s \in \mathbb{R}$, the single-layer operator S_k defines a bounded linear operator $S_k : \widetilde{H}^s(\Gamma) \to H^{s+1}(\Gamma)$, and there exists a constant $C > 0$, independent of k and Γ, such that, for all $\phi \in \widetilde{H}^s(\Gamma)$, and with $L := \operatorname{diam}\Gamma$,

$$\|S_k\phi\|_{H_k^{s+1}(\Gamma)} \le \begin{cases} C(1+(kL)^{1/2})\|\phi\|_{\widetilde{H}_k^s(\Gamma)}, & n = 3, \\ C\log(2+(kL)^{-1})(1+(kL)^{1/2})\|\phi\|_{\widetilde{H}_k^s(\Gamma)}, & n = 2, \end{cases} \qquad k > 0. \tag{14}$$

Theorem 1.7. The sesquilinear form on $\tilde{H}$

very mildly as $k \to \infty$. This aim is provably achieved in certain cases, mainly 2D so far; see, e.g., [22, 8] and the recent review [5]. For 2D screen and aperture problems we recently proposed in [21] an HNA BEM which provably achieves a fixed accuracy of approximation with N growing at worst like $\log^2 k$ as $k \to \infty$, our numerical analysis using the wavenumber-explicit estimates of the current paper. Numerical experiments demonstrating the effectiveness of HNA approximation spaces for a 3D screen problem have been presented in [57, 6].

Clearly the results in this paper are a contribution to this endeavour. In particular, Theorems 1.6 and 1.8 provide upper bounds on $\|S_k\|_{\tilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)}$ and $\|T_k\|_{\tilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}$ (with $\tilde{H}^{\pm 1/2}(\Gamma)$ and $H^{\pm 1/2}(\Gamma)$ equipped with the wavenumber-dependent norms specified in §2). Further, as noted generically above, through providing lower bounds on the coercivity constant γ, Theorems 1.7 and 1.9 bound the inverses ximat6b5 Td [( )]TJ/F7 6.9.o586 2

$H^s(\mathbb{R}^n)$ defined by

$$\|u\|^2_{H^s_k(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (k^2+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi. \tag{26}$$

We emphasize that $\|\cdot\|_{H^s(\mathbb{R}^n)} := \|\cdot\|_{H^s_1(\mathbb{R}^n)}$ is the standard norm on $H^s(\mathbb{R}^n)$, and that, for $k > 0$, $\|\cdot\|_{H^s_k(\mathbb{R}^n)}$ is another, equivalent, norm on $H^s(\mathbb{R}^n)$. Explicitly,

$$\min\{1, k^s\}\|u\|_{H^s(\mathbb{R}^n)} \le \|u\|_{H^s_k(\mathbb{R}^n)} \le \max\{1, k^s\}\|u\|_{H^s(\mathbb{R}^n)}, \quad \text{for } u \in H^s(\mathbb{R}^n). \tag{27}$$

It is standard that $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. It is also standard (see, e.g., [26]) that $H^{-s}(\mathbb{R}^n)$ is a natural isometric realisation of $(H^s(\mathbb{R}^n))^*$, the dual space of bounded antilinear functionals on $H^s(\mathbb{R}^n)$, in the sense that the mapping $u \mapsto u^*$ from $H^{-s}(\mathbb{R}^n)$ to $(H^s(\mathbb{R}^n))^*$, defined by

$$u^*(v) := \langle u, v\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi, \quad v \in H^s(\mathbb{R}^n), \tag{28}$$

is a unitary isomorphism. The duality pairing $\langle\cdot,\cdot\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}$ defined in (28) represents a natural extension of the $L^2(\mathbb{R}^n)$ inner product in the sense that if $u_j, v_j \in L^2(\mathbb{R}^n)$ for each j and $u_j \to u$ and $v_j \to v$ as $j \to \infty$, with respect to the norms on $H^{-s}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ respectively, then $\langle u, v\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} = \lim_{j\to\infty}(u_j, v_j)_{L^2(\mathbb{R}^n)}$.

We define two Sobolev spaces on Ω when Ω is a non-empty open subset of $\mathbb{R}^n$. First, let $\mathscr{D}(\Omega) := C_0^\infty(\Omega) = \{u \in \mathscr{D}(\mathbb{R}^n) : \operatorname{supp} u \subset \Omega\}$, and let $\mathscr{D}^*(\Omega)$ denote the associated space of distributions (continuous antilinear functionals on $\mathscr{D}(\Omega)$). We set

$$H^s(\Omega) := \{u \in \mathscr{D}^*(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\},$$

where $U|_\Omega$ denotes the restriction of the distribution U to Ω (cf. [26, p. 66]), with norm

$$\|u\|_{H^s_k(\Omega)} := \inf_{U \in H^s(\mathbb{R}^n),\, U|_\Omega = u} \|U\|_{H^s_k(\mathbb{R}^n)}.$$

Then $\mathscr{D}(\overline{\Omega}) := \{u \in C^\infty(\Omega) : u = U|_\Omega \text{ for some } U \in \mathscr{D}(\mathbb{R}^n)\}$ is dense in $H^s(\Omega)$. Second, let

$$\widetilde{H}^s(\Omega) := \overline{\mathscr{D}(\Omega)}^{H^s(\mathbb{R}^n)}$$

denote the closure of $\mathscr{D}(\Omega)$ in the space $H^s(\mathbb{R}^n)$, equipped with the norm $\|\cdot\|_{\widetilde{H}^s_k(\Omega)} := \|\cdot\|_{H^s_k(\mathbb{R}^n)}$. When Ω is sufficiently regular (e.g. when Ω is C^0, cf. [26, Thm 3.29]) we have that $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}} := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\}$. (But for non-regular Ω $\widetilde{H}^s(\Omega)$ may be a proper subset of $H^s_{\overline{\Omega}}$, see [9].)

For $s \in \mathbb{R}$ and Ω any open, non-empty subset of $\mathbb{R}^n$ it holds that

H s s , with respe16 Td [(it)-333(holds)-334(that)]TJ/F11 9.9626 Tf in the sense that the natural embeddings $I: H^{-s}(\Omega) \to (\widetilde{H}^{s}(\Omega))^*$ and $I^*: \widetilde{H}^{s}(\Omega) \to (H^{-s}(\Omega))^*$,

$$(Iu)(v) := \langle u, v\rangle_{H^{-s}(\Omega)\times\widetilde{H}^{s}(\Omega)} := \langle U, v\rangle_{H^{-s}(\mathbb{R}^n)\times H^{s}(\mathbb{R}^n)},$$
$$(I^* v)(u) := \langle v, u\rangle_{\widetilde{H}^{s}(\Omega)\times H^{-s}(\Omega)} := \langle v, U\rangle_{H^{s}(\mathbb{R}^n)\times H^{-s}(\mathbb{R}^n)},$$

where $U \in H^{-s}(\mathbb{R}^n)$ is any extension of $u \in H^{-s}(\Omega)$ with $U|_\Omega = u$, are unitary isomorphisms. We remark that the representations (29) for the dual spaces are well known when Ω is sufficiently regular. However, it does not appear to be widely appreciated, at least in the numerical analysis for PDEs community, that (29) holds without constraint on the geometry of Ω, proof of this given recently in [7, Thm 2.1].

Sobolev spaces can also be defined, for $s \geq 0$, as subspaces of $L^2(\mathbb{R}^n)$ satisfying constraints on weak derivatives. In particular, given a non-empty open subset Ω of $\mathbb{R}^n$, let

$$W^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\},$$

where ∇u is the weak gradient. Note that $W^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ with

$$\|u\|^2_{H^1_k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + k^2|u(x)|^2\right) dx.$$

Further [26, Theorem 3.30], $W^1(\Omega) = H^1(\Omega)$ whenever Ω is a Lipschitz open set. It is convenient to define

$$W^1_{\mathrm{loc}}(\Omega) := \{u \in L^2_{\mathrm{loc}}(\Omega) : \nabla u \in L^2_{\mathrm{loc}}(\Omega)\},$$

where $L^2_{\mathrm{loc}}(\Omega)$ denotes the set of locally integrable functions u on Ω for which $\int_G |u(x)|^2\,dx < \infty$ for every bounded measurable $G \subset \Omega$.

To define Sobolev spaces on the screen $\Gamma_\infty := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$ we make the natural associations of Γ_∞ with $\mathbb{R}^{n-1}$ and of Γ with $\widetilde{\Gamma} := \{\tilde{x} \in \mathbb{R}^{n-1} : (\tilde{x}, 0) \in \Gamma\}$ and set $H^s(\Gamma_\infty) := H^s(\mathbb{R}^{n-1})$, $H^s(\Gamma) := H^s(\widetilde{\Gamma})$ and $\widetilde{H}^s(\Gamma) := \widetilde{H}^s(\widetilde{\Gamma})$ (with $C^\infty(\Gamma_\infty)$, $\mathscr{D}(\Gamma_\infty)$, $\mathscr{D}(\overline{\Gamma})$ and $\mathscr{D}(\Gamma)$ defined analogously).

2.2. Traces, jumps and boundary conditions

Letting $U^+ := \{x \in \mathbb{R}^n : x_n > 0\}$ and $U^- := \mathbb{R}^n \setminus \overline{U^+}$ denote the upper and lower half-spaces, respectively, we define trace operators $\gamma^\pm : \mathscr{D}(\ \ H$

for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. We then define

$$[u] := \gamma^+(\chi u) - \gamma^-(\chi u) \in \widetilde{H}^{1/2}(\Gamma)$$

$$[\partial u/\partial n] := \partial_n^+(\chi u) - \partial_n^-(\chi u) \in \widetilde{H}^{-1/2}(\Gamma),$$

where χ is any element of $\mathcal{D}_{1,\Gamma}(\mathbb{R}^n) := \{\phi \in \mathcal{D}(\mathbb{R}^n) : \phi = 1$ in some neighbourhood of $\Gamma\}$.

The boundary conditions (3) and (5) can now be stated more precisely: by (3) and (5) we mean that

$$\gamma^\pm(\chi u)|_\Gamma = g_D; \quad \text{and} \quad \partial_n^\pm(\chi u)|_\Gamma = g_N; \quad \text{for every } \chi \in \mathcal{D}_{1,\Gamma}(\mathbb{R}^n).$$

2.3. Layer potentials and boundary integral operators

We can now give precise definitions for the single and double layer potentials

$$\mathcal{S}_k : \widetilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^1_{\mathrm{loc}}(D), \qquad \mathcal{D}_k : \widetilde{H}^{1/2}(\Gamma) \to C^2(D) \cap W^1_{\mathrm{loc}}(D),$$

namely

$$\mathcal{S}_k\phi(x) := \left\langle \gamma^\pm(\chi\Phi_k(x,\cdot)), \overline{\phi} \right\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)}, \quad x \in D, \quad \phi \in \widetilde{H}^{-1/2}(\Gamma),$$

$$\mathcal{D}_k\psi(x) := \left\langle \psi, \overline{\partial_n^\pm(\chi\Phi_k(x,\cdot))} \right\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)}, \quad x \in D, \quad \psi \in \widetilde{H}^{1/2}(\Gamma),$$

where χ is any element of $\mathcal{D}_{1,\Gamma}(\mathbb{R}^n)$ with $x \notin \operatorname{supp}\chi$. The single-layer and hypersingular boundary integral operators

$$\mathrm{S}_k : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad \mathrm{T}_k : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

are then defined by

$$\mathrm{S}_k := \gamma^\pm(\chi\mathcal{S}_k)|_\Gamma :=$$

To evaluate $\hat{\Phi}_c$ we note that for a function $f(x) = F(r)$, where $r = |x|$ for $x \in \mathbb{R}^d$, $d = 1, 2$, the Fourier transform of f is given by (cf. [15, §B.5]) [4]

$$\hat{f}(\xi) = \begin{cases} \displaystyle\int_0^\infty F(r) J_0(|\xi| r)\, r\, dr, & d = 2, \\ \displaystyle\sqrt{\frac{2}{\pi}} \int_0^\infty F(r) \cos(\xi r)\, dr, & d = 1. \end{cases} \tag{38}$$

This result, combined with the identities [14, (6.677), (6.737)] and [1, (10.16.1), (10.39.2)], gives

$$\hat{\Phi}_c(\xi; x_n) = \frac{i\, e^{i|x_n| Z(\xi)}}{2(2\pi)^{(n-1)/2} Z(\xi)},$$

where $Z(\xi)$ is defined as in (32). The representation (33) is then obtained by Fourier inversion.

The representation (31) for $\mathcal{D}_k$ can be then obtained from (30) by noting that

$$\frac{\partial \Phi(x, y)}{\partial n(y)} = \frac{\partial \Phi(x, y)}{\partial y_n} = -\frac{\partial \Phi(x, y)}{\partial x_n}, \qquad x \in D,\ y \in \Gamma,$$

and the representations for S_k and T_k follow from taking the appropriate traces of (30) and (31).

Finally, (35) and (36) follow from viewing $S_k^\infty \phi$ and $T_k^\infty \phi$ as elements of $C^\infty(\mathbb{R}^{n-1}) \cap \mathcal{S}^*(\mathbb{R}^{n-1})$ and recalling the definition of the Fourier transform of a distribution, e.g., for S_k,

$$\begin{aligned}(S_k \phi, \psi)_{L^2(\Gamma)} &= \int_{\mathbb{R}^{n-1}} S_k^\infty \phi(\tilde{x})\, \overline{\psi(\tilde{x})}\, d\tilde{x} = \int_{\mathbb{R}^{n-1}} \widehat{S_k^\infty \phi}(\xi)\, \overline{\hat{\psi}(\xi)}\, d\xi \\ &= \frac{i}{2} \int_{\mathbb{R}^{n-1}} \frac{1}{Z(\xi)} \hat{\phi}(\xi)\, \overline{\hat{\psi}(\xi)}\, d\xi.\end{aligned}$$

4. k-explicit analysis of S_k

Our k-explicit analysis of the single-layer operator S_k makes use of the following lemma.

Lemma 4.1. Given $L > 0$ let

$$\Phi_L(\tilde{x}; x_n) := \begin{cases} \Phi_c(\tilde{x}; x_n), & |\tilde{x}| \le L, \\ 0, & |\tilde{x}| > L, \end{cases} \tag{39}$$

where Φ_c is defined as in the proof of Theorem 3.1. Then there exists a constant $C > 0$, independent of k, L, Γ and x_n, such that, for all $k > 0$,

[4]Strictly speaking, [15, §B.5] only provides (38) for $f \in L^1(\mathbb{R}^d)$. But for the functions $f = \Phi_c(\cdot; x_n)$ one can check using the dominated convergence theorem that (38) holds.

$\xi \in \mathbb{R}^{n-1}$, and $x_n \in \mathbb{R}$,

$$|\hat{c}_L(\xi; x_n)|\sqrt{k^2+|\xi|^2} \leq \begin{cases} C(1+(kL)^{1/2}), & n=3,\\ C\left(\log(2+(kL)^{-1})+(kL)^{1/2}\right), & n=2.\end{cases} \quad (40)$$

Proof. It is convenient to introduce the notation $\rho := |\xi|$, and by $C>0$ we denote an arbitrary constant, independent of k, L, ρ, and x_n, which may change from occurrence to occurrence. To prove (40) we proceed by estimating $|\hat{c}_L(\xi; x_n)|$ directly, using the formula (38). We treat the cases $n=3$ and $n=2$ separately. We will make use of the following well-known properties of the Bessel functions (cf. [1, Sections 10.6, 10.14, 10.17]), where $\mathcal{B}_n$ represents either J_n or $H_n^{(1)}$:

$$|J_n(z)| \leq 1, \qquad n \in \mathbb{N},\ z>0; \qquad (41)$$

$$|H_0^{(1)}(z)| \leq C(1+|\log z|), \qquad 0<z\leq 1 \qquad (42)$$

$$|H_1^{(1)}(z)| \leq Cz^{-1}, \qquad 0<z\leq 1 \qquad (43)$$

$$\left|H_1^{(1)}(z)+\frac{2i}{\pi z}\right| \leq Cz^{-1/2}, \qquad z>0; \qquad (44)$$

$$|\mathcal{B}_n(z)| \leq Cz^{-1/2}, \qquad n=0,1;\ z>1; \qquad (45)$$

$$\mathcal{B}_0'(z) = -\mathcal{B}_1(z), \qquad z>0; \qquad (46)$$

$$\frac{d}{dz}(z\mathcal{B}_1(z)) = z\mathcal{B}_0(z), \qquad z>0. \qquad (47)$$

(i) In the case $n=3$, $|\hat{c}_L(\xi; x_3)| \leq |I(L)|/(4\pi)$, where

$$I(L) := \int_0^L \frac{e^{ik\sqrt{r^2+x_3^2}}}{\sqrt{r^2+x_3^2}} J_0(\rho r)\, r\, dr, \quad \text{for } L>0.$$

Using (41), we see that $|I(L)| \leq 1/\rho$, if $L \leq 1/\rho$. If $L>1/\rho$ then, integrating by parts using the relation (47),

$$I(L) - I(1/\rho) = \frac{1}{\rho}\left[\frac{r e^{ik\sqrt{r^2+x_3^2}}}{\sqrt{r^2+x_3^2}} J_1(\rho r)\right]_{1/\rho}^{L} - \frac{1}{\rho}\int_{1/\rho}^{L} r^2 e^{ik\sqrt{r^2+x_3^2}}\left(\frac{ik}{r^2+x_3^2} - \frac{1}{(r^2+x_3^2)^{3/2}}\right) J_1(\rho r)\, dr,$$

so that, substituting $t=\rho r$ and using (41),

$$|I(L)| \leq |I(1/\rho)| + |I(L)-I(1/\rho)| \leq \frac{3}{\rho} + \frac{1}{\rho}\int_1^{\rho L}\left(\frac{k}{\rho}+t^{-1}\right)|J_1(t)|\, dt. \quad (48)$$

Using the bound (45) in (48), it follows that

$$|I(L)| \leq \frac{3}{\rho} + \frac{C}{\rho}\left(\frac{kL^{1/2}}{\rho^{1/2}}+1\right), \qquad L>0;$$

so that

$$|c_1(\ ; x_3)| \sqrt{k^2 + \ +}$$

so that

$$|I_0(L)| \quad \frac{2}{\ } \left(3 + \int_1^L \frac{1}{t^2}\,dt \right) < \frac{8}{\ }:$$

Using these bounds on $I_0(L)$ and the bound (44) on $F_1(z)$, we see that

$$|I(L)|$$

since $|k^2 - |\xi|^2| \geq 2k|\xi| + |\xi|^2 \geq 3k\epsilon^2$, for $|\xi| \geq \epsilon$. Also, for the same choice of ϵ and all $t \geq 0$,

$$\|\phi_\epsilon\|^2_{\tilde{H}_k^{-t}(\Gamma)} \leq \frac{1}{k^{2t}} \int_{\mathbb{R}^{n-1}} |\hat{\phi}_\epsilon(\xi)|^2 \, d\xi = \frac{1}{k^{2t}} \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi)|^2 \, d\xi \leq 2k^{-2t} I(\epsilon); \tag{54}$$

for ϵ sufficiently large. Further, for ϵ sufficiently large and $k > 1$,

$$I(\epsilon) \geq \int_{|\xi|>\epsilon} (2+2|\xi|+|\xi|^2)^t |\hat{\phi}(\xi)|^2 \, d\xi \geq \int_{|\xi|>\epsilon} \left(1 + \frac{\epsilon}{k} + \epsilon^2\right)^t |\hat{\phi}(\xi)|^2 \, d\xi;$$

so that, since $k^2 + (|\xi| + k\epsilon)^2 \leq k^2 + (\epsilon + k)^2 \leq 5k^2\epsilon^2$ for $|\xi| \leq \epsilon$,

$$\|\phi_\epsilon\|^2_{\tilde{H}_k^{t}(\Gamma)} = \int_{\mathbb{R}^{n-1}} \left(k^2 + |\xi| + k\epsilon^2\right)^t |\hat{\phi}_\epsilon(\xi)|^2 \, d\xi \leq 6k^{2t}\epsilon^{2t} I(\epsilon): \tag{55}$$

Combining (53), (54) and (55) we see that, for every $s \in \mathbb{R}$, if $\epsilon \geq 1$ is sufficiently large, there exists $C > 0$, depending on ϵ and s but independent of k, such that

$$|(S_k\phi_\epsilon, \phi_\epsilon)_{L^2(\Gamma)}| \geq Ck^{1/2}\|\phi_\epsilon\|_{\tilde{H}_k^{-(s+1)}(\Gamma)}\|\phi_\epsilon\|_{\tilde{H}_k^{s}(\Gamma)}:$$

But, on the other hand,

$$|(S_k\phi_\epsilon, \phi_\epsilon)_{L^2(\Gamma)}| = |\langle S_k\phi_\epsilon, \phi_\epsilon\rangle_{H^{s+1}(\Gamma)\times\tilde{H}^{-(s+1)}(\Gamma)}| \leq \|S_k\phi_\epsilon\|_{H_k^{s+1}(\Gamma)}\|\phi_\epsilon\|_{\tilde{H}_k^{-(s+1)}(\Gamma)};$$

so that, for this particular choice of ϕ,

$$\|S_k\phi_\epsilon\|_{H_k^{s+1}(\Gamma)} \geq Ck^{1/2}\|\phi_\epsilon\|_{\tilde{H}_k^{s}(\Gamma)};$$

which demonstrates the sharpness of (14) in the limit $k \to \infty$.

Remark 4.3. Theorem 1.6 bounds $S_k : \tilde{H}^s(\Gamma) \to H^{s+1}(\Gamma)$. We can also bound S_k as a mapping $S_k : \tilde{H}^s(\Gamma) \to H^s(\Gamma)$. Since $\|\phi\|_{\tilde{H}^{s-1}(\Gamma)} \leq k^{-1}\|\phi\|_{\tilde{H}^{s}(\Gamma)}$ for $\phi \in \tilde{H}^s(\Gamma)$, it follows from Theorem 1.6 that, for $kL \geq 1$, $\|k^{-1}\phi\|$ …

Remark 4.4. We can also show that the bound in Theorem 1.7 is sharp in its dependence on k as $k \to \infty$. Let $0 \neq \quad \in D(\quad)$ be independent of k. Then, by (35), and since $\hat{}(\quad)$ is rapidly decreasing as $k \to \infty$,

$$|a_D(\ ;\ )| \quad 1$$

Defining

$$J := \|\psi\|^2_{H_k^{-1/2}(\Gamma)} = \int_{\mathbb{R}^{n-1}} (k^2 + |\xi|^2)^{-1/2} |\hat{\psi}(\xi)|^2 \, d\xi;$$

the problem of proving (17) reduces to that of proving

$$I \geq Ck\,J, \qquad k \geq k_0; \tag{63}$$

for some $C > 0$ depending only on k_0.

The difficulty in proving (63) is that the factor $|Z(\xi)|$ in I vanishes when $|\xi| = k$. To deal with this, we write the integrals I and J as

$$I = I_1 + I_2 + I_3 + I_4; \qquad J = J_1 + J_2 + J_3 + J_4;$$

corresponding to the decomposition

$$\int_{\mathbb{R}^{n-1}} = \int_{0<|\xi|<k-\varepsilon} + \int_{k-\varepsilon<|\xi|<k} + \int_{k<|\xi|<k+\varepsilon} + \int_{|\xi|>k+\varepsilon};$$

where $0 < \varepsilon \leq k$ is to be specified later. We then proceed to estimate the integrals $J_1, \ldots, J_4$ separately. Throughout the remainder of the proof $c > 0$ denotes an absolute constant whose value may change from occurrence to occurrence.

We first observe that, for $0 < |\xi| < k - \varepsilon$,

$$\frac{k^2 + |\xi|^2}{k^2 - |\xi|^2} \leq \frac{k^2 + (k-\varepsilon)^2}{k^2 - (k-\varepsilon)^2} \leq \frac{2k^2}{\varepsilon(2k-\varepsilon)} \leq \frac{2k}{\varepsilon};$$

so that

$$\begin{aligned} J_1 &:= \int_{0<|\xi|<k-\varepsilon} (k^2 + |\xi|^2)^{-1/2} |\hat{\psi}(\xi)|^2 \, d\xi \\ &= \int_{0<|\xi|<k-\varepsilon} \frac{(k^2 + |\xi|^2)^{1/2}}{(k^2 - |\xi|^2)^{1/2}} |Z(\xi)| |\hat{\psi}(\xi)|^2 \, d\xi \\ &\leq c\sqrt{\frac{k}{\varepsilon}}\, I_1. \end{aligned}$$

Similarly, for $|\xi| > k + \varepsilon$,

$$\frac{k^2 + |\xi|^2}{|\xi|^2 - k^2} \leq \frac{k^2 + (k+\varepsilon)^2}{(k+\varepsilon)^2 - k^2} \leq \frac{5k}{2\varepsilon}$$

$4\sin^2(t=2)$ t^2, so that, for $_1$;

Arguing similarly, again assuming that $0 < \varepsilon < k/3$, but using (64) and (66) with the plus rather than the minus sign, gives

$$J_3 := \int_{k<|\xi|<k+\varepsilon} (k^2 + |\xi|^2)^{1/2} |\hat{\phi}(\xi)|^2 \, d\xi \le c\sqrt{\frac{k}{\varepsilon}}\, I_4 + c\varepsilon^3 k^{n-2} J.$$

Combining the above estimates we see that, for $0 < \varepsilon < k/3$,

$$J = J_1 + J_2 + J_3 + J_4 \le c\sqrt{\frac{k}{\varepsilon}}\,(I_1 + I_4) + c\varepsilon^3 k^{n-2} J;$$

which implies that

$$J\left(1 - c\varepsilon^3 k^{n-2}\right) \le c\sqrt{\frac{k}{\varepsilon}}\, I. \qquad (67)$$

Now, given $k_0 > 0$, choose $\tilde{c} > 0$ such that $\tilde{c} < 1/(2c)$ and $\tilde{c}k^{-(n-2)/3} \le k/3$, for $k \ge k_0$. Then, for $k \ge k_0$, setting $\varepsilon = \tilde{c}k^{-(n-2)/3}$ in (67), it follows from (67) that

$$J \le c\tilde{c}^{-1/2} k^{\gamma} I;$$

where $\gamma = 1/2 + (n-2)/6$. Thus (63) holds, which completes the proof.

Remark 5.1. We can show that the bounds established in Theorem 1.8 are sharp in their dependence on k as $k \to \infty$. Let $0 \neq \phi \in D(\Gamma)$ be independent of k. Then, by (62), and since $a_N(\phi, \phi) = (T_k\phi, \phi)_{L^2(\Gamma)}$ and $\hat{\phi}(\xi)$ is rapidly decreasing,

$$|(T_k\phi, \phi)_{L^2(\Gamma)}| \ge \frac{1}{2\sqrt{2}} \int_{\mathbb{R}^{n-1}} \sqrt{|k^2 - |\xi|^2|}\, |\hat{\phi}(\xi)|^2 \, d\xi \sim \frac{k}{2\sqrt{2}} \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi)|^2 \, d\xi, \qquad (68)$$

as $k \to \infty$. Also, for every $s \in \mathbb{R}$, $|(T_k\phi, \phi)_{L^2(\Gamma)}| \le \|T_k\phi\|_{H^{s-1}_k(\Gamma)} \|\phi\|_{\tilde{H}^{1-s}_k(\Gamma)}$ and

$$\|\phi\|^2_{\tilde{H}^s_k(\Gamma)} = \int_{\mathbb{R}^{n-1}} (k^2 + |\xi|^2)^s |\hat{\phi}(\xi)|^2 \, d\xi \sim k^{2s} \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi)|^2 \, d\xi, \qquad (69)$$

as $k \to \infty$. Combining (68) and (69) we see that, for every $s \in \mathbb{R}$ and $C < 1/(2\sqrt{2})$, it holds for all sufficiently large k that $|(T_k\phi, \phi)_{L^2(\Gamma)}| \ge Ck\|\phi\|_{\tilde{H}^{1-s}_k(\Gamma)} \|\phi\|_{\tilde{H}^s_k(\Gamma)}$, so that

$$\|T_k\phi\|_{H^{s-1}_k(\Gamma)} \ge Ck\|\phi\|_{\tilde{H}^s_k(\Gamma)};$$

for all k sufficiently large, which demonstrates the sharpness of (16) in the limit $k \to \infty$.

Remark 5.2. We can also show that the bound established in Theorem 1.9 is sharp in its dependence on k as $k \to \infty$, in the case $n = 2$. As in Remark 4.2, let $\phi(x) := e^{\dots}(x)$, 34 - 0 0 G=/F36 9.9626 Tf -54.638 4(b)] its dependencesharp

$\|T_k$

independent of k, so that $\hat\psi(\xi) = \hat\phi(\eta)$, where $\eta = \xi - k\tilde d$. Since $|k^2 - |\xi|^2| \leq 2k|\eta| + |\eta|^2$, and since $\hat\phi(\eta)$ is rapidly decreasing,

$$|a_N(\psi;\psi)| \leq \frac{1}{2}\int_{\mathbb{R}^{n-1}} \sqrt{2k|\eta| + |\eta|^2}\,|\hat\phi(\eta)|^2\,d\eta \sim \frac{k^{1/2}}{\sqrt{2}}\int_{\mathbb{R}^{n-1}} |\eta|^{1/2}|\hat\phi(\eta)|^2\,d\eta, \tag{70}$$

as $k \to \infty$. Further,

$$\|\psi\|^2_{\tilde H_k^{-1/2}(\Gamma)} = \int_{\mathbb{R}^{n-1}} \left(2k^2 + 2k\,\eta\cdot\tilde d + |\eta|^2\right)^{-1/2}|\hat\phi(\eta)|^2\,d\eta \sim \frac{1}{\sqrt{2}k}\int_{\mathbb{R}^{n-1}} |\hat\phi(\eta)|^2\,d\eta, \tag{71}$$

as $k \to \infty$. Combining (70) and (71) we see that, for some constant $C > 0$ independent of k,

$$|a_N(\psi;\psi)| \leq C k^{1/2}\|\psi\|^2_{\tilde H_k^{-1/2}(\Gamma)},$$

for all sufficiently large k. This demonstrates the sharpness of (17) in the limit $k \to \infty$, for the case $n = 2$. In the case $n = 3$ it may be that (17) holds with the value of β increased from $-2/3$ to $-1/2$, i.e., to its value for $n = 2$.

6. Norm estimates in $H^{-1/2}(\Gamma)$

In this section we derive k-explicit estimates of the norms of certain functions in $H^{-1/2}(\Gamma)$, which are of relevance to the numerical solution of the Dirichlet boundary value problem D, when it is solved via the integral equation formulation (11). For an application of the results presented here see [21].

The motivation for the estimates we prove in Lemma 6.1 below comes

Given an estimate of $\|v\|_{\tilde{H}_k^{-1/2}(\;)}$, an estimate of $|I|$ follows from

$$|I| = |\langle w, \nabla \rangle_{H^{1/2}(\;)\times\tilde{H}^{-1/2}(\;)}| \quad \|w\|_{H_k^{1/2}(\;)}\|v\|_{\tilde{H}_k^{-1/2}(\;)},$$

provided we can bound $\|w\|_{H_k^{1/2}(\;)}$. We now do this for the choices of w noted above.

Lemma 6.1. Let $k > 0$, let be an arbitrary nonempty relatively open subset of ${}_1$, and let $L := \operatorname{diam}$.

(i) Let $d \in \mathbb{R}^n$ with $|d|$ 1. Then, for s 0, there exists $C_s > 0$, dependent only on s, such that

$$\|e^{ikd\cdot(\;)}\|_{H_k^s(\;)} \quad C_s L^{(n-1-2s)/2}(1 + kL)^s. \tag{73}$$

(ii) Let $x \in D := \mathbb{R}^n \setminus \overline{\;}$. Then there exists $C > 0$, independent of k, , and x, such that

k

and, using that $|F(z)|$ is monotonic for $z > 0$,

$$\|u - r_L\|^2_{L^2(\mathbb{R}^{n-1})} \le Ck^{2n-4}\left(d^{n-3}|F(kd/2)|^2 + L^{-2}\int_{S_{5L}\setminus S_d} |F(kr(y))|^2 \,dy\right)$$
$$\le Ck^{2n-4}d^{n-3}|F(kd/2)|^2 + Ck^{n-3}L^{-2}\log(5L/d). \qquad (82)$$

Combining (78) and (80)–(82), we see that, for $d < 4L$,

$$\|u\|_{H^1_k(\Gamma)} \le Ck^{(n-3)/2}(k + L^{-1})\log^{1/2}(5L/d)$$
$$+ Cd^{(1-n)/2} + Ck^{n-2}d^{(n-3)/2}|F(kd/2)|. \qquad (83)$$

Now, in the case $n = 3$, for which $|F(kd/2)| \le C/(kd)$, it follows from (79) and (83), and noting that $\log^{1/2}(5L/d) \le CP_3(L/d) \le CL/d$ for $4L > d$, that

$$\|u\|_{H^1_k(\Gamma)} \le \frac{C}{d}\left(1 + kd\,P_3(L/d)\right). \qquad (84)$$

For $n = 2$, $F(kd/2) \le C\log(2+(kd)^{-1})(1+kd)^{-1/2} \le C(kd)^{-1/2}$, by (42) and (45). Hence, and by (79) and (83) and as $\log^{1/2}(5L/d) \le CP_2(L/d) \le CL/d$ for $4L > d$,

$$\|u\|_{H^1_k(\Gamma)} \le \frac{C}{d^{1/2}}\left((kL)^{-1/2} + \log(2 + (kd)^{-1}) + (kd)^{1/2}P_2(L/d)\right). \qquad (85)$$

Part (ii) of the lemma then follows from (76), (84), and (85).

As an application we use Lemma 6.1 to prove a k-explicit pointwise bound on the solution of the sound-soft screen scattering problem considered in Example 1.3.

Corollary 6.2. The solution u of problem D, with $g_D = -u^i|_\Gamma$, satisfies the pointwise bound

$$|u(x)| \le \begin{cases} C\sqrt{kL}\sqrt{1+kL}\left(\dfrac{1}{kd} + P_3(L/d)\right), & n = 3;\\[2ex] C\sqrt{1+kL}\left(\dfrac{1}{\sqrt{kd}}\left(\dfrac{1}{\sqrt{kL}} + \log\left(2 + \dfrac{1}{kd}\right)\right) + P_2(L/d)\right), & n = 2; \end{cases}$$

where $x \in D$, $d := \operatorname{dist}(x, \Gamma)$, $L := \operatorname{diam}\Gamma$, and $C > 0$ is independent of

[15] L. Grafakos, Classical Fourier Analysis, Springer, 2008.

[16] I. G. Graham, M. Löhndorf, J. M. Melenk, and E. A. Spence, When is the error in the h-BEM for solving the Helmholtz equation bounded independently of k? BIT Num. Math., to appear.

[17] T. Ha-Duong, On the transient acoustic scattering by a flat object, Japan Journal of Industrial and Applied Mathematics, 7 (1990), pp. 489–513.

[18] ———, On the boundary integral equations for the crack opening displacement of flat cracks, Integral Equations and Operator Theory, 15 (1992), pp. 427–453.

[19] X. Han and M. Tacy, Semiclassical single and double layer potentials: boundedness and sharpness, 2014. With an Appendix by J. Galkowski, http://arxiv.org/abs/1403.6576.

[20] D. P. Hewett and S. N. Chandler-Wilde, Wavenumber-explicit coercivity estimates in scattering by screens, in Proc. 11th Int. Conf. on Mathematical and Numerical Aspects of Waves, Gammarth, Tunisia, Available from http://www.enit.rnu.tn/waves2013, 2013, pp. 253–254.

[21] D. P. Hewett, S. Langdon, and S. N. Chandler-Wilde, A frequency-independent boundary element method for scattering by two-dimensional screens and apertures, IMA J. Numer. Anal., to appear.

[22] D. P. Hewett, S. L. Langdon, and J. M. Melenk, A high frequency hp boundary element method for scattering by convex polygons, SIAM J. Numer. Anal., 51(1) (2013), pp. 629–653.

[23] H. Holm, M. Maischak, and E. P. Stephan, The hp-version of the boundary element method for Helmholtz screen problems