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by

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Abstract

This paper provides a new proof of a theorem of Chandler-Wilde, Chonchaiya and Lindner that the spectra of a certain class of infinite, random, tridiagonal matrices contain the unit disc almost surely. It also obtains an analogous result for a more general class of random matrices whose spectra contain a hole around the origin. The presence of the hole forces substantial changes to the analysis.

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Key Words: spectrum, random matrix, hopping model, tridiagonal matrix, non-self-adjoint operator.

1 Introduction

Over the last fifteen years there have been many studies of the spectral properties of non-self-adjoint, random, tridiagonal matrices A, some of them cited in [8, 13, 14, 17]. It has become clear that if all of the off-diagonal entries $A_{i,j}$ with $i - j = \pm 1$ of the matrices concerned are positive, the almost sure limit as $N \to \infty$ of the spectra of random $N \times N$ matrices subject to periodic boundary conditions can be quite different from the spectral behaviour of the corresponding infinite random matrix, [9, 10, 15, 16]. Indeed the limit in the first case can be the union of a small number of simple curves, while the second limit has a non-empty interior.

Numerical calculations suggest that the situation is quite different if the off-diagonal entries have variable signs, but much less has been proved in this situation, which is the one that we consider here. In a recent paper, [5], Chandler-Wilde, Chonchaiya and Lindner made important progress in determining the almost sure spectrum of a remarkably interesting class of non-self-adjoint, random, tridiagonal matrices introduced by Feinberg and Zee in [13], and sometimes called random hopping matrices, because the diagonal entries all vanish. Specifically they proved that, contrary to earlier conjectures, the infinite, tridiagonal matrix

$$A_c = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & c_{n-1} & 0 & 1 & & \\ & & c_n & 0 & 1 & \\ & & & c_{n+1} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

has spectrum that contains the unit disc almost surely, [5]. The paper assumed that the entries c_n are independent and identically distributed with values in $\{\pm 1\}$.

In the present paper we assume that the entries c_n are independent and identically distributed with values in $\{\pm\sigma\}$ for some fixed $\sigma \in (0, 1]$. We assume that the probability p that $c_n = \sigma$ satisfies $0 < p < 1$; the corresponding probability measure on $\Omega = \{\pm\sigma\}^{\mathbb{Z}}$ is denoted by μ. The matrix A_c is identified with the bounded operator acting in the natural manner on $\ell^2(\mathbb{Z})$.

In Lemma 26 we prove that

$$\mathrm{Spec}(A_c) \subseteq \{\lambda : 1 - \sigma \leq |\lambda| \leq 1 + \sigma\}$$

by a perturbation argument. We also prove that

$$\mathrm{Spec}(A_c) \subseteq \{x + iy : |x| + |y| \leq \sqrt{2(1 + \sigma^2)}\}$$

by obtaining a bound on the numerical range of A_c. There are currently no general techniques for identifying the precise forms of holes in the spectra of non-self-adjoint operators, and we have not done so here, but numerical calculations are consistent with the hypothesis that it is the intersection, H_σ, of two elliptical regions as defined in (36); see the figures at the end of Section 7. Little is known about the part of the spectrum of A_c outside the unit disc even in the case $\sigma = 1$, but numerical studies suggest that the boundary of the spectrum has a self-similar fractal structure in that case; [5, 17].

The main result of [5], that the spectrum contains the unit disc almost surely, is for the case that $\sigma = 1$, when there is no hole in the spectrum. It depends upon the identification of a particular sequence $c \in \ell^1$ such that the equation $A_c f = \lambda f$ has a bounded solution f for every $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$.

Our Theorem 7 rederives the main result of [5], in which $\lambda = 1$, but depends

As is well-known, the set $_{N;}$ is the union of eigenvalues of N N matrices. (Precisely, it is the union, over all sequences c and all j $j = 1$, of the eigenvalues of the matrix $A_{c;}^{(N;\mathrm{per})}$

This implies that $B = XY$ and $M = YX$. The second identity in (3) follows by some simple algebra that holds for any pair of bounded operators X and Y, and the first identity is a trivial consequence.

If A is invertible then (5) implies that B and M are also invertible; therefore (4) is equivalent to (3). □

Theorem 3 *Let $H = \ell^2(\mathbb{Z})$, let H_e be the closed subspace of sequences whose supports are contained in the set of even integers, and let H_o be the closed subspace of sequences whose supports are contained in the set of odd integers. Let A be a bounded operator on H whose matrix satisfies $A_{r,s} = 0$ for all r, s such that $|r - s| \neq 1$. Then $A(H_e) \subseteq H_o$ and $A(H_o) \subseteq H_e$. Moreover the identities*

$$\mathrm{Spec}(A^2) = \mathrm{Spec}(B) = \mathrm{Spec}(M)$$

are valid in either of the following two cases.

1. *$|A_{r,s}| = 1$ for all r, s such that $|r - s| = 1$;*
2. *There exist constants α, β such that $0 < \alpha < \beta < 1$ and $|A_{r,s}| \geq \beta$ if $r - s = 1$ while $|A_{r,s}| \leq \alpha$ if $r - s = -1$.*

Proof

Case 1. An elementary calculation establishes that there exists a sequence $f : \mathbb{Z} \to \mathbb{C}$ such that $Af = 0$, $|f_{2n}| = 1$ for all n and $f_{2n+1} = 0$ for all n.6TJ/F35 7.9701 TTf 15

3 The case = 1

The following lemma was noted in [5].

Lemma 4 *If c 2 then* Spec(A_c) *is invariant with respect to both of the maps ! $\bar{}$ and ! . If 2 S then $\bar{}$ and i lie in S. Hence S is invariant under the dihedral symmetry group D_2 generated by these two maps.*

Proof The invariance of Spec(A_c) under complex conjugation follows directly from the fact that A_c has real entries. If D is the diagonal matrix with entries $D_{r;r} = (\ i)^r$ for all r 2 Z then $DA_cD^{\ 1} = iA_{\ c}$, so

$$\mathrm{Spec}(A_c) = i\mathrm{Spec}(A_{\ c}): \tag{6}$$

Iterating this identity yields Spec(A_c) = Spec(A_c). This proves the rst part of the lemma. The second part follows once one observes that c 2 E if and only if c 2 E. □

The operator A_c^2 has two invariant subspaces

$$H_e = \{f \in \ell^2(\mathbb{Z}) : f_{2n+1} = 0 \text{ for all } n \in \mathbb{Z}\}$$

and $H_o = \ell^2(\mathbb{Z}) \ominus H_e$. After an obvious relabeling of the subscripts, the restriction of A_c to H_e equals A_b while the restriction of A_c to H_o is equal to M_b, as defined in (8). The final statement of the lemma is now an application of Theorem 3, case 1. □

We will exploit extensively the formula $\mathrm{Spec}(A_c^2) = \mathrm{Spec}(A_b)$ which appears in the above lemma. The equation $\mathrm{Spec}(A_b) = \mathrm{Spec}(M_b)$ will not play a role in our subsequent arguments, but makes an intriguing connection between spectra of rather different tridiagonal operators. Extending this connection slightly, for $b \in$ define $c = {}_{+}(b)$ and $\tilde{M}_b$ by

$$(\tilde{M}_b f)_n = f_{n-1} + i^n(c_{2n+1} +$$

Theorem 7 *The set S contains*

$$\bigcup_{n\in\mathbb{Z}_+;\ r\in\{0,\dots,2^{n+2}\}} e^{i\pi r/2^{n+1}}[0,2^{1/2^n}]. \tag{9}$$

Hence S contains the unit disc in $\mathbb{C}$.

Proof For $n = 0$ the theorem states that

$$[0,2]\cdot\{1,i,-1,-i\}\subseteq S.$$

This follows by combining Lemma 4 with direct calculations of $\mathrm{Spec}(A_c)$ when $c_n = 1$ for all $n \in \mathbb{Z}$ (in which case $\mathrm{Spec}(A_c) = [-2,2]$) and when $c_n = -1$ for all $n \in \mathbb{Z}$ (in which case $\mathrm{Spec}(A_c) = i[-2,2]$). For larger n the first statement of the theorem follows by applying Lemma 6 inductively. The second statement is now a consequence of the fact that the set (9) is dense in the unit disc. □

4 The maps $\Phi_\pm$

A crucial role has been played in the proofs above by the nonlinear map Φ_+ on Ω introduced in Lemma 5, and this map will be key to the arguments that we make throughout this paper. And in fact a sequence which is almost a fixed point of Φ_+ (in a sense made precise below Lemma 8) is central to the proof of Theorem 7 in [5], though the proof is quite different and no mapping Φ_+ appears in [5].

The relationship between the above proof of Theorem 7 and that in [5] is clarified to some extent by the following. Building on the definition of Φ_+ made above, let us define maps $\Phi_\pm : \Omega \to \Omega$ by $\Phi_\pm(b) = c$ where

$$c_0 = \pm 1; \qquad c_{2n} + c_{2n+1} = 0; \qquad c_{2n}c_{2n-1} = b_n; \tag{10}$$

for all $n \in \mathbb{Z}$. We also define the space inversion symmetry $b \to \hat{b}$ by $\hat{b}_n = b_{1-n}$ for all $n \in \mathbb{Z}$.

Lemma 8 *If $\Phi_\pm(b) = c$ then $\Phi_\mp(\hat{b}) = \hat{c}$. In particular $\Phi_\pm(c) = c$ if and only if $\Phi_\mp(\hat{c}) = \hat{c}$. Each of the equations $\Phi_\pm(c) = c$ has exactly one solution.*

Proof Let $c = \Phi_+(b)$ and $d = \Phi_-(\hat{b})$. Then $d_0 = -1$, $d_{2n} + d_{2n+1} = 0$ and $d_{2n}d_{2n-1} = \hat{b}_n = b_{1-n}$ for all $n \in \mathbb{Z}$. Therefore $\hat{d}_0 = d_1 = 1$. Also

$$\hat{d}_{2n+1} + \hat{d}_{2n} = d_{1-(2n+1)} + d_{1-2n} = d_{-2n} + d_{1-2n} = 0$$

and

$$\hat{d}_{2n}\hat{d}_{2n-1} = d_{1-2n}d_{1-(2n-1)} = d_{2(1-n)-1}d_{2(1-n)} = \hat{b}_{1-n} = b_n$$

for all $n \in \mathbb{Z}$. Therefore $\hat{d} = {}_+(b) = c$ and $d = \hat{b}$.

The proof that $c = {}_-(b)$ implies $d = {}_+(\hat{b})$ is similar. The other statements of the lemma follow immediately. $\square$

This paper and [5] use three different special sequences. The sequences $c_\pm$ are defined by ${}_\pm(c_\pm) = c_\pm$. It follows directly from their definitions that $c_{+;0} = 1$ and $c_{+;1} = -1$ while $c_{-;0} = -1$ and $c_{-;1} = 1$. However

$$c_{+;n} = c_{-;n} = c_{+;1-n} = c_{-;1-n}$$

for all $n \neq 0; 1$. The paper [5] uses the sequence c_e such that $c_{e;0} = c_{e;1} = 1$, while $c_{e;n} = c_{\pm;n}$ for all other n. Because of the space inversion symmetry the use of c_+ or c_- in any proof is really a matter of convenience.

We now turn to the solution of the equation $A_c u = \lambda u$ where $u : \mathbb{Z} \to \mathbb{C}$ is an arbitrary sequence. The eigenvalue equation is equivalent to the second order recurrence equation

$$u_{n+1} + c_n u_{n-1} = \lambda u_n.$$

Lemma 9 *Suppose that* $c \in$ *and* $\hat{c}_n = c_{1-n}$ *for all* $n \in \mathbb{Z}$*; that* $u_{n+1} + c_n u_{n-1} = \lambda u_n$ *for some* $\lambda \in \mathbb{C}$ *and all* $n \in \mathbb{Z}$ *and* $u_0 = 0$, $u_1 = 1$*; and that* $\hat{u}_{n+1} + \hat{c}_n\hat{u}_{n-1} = \lambda\hat{u}_n$ *for all* $n \in \mathbb{Z}$ *and* $\hat{u}_0 = 0$, $\hat{u}_1$

Proof

Lemma 14 *If $b \in$ is periodic with period N, i.e. $b_{n+N} = b_n$, $n \in \mathbb{Z}$, then $c = {}_+(b)$ is $4N$-periodic. Conversely, if $b \in$, $c = {}_+(b)$, and c is $2N$-periodic for some $N \in \mathbb{N}$, then b is N-periodic.*

Proof First note that, if $c = {}_+(b)$ and one defines $\tilde{c} \in$ by $\tilde{c}_n = c_{2n}$, $n \in \mathbb{Z}$, then

$$c = {}_+(b) \iff (\tilde{c}_0 = 1; \quad \tilde{c}_n = -b_n \tilde{c}_{n-1}; \; c_{2n+1} = -\tilde{c}_n; \; n \in \mathbb{Z}). \tag{12}$$

Therefore

$$\tilde{c}_{m+n} = \tilde{c}_m (-1)^n \prod_{j=1}^{n} b_{m+j} \tag{13}$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. If b is N-periodic, then

$$\tilde{c}_{m+2N} = \tilde{c}_m \prod_{j=1}^{2N} b_{m+j} = \tilde{c}_m \prod_{j=1}^{N} b^2_{m+j} = \tilde{c}_m;$$

for all $m \in \mathbb{Z}$. Therefore c is $4N$-periodic.

Conversely, if $c = {}_+(b)$, for some $b \in$, and c is $2N$-periodic for some $N \in \mathbb{N}$, then $\tilde{c}$ is N-periodic and, from (12), it follows that b is N-periodic. □

To illustrate the above lemma, define $c^-, c^+ \in$ by $c^-_n = -1$, $c^+_n = 1$, for $n \in \mathbb{Z}$, and define the sequences $c^{(m,+)}, c^{(m,-)} \in$, for $m = 0, 1, \ldots$, by

$$c^{(0,\pm)} = c^\pm, \quad c^{(m,\pm)} = {}_+(c^{(m-1,\pm)}), \quad m \in \mathbb{N}. \tag{14}$$

Then explicit calculations of the action of $_+$ yield that $c^{(1,+)} =$ TJ/ehC.527lations of the actio

and

$$\mathrm{Spec}(A_{c^{(m,-)}}) = e^{i\pi/2^{m+1}}\,\mathrm{Spec}(A_{c^{(m,+)}}). \tag{18}$$

Combining equations (15), (17) and (18), we see that we have shown that

$$\left\{ r e^{i\pi j/2^m} : 0 \le r \le 2^{1/2^{m+1}},\ j \in \{0,\dots,2^{m+2}-1\} \right\} \subset \pi_{4^m}, \quad m = 0,1,\dots.$$

Thus we have shown the following modification of Theorem 7 which, of course, by (2), has Theorem 7 as a corollary.

Theorem 15 *The set π_∞ contains the set (9), and so is dense in the unit disc in $\mathbb{C}$.*

We know $\mathrm{Spec}(A_{c^{(m,-)}})$ explicitly, but do not have explicit formulae for the sequences $c^{(m,-)}$. However we can show that $c^{(m,-)}$ converges pointwise to the sequence c_+, the unique fixed point of Φ_+, as $m \to \infty$. This is the content of the next two lemmas. We omit a proof of the first of these lemmas which is an easy consequence, by simple induction arguments, of the definition of Φ_+.

Lemma 16 *If $b \in \Omega$ and $c = \Phi_+(b)$, then $c_0 = c_{+,0}$ and $c_1 = c_{+,1}$. If, for some $N \in \mathbb{N}$, $b_m = c_{+,m}$ for $m = 1,\dots,N$, then also $c_m = c_{+,m}$ for $m = 2,\dots,2N+1$. If, for some $N \in \mathbb{Z}_+$, $b_{-m} = c_{+,-m}$ for $m = 0,1,\dots,N$, then $b_{-m} = c_{+,-m}$ for $m = 1,2,\dots,2N+2$.*

Lemma 17 *Let $b \in \Omega$, and define $c^{(n)} \in \Omega$ for $n \in \mathbb{N}$ by $c^{(1)} = \Phi_+(b)$ and $c^{(n+1)} = \Phi_+(c^{(n)})$, $n \in \mathbb{N}$. Then, for $n \in \mathbb{N}$,*

$$c^{(n)}_m = c_{+,m}, \quad m = 2-2^n, 3-2^n, \dots, 2^n-1,$$

so that $c^{(n)} \to c_+$ pointwise and $A_{c^{(n)}}$ converges strongly to A_{c_+} as $n \to \infty$. Further,

$$\mathrm{Spec}(A_{c^{(n)}}) \subset \{\lambda : |\lambda| \le 2^{1/2^n}\}.$$

Proof The first equation follows by induction from Lemma 16. The second equation follows by induction from (16) and the trivial bound that $\mathrm{Spec}(A_b) \subset \{\lambda : |\lambda| \le 2\}$, which holds for all $b \in \Omega$. □

5 The mapping $\Phi_{\sigma,+}$

For the rest of the paper we consider operators A_c for which the coefficients c_n take values in $\{\pm\sigma\}$, where $0 < \sigma \le 1$; that is, in the notation we have introduced in the introduction, we assume that $c \in \Omega_\sigma$, for some $\sigma \in (0,1]$.

The mapping $_{+}$ that we have introduced continues to play an important role. We extend the mapping so that it operates on $_{2}$, defining, for $\in (0,1]$, $_{,+} : {}_{2} \to$ by

$$_{,+}(c) = {}_{+}({}^{2}c).$$

developed to a high degree of sophistication; see [1, 3, 2, 4] and the references therein. We need only a small part of this theory, and it is easy to develop this from first principles. We do this in a short Lemma 25 below, inspired by earlier analysis in [12, 9, 10], and particularly [10, Theorem 12]. Both the proof of Lemma 25, and the effective application of this lemma to prove Theorem 27, depend on the next two lemmas which describe properties of the spectra and eigenfunctions of periodic operators.

We assume throughout this section that the parameter $\in (0, 1)$.

Lemma 19 *Let*

$$(\; ; \;) = \frac{\mathrm{Re}(\;)^2}{(1 + \;)^2} + \frac{\mathrm{Im}(\;)^2}{(1 \quad)^2} \tag{25}$$

where $\in \mathbb{C}$ *and* $1 < \quad < 1$. *Then the quadratic equation*

$$z^2 \quad z + \quad = 0 \tag{26}$$

has a solution satisfying $|z| = 1$ *if and only if* $= 1$. *If* < 1 *then both solutions satisfy* $|z| < 1$. *If* > 1 *then one solution satisfies* $|z| < 1$ *and the other satisfies* $|z| > 1$.

Proof For $\in \mathbb{R}$, $z = e^{i}$ is a solution of (26) if and only if

$$\cos(\;) = \frac{\mathrm{Re}(\;)}{1 + }; \qquad \sin(\;) = \frac{\mathrm{Im}(\;)}{1 };$$

so that (26) has a solution satisfying $|z| = 1$ if and only if $(\; ; \;) = 1$.

The set $U = \{ \in \mathbb{C} : \quad < 1\}$ is connected and contains the origin. Since the solutions of (26) depend continuously on , and both solutions satisfy $|z| < 1$ if $= 0$, it follows that both satisfy $|z| < 1$ for all $\in U$. The case > 1 is similar. □

The following lemma is closely related to a similar result for the non-self-adjoint Anderson model in [10, Theorem 11].

Lemma 20 *If* $c \in$ *and* $\in \mathbb{C}$ *then the space of all solutions of* $A_c f = \; f$ *is two-dimensional. If* c *is periodic with period* p *then the asymptotic behaviour as* $n \to \quad 1$ *of the solutions is determined by the solutions* $z_1; z_2$ *of the polynomial* $z^2 \quad (\;) z + \quad = 0$, *where* $(\;)$ *is a monic polynomial in* *with degree* p, *given by* $(\;) = \mathrm{tr}(T_p)$, *where* $T_p = X_p X_{p \; 1} \ldots X_1$ *and*

$$X_n = \begin{pmatrix} 0 & 1 \\ & c_n \end{pmatrix};$$

and $= \det(T_p) = \quad ^p$. *Ordering the two solutions so that* $|z_1| \quad |z_2|$, *there are three cases:*

1. *λ lies in the closed set*

$$B_c = \{\lambda : |z_1| = 1 \text{ and } |z_2| = {}^p\}.$$

This set is the spectrum of A_c, equivalently, the set of λ for which $A_c f = \lambda f$ has a bounded solution.

2. *λ lies in the open set*

$$I_c = \{\lambda : 1 > |z_1| \geq |z_2| > {}^p\}.$$

This is the case if and only if all solutions of $A_c f = \lambda f$ decay exponentially as $n \to +\infty$.

3. *λ lies in the open set*

$$O_c = \{\lambda : |z_1| > 1 \text{ and } |z_2| < {}^p\}.$$

This is the case if and only if there exists a solution of $A_c f = \lambda f$ that decays exponentially as $n \to +\infty$ and grows exponentially as $n \to -\infty$, and another solution that decays exponentially as $n \to -\infty$ and grows exponentially as $n \to +\infty$.

Proof The sequence $f : \mathbb{Z} \to \mathbb{C}$ is a solution of $A_c f = \lambda f$ if and only if $f_{n+1} + c_n f_{n-1} = \lambda f_n$ for all $n \in \mathbb{Z}$. This recurrence relation can be rewritten in the form

$$\begin{aligned} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -c_n & \lambda \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \\ &= X_n \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \\ &= T_n \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \end{aligned}$$

where $T_n = X_n X_{n-1} \ldots X_1$. If c is periodic with period p, then the asymptotic behaviour of the two-dimensional space of eigenfunctions f is determined by the magnitude of the eigenvalues z_1, z_2 of T_p. These are the solutions of the equation $z^2 - \alpha z + \beta = 0$ where $\alpha = \mathrm{tr}(T_p)$ and $\beta = \det(T_p)$. A simple induction establishes that the (i,j)-th entry of T_p is a polynomial in λ with degree less than p unless $i = j = 2$ in which case it is a monic polynomial with degree p. Therefore α is a monic polynomial in λ with degree p. However

$$\det(T_p) = \prod_{r=1}^{p} \det(X_r) = c_1 \ldots c_p = {}^p$$

does not depend on . The continuous dependence of the roots of a polynomial on its coe cients implies that B_c is closed while I_c and O_c are open. An application of Lemma 19 now completes the proof. One sees, in particular, that

$$\mathrm{Spec}(A_c) = B_c = f \quad : \quad (\; ; \quad) = 1g:$$

□

Our next lemma enables us to determine the sets I_c and O_c for certain important periodic sequences c, and to determine the spectra of certain paired periodic operators. We continue with the assumptions and notation of Lemma 20.

Lemma 21 *If V is a connected component of* CnB_c *then $V \quad I_c$ or $V \quad O_c$. If V is unbounded then $V \quad O_c$, and if* 0 *2 V then $V \quad I_c$. If* CnB_c *has exactly two components then the bounded component equals I_c and the unbounded component equals O_c.*

Proof We rst observe that V, I_c and O_c are all open sets and that their de nitions imply directly that $I_c; O_c$ are disjoint. Therefore $V = (V \setminus I_c)\;[\;(V \setminus O_c)$, where the two intersections on the right-hand side are disjoint. Since V is connected, it follows that $V = V \setminus I_c$ or $V = V \setminus O_c$ This completes the proof of the rst statement.

Lemma 20 case 1 implies that

$$B \quad = \mathrm{Spec}(A_c) \quad f \quad : j \;\; j \quad 1 + \quad g:$$

Therefore CnB_c has only one unbounded component V and it contains $f \quad : j \;\; j > 1 + \quad g$. To prove that $V \quad O_c$ it is su cient by the rst part of this proof to nd a single point $\quad 2\; V \setminus O_c$. The fact that is a polynomial with degree p implies that $j \;(\;)j \;! \;1$ as $j \;\; j \;! \;1$. This implies that the solutions of $z^2 \quad (\;)z + \quad = 0$, where $= \quad ^p$, are $z \quad (\;)$ and $z \quad = (\;)$ to leading order for all large enough $j \;\; j$. Therefore $\quad 2\; O_c$ for all such .

The proof is completed by proving that $0 \;2\; I_c$. For $= 0$ one has $T_p = X_p X_{p \; 1} ::: X_1$ where each X_r is of the form $\begin{matrix} 0 & 1 \\ & 0 \end{matrix}$. If $p = 2m$ it follows that $T_p = \begin{matrix} ^m & 0 \\ 0 & _m \end{matrix}$. The fundamental equation must therefore take one of the forms $z^2 \quad 2 \; ^m z + \quad ^{2m} = 0$, $z^2 + 2 \; ^m z + \quad ^{2m} = 0$ or $z^2 \quad ^{2m} = 0$. In each case both solutions have modulus $^{p=2} < 1$. The same holds if $p = 2m + 1$.

The nal statement of the lemma follows from the following observations. There must be a component of C n B_c that contains 0 and there must be an unbounded component. The rst part of the proof shows that these are distinct, and the extra hypothesis is that there are no other components. □

Our next task is to determine the sets $B_c; I_c$ and O_c for certain particular periodic sequences.

Lemma 22 *If* $c_n =$ *for all* n 2 Z *then* Spec(A_c) *is the ellipse*

$$\mathrm{Spec}(A_c) = \left(u + iv : \frac{u^2}{(1 + \ )^2} + \frac{v^2}{(1 \ \ )^2} = 1 \right) \tag{27}$$

$$= \ \mathrm{e}^{i} \ : \ = \mathrm{p}\frac{1 \quad ^2}{1 + \ ^2 \ \ 2 \ \cos(2 \ )} \ : \tag{28}$$

Moreover the interior U

Moreover,

$$I_c = \text{e}^{i} : 0 < {}_n(;)$$

and

$$O_c = \text{e}^{i} : > {}_n(;) :$$

Proof Our proof of (31) is by induction. We note rst that (31) holds for $n = 0$ by Lemmas 22 and 23. Suppose now that (31) holds for some n 0 and all $0 < \ < 1$. Then

$$\text{Spec}(A_{\ ^2c^{(n;)}}) = f \ \text{e}^{\text{i}} : = {}_n(; \ ^2)g = \text{e}^{\text{i}} : = {}_0(2^n ; 2^{n+1})^{1=2^n} :$$

Further, since $c^{(n+1;)} = {}_+(c^{(n;+)}) = {}_{;+}(\ ^2c^{(n;+)})$, it follows from Lemma 18 that

$$\text{Spec}(A_{c^{(n+1;)}}) = \text{n} \ p\_ : \ 2 \ \text{Spec}(A_{\ ^2c^{(n;)}}) \ \text{o} :$$

Combining these equations, we see that (31) holds with n replaced by $n + 1$. Thus (31) follows by induction.

The formulae for I_c and O_c follow from (31) and Lemma 21. □

We remark that ${}_n(;) = {}^+_n(=2^{n+1};)$, so that the spectra of A_c in the above lemma are related by

$$\text{Spec}(A_{c^+}) = \text{e}^{i =2^{n+1}} \text{Spec}(A_c):$$

This is a symmetry which is surprising from an inspection of the sequences c , which need not even have the same period. (For example, as observed in Section 4, c^+ has period 4 and c period 2 in the case $n = 1$.)

In principle, since c is periodic, (31) should be computable alternatively from the characterisation of the spectrum for general periodic sequences in Lemma 20. As an example of this, for the sequence $c = c^{(1;)}$ which has period 2, with $c_n = (1)^n$, the transfer matrix T_2 is given by

$$T_2 = X_2X_1 = \begin{matrix} 0 & 1 \end{matrix} \quad \begin{matrix} 0 & 1 \end{matrix} = \quad + \ ^2 :$$

Applying Lemmas 19 and 20 with $= \ ^2$ and $= \ ^2$, we nd that Spec(A_c) is the set of all $= u + iv$ for which

$$\frac{(u^2 \ v^2)^2}{(1 \ \ ^2)^2} + \frac{(2uv)^2}{(1 + \ ^2)^2} = 1:$$

If one puts $= \text{e}^{i}$, then this may be rewritten in the form (31).

The main point of the above theory and calculations are to prove and prepare the use of the following result on operators A_c that are paired periodic operators. To state this result let us introduce the notations

$$E \quad = \quad x + iy : \frac{x^2}{(1 + \quad)^2} + \frac{y^2}{(1 \quad\quad)^2} < 1 \qquad (32)$$

and

$$E \quad = \quad x + iy : \frac{x^2}{(1 \quad\quad)^2} + \frac{y^2}{(1 + \quad)^2} < 1 \quad ; \qquad (33)$$

so that E and E 2 T48f003f003sta003 02([10,31(TdT(v)2emta003)12],31(Td)-432431(airedta

and

Proof We regard V_cR as a small perturbation of L in the identity $A_c = V_cR + L$, noted in the proof of Lemma 5. Since L is a unitary operator with spectrum $\{z : |z| = 1\}$, we have

$$\|(L - zI)^{-1}\| = |1 - |z||^{-1}$$

for all z not on the unit circle. The inclusion (34) now follows from $\|V_cR\| = \gamma$ by a perturbation argument; see [11, Th. 9.2.13].

The inclusion (35) depends on an estimate of the numerical range of A_c. Following [11, Section 9.3], $x + iy \in \mathrm{Num}(A_c)$ if there exists $f \in \ell^2(\mathbb{Z})$ such that $\|f\| = 1$ and $x + iy = \langle A_c f, f\rangle$. This implies that

$$x = \frac{1}{2}\langle (A_c + A_c^*)f, f\rangle; \qquad y = -\frac{i}{2}\langle (A_c - A_c^*)f, f\rangle.$$

Therefore

$$x + y = \frac{1}{2}\langle Bf, f\rangle$$

where

$$B = (A_c + A_c^*) - i(A_c - A_c^*).$$

A simple calculation shows that $B_{m,n} = 0$ unless $|m - n| = 1$, while

$$B_{n,n+1} = \overline{B_{n+1,n}} = (1 - \gamma) - i(1 + \gamma).$$

Therefore $|B_{n+1,n}| = |B_{n,n+1}| = \sqrt{2(1 + \gamma^2)}$ for all $n \in \mathbb{Z}$ and

$$x + y \leq \frac{1}{2}\|B\| \leq \sqrt{2(1 + \gamma^2)}.$$

The other three steps in the proof of the bound for $|x| + |y|$ are similar. □

The statement of our main theorem refers to the open set

$$H_\gamma = E_\gamma \cap E_{-\gamma}, \tag{36}$$

the intersection of the ellipses E_γ and $E_{-\gamma}$. This set satisfies

$$[\![0, 1 - \gamma]\!] \subseteq \overline{H_\gamma} \subseteq [\![0, r_\gamma]\!] \tag{37}$$

where

$$r_\gamma = \frac{1 - \gamma^2}{\sqrt{1 + \gamma^2}}. \tag{38}$$

Theorem 27 *If* $0 < \gamma < 1$ *then*

$$\{\lambda : |\lambda| \leq 1\} \backslash H_\gamma \subseteq S_\gamma.$$

Proof Note rst that if c and $_n(;)$ are de ned as in Lemma 24, then

$$_n(;) \quad _{;n} = \left(\frac{1}{1 + \quad 2^n} \quad 2^{n+1} \right)^{1=2^n} ;$$

for all 2 R, so that $I_c \quad f \ : j \ j < \quad _{;n} g$. Thus, de ning $c \ 2$ as in Lemma 25, with $c = c^+$ or c and $=$, we see from Lemma 25 that

$$\mathrm{Spec}(A_c \ ) \quad \overline{I_c} \, n \, E \qquad f \ : j \ j \qquad _{;n} g \, n \, E \ : \qquad (39)$$

Applying Proposition 1, it follows that, for all n 2 N,

$$S \qquad f \ : j \ j \qquad _{;n} g \, n \, H \ :$$

The theorem follows since $\sup_n \quad _{;n} = 1$ and S is closed. □

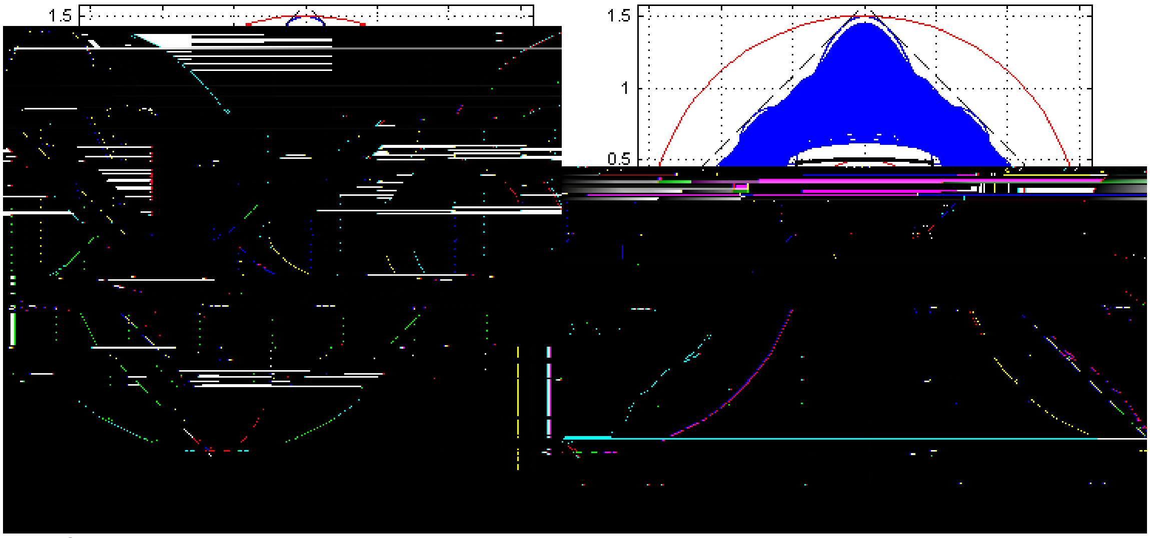

Figure 1: Plots of Spec(A_c) for the case when c is periodic and $= 0{:}5$. The two plots show the sets $_{N;} \quad S$, the union of the spectra for all sequences c of period N, for $N = 2$ (left) and $N = 12$ (right). The two ellipses visible in the left-hand plot, the boundaries of E and E de ned in (32) and (33), are the components of $_{1;}$; the other closed curve is Spec(A_c) for $c = \ c^{(1; \ )}$, i.e. c

numerical computations we have been able to carry out are consistent with a hypothesis that the hole is precisely the set H , i.e. they are consistent with a hypothesis that $\mathrm{Spec}(A_c) \cap H = \emptyset$ for every $c \in$, and hence for every

It is not feasible to calculate $_{N;}$, the union of all 2^{N} 1

n	e_n	u_n	v_n
1	1	1	0
2	1		1
3	1	^{2} 1	
4	1	^{3}	^{2} 1
5	1	^{4} + ^{2} 1	^{3} 2
6	1	^{5}	^{4} ^{2} + 1
7	1	^{6} + ^{4} 1	^{5} 2 ^{3}
8	1	^{7}	^{6} ^{4} 1
9	1	^{8} + ^{6} + ^{4} 1	^{7} 2 +8 Td [(4)]TJ/F37 11.9552 Tf 7.3 610

n	$\mathrm{tr}(T_{n;\ })=u_{n+1}+v_n$
1	
2	$^{2}\quad 2$
3	^{3}
4	$^{4}\quad 2$
5	$^{5}\quad ^{3}\quad 3$
6	$^{6}\quad ^{2}$
7	$^{7}\quad ^{5}\quad 2\ ^{3}$
8	$^{8}\quad 2$

Table 2: Values of $\mathrm{tr}(T_{n;\ })$ for $1\quad n\quad 8$.

for all $r \quad 1$. *If* $\ ^2 = 4\ $ *then for every* $" > 0$ *there exists a constant* $b_"$ *such that*

$$kT^r k \quad b_"(\quad + ")^r \tag{44}$$

for all $r \quad 1$.

Proof The eigenvalues z of T are the roots z of $z^2 \quad z + \quad = 0$. The condition $\ ^2 \neq 4\ $ implies that the eigenvalues are distinct, so T is diagonalizable { there exists an invertible matrix B such that

$$T = B \begin{pmatrix} z_+ & 0 \\ 0 & z \end{pmatrix} B^{\ 1}:$$

Therefore

$$kT^r k = kB \begin{pmatrix} z_+^r & 0 \\ 0 & z^r \end{pmatrix} B^{\ 1}k \quad kBkkB^{\ 1}k^{\ r}:$$

The slightly worse bound (44) is obtained when $\ ^2 = 4\ $ because one has to use the Jordan canonical form for T. □

Lemma 29 *The identity* $\det(T_{2^n;\ }) = 1$ *holds for all* $n \quad 1$ *and all* $\ 2\ \mathbb{C}$.

Proof If $m\ 2\ \mathbb{N}$ then (42) and (43) imply

$$\begin{aligned}\det(T_{2m;\ }) &= \prod_{r=1}^{m} \det(X_{2r}X_{2r\ 1}) \\ &= \prod_{r=1}^{m} (e_{2r}e_{2r\ 1})\end{aligned}$$

$$= \prod_{r=1}^{m} e_r$$

$$= \det(T_{m;}\ ):$$

It follows by induction that

$$\det(T_{2^n;}\ ) = \det(T_{1;}\ ) = e_1 = 1:$$

□

The following lemma depends on Proposition 11 above, abstracted from [5], which notes properties of the integer coe cients $p_{i;j}$ of the polynomials

$$u_i = \sum_{j=1}^{i} p_{i;j} \quad ^{j \quad 1}:$$

Lemma 30 *The polynomial u_m is even for odd m and odd for even m. Its leading term is $^{m \quad 1}$. If $m = 2^n$ and $n \quad 2$ then*

$$u_m = \quad ^{m \quad 1}; \tag{45}$$

$$u_{m+1} = \quad 1 + \quad ^{m=2} \sum_{r=0}^{m=4} \quad _r \quad ^{2r} \tag{46}$$

where $_r$ 2 $f0; 1; \quad 1g$ for all r.

ProPro(1.9552 Tf 5.244 0 Td [(1)]TJ/F34 11.9552 Tf 5.853 0 Td [(;)]T1.70(e)]TJ/F34 1

for all m 2 N. Therefore $_5$ = $_3$ = 1 and

$$
\begin{aligned}
{}_{8m+1} &= \prod_{r=1}^{m} (e_{8r} e_{8r\ 2} e_{8r\ 4} e_{8r\ 6}) \\
&= \prod_{r=1}^{m} (e_{4r} e_{8r\ 1} e_{8r\ 2} e_{4r\ 2} e_{8r\ 5} e_{8r\ 6}) \\
&= \prod_{r=1}^{m} (e_{4r} e_{4r\ 2}) \\
&= {}_{4m+1}
\end{aligned}
$$

for all m 2 N. A simple induction now implies that $_m$ = 1 for $m = 2^n + 1$ and all n 1. □

Lemma 31 *If $m = 2^n$ and n 2 then*

$$ = \mathrm{tr}(T_{m;\ }) = v_m + u_{m+1} = \ ^m \ \ 2 \tag{47}$$

for all 2 C.

Proof The proof uses the identity $v_m u_{m+1} \quad u_m v_{m+1} = 1$ of Lemma 29 together with the two identities proved in Lemma 30. These are identities within the commutative ring Z( ) of all polynomials with integer coe cents in the indeterminate quantity , but they imply similar identities in the commutative ring Z(b) of all polynomials with integer coe cients in an indeterminate quantity b that satis es the identity $b^{m\ 1} = 0$. (Equivalently one may start by disregarding all terms in the identities that involve ^{r} with r m 1.) The

identities /F34 Tf [(b)] TJ /F34 11.9552 Tf -0.393 -3.0] TJ /F34 11 may 1.9552 T 8.3(5) TJ/(6)

Theorem 34 *One has*

$$\{ \ : | \ | \ \ 1\} \setminus H \quad S \tag{48}$$

for all $\ \in (0, 1)$.

Proof Given $\ \in (0, 1)$ we put $m = 2^d$ where $d \in \mathbb{N}$ is large enough to yield

$$^{1/2} < h = 4 \ ^{1/m}. \tag{49}$$

We use the identities

$$ = \det(T_{m,}\ ) = 1$$

and

$$ = \mathrm{tr}(T_{m,}\ ) = \ ^m \ \ 2$$

proved in Lemmas 29 and 31 and valid for all $\ \in \mathbb{C}$. Let $c \in$ be the periodic sequence with period m such that $c_n = \tilde{c}_n$ for all $1 \ \ n \ \ m$. The main task is to prove that if $| \ | < h$ then all solutions $\ : \mathbb{Z} \to \mathbb{C}$ of

$$_{n+1} = \ _n \ \ c_n \ _{n-1} \tag{50}$$

decay exponentially as $n \to +\infty$. This will imply, by Lemma 20, and using the notations of that lemma, that

$$I_c \ \ \{ \ : | \ | < h\}.$$

Arguing as in the proof of Theorem 27, it will then follow from Lemma 25 and Proposition 1 that

$$S \ \ \{ \ : | \ | \ \ h\} \setminus H \ ;$$

this holding for any $h = 4 \ ^{1/m}$ such that (49) holds and $m = 2^d$, so that

$$S \ \ \{ \ : | \ | < 1\} \setminus H \ .$$

Since S is closed, (48) will follow.

Thus it remains only to show that all solutions of (50) decay exponentially at $+\infty$. To see that this holds, define $x_n = \ ^{n/2} \ _n$ and $\ = \ ^{1/2}$ so that (50) may be rewritten in the form

$$x_{n+1} = \ x_n \ \ \tilde{c}_n x_{n-1}$$

for $1 \ \ n \ \ m$. Where $\ = \max(1, | \ |)$, Lemma 32 now yields

$$\|(T_{m,}\ )^r\| \ \ b 4^r \ ^{rm}$$

for all $r \in \mathbb{N}$. Lemma 33 with $\ = 4 \ ^m$ implies

$$\|T_{r,}\ \| \ \ b 4^{r/m} \ ^r;$$

and hence

$$|x_r| \le b_3 4^{r/m - r};$$

again for all $r \in \mathbb{N}$. Hence, where $\ = \max(\ ^{1/2}, |\ |)$,

$$|\ _r| \le b_3 4^{r/m - r}\ ^{r/2} = b_3 \left(\ h^{-1}\right)^r$$

for all $r \in \mathbb{N}$. Since $0 < \ < h$, it follows that decays exponentially. □

9 Semi-infinite and finite matrices

All our results so far have focused on calculations of the spectrum of the bi-infinite matrix A_c. In this final section we say something about the spectrum of the semi-infinite matrix

$$A_c^+ = \begin{pmatrix} 0 & 1 & & \\ c_1 & 0 & 1 & \\ & c_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

in the case that $c = (c_1, c_2, \ldots) \in \{\ \}^{\mathbb{N}}$ is pseudo-ergodic (contains every finite sequence of 's as a consecutive sequence). We also say something (though have mainly unanswered questions) about the finite $N \times N$ matrices

$$A_c^{(N)} = \begin{pmatrix} 0 & 1 & & & \\ c_1 & 0 & 1 & & \\ & c_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & c_{N-1} & 0 \end{pmatrix}$$

and

$$A_{c,}^{(N,\mathrm{per})} = \begin{pmatrix} 0 & 1 & & & c_N \\ c_1 & 0 & 1 & & \\ & c_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ \ ^{-1} & & & c_{N-1} & 0 \end{pmatrix}. \tag{51}$$

Here $A_c^{(N)}$ is tridiagonal, $A_c^{(N,\mathrm{per})}$ is tridiagonal except for "periodising" entries in row 1 column N and row N column 1 (in these entries we assume that $|\ | = 1$), and each $c_j = \ $: we have in mind particularly the random case where the c_j's are independent and identically distributed random variables taking the values .

Our main result on the spectrum of A_c, proved in the previous sections, is that it contains the set $f : j j 1g n H$. We suspect that H is a genuine hole in the spectrum for $0 < < 1$, i.e. that $H \ S = ;$. We have not shown this result but have shown in Lemma 26 the weaker result that $f : j j < 1 g \ S = ;$. Our rst result in this section is that this hole is not present in the spectrum of the semi-in nite matrix. The proof depends on recent results on semi-in nite pseudo-ergodic operators due to Lindner and Roch [19], derived using characterisations of the index of Fredholm operators, whose matrix representations are banded semi-in nite matrices, in terms of so-called \plus indices" of limit operators, these characterisations derived using K-theory results for C -algebras in [20].

Theorem 35 *Suppose c 2 f $g^{\mathbf{N}}$ is pseudo-ergodic. If = 1 then* $\mathrm{Spec}(A_c^+) = S$. *For all 2 (0; 1], $f : j j 1g$ $\mathrm{Spec}(A_c^+)$.*

Proof In the case that = 1 it is shown in [6] that $\mathrm{Spec}(A_c^+) = S$. Thus, for = 1,

$$f : j j 1g \quad S = \mathrm{Spec}(A_c^+)$$

follows from Theorem 7 (or [5, Theorem 2.3]). For all 2 (0; 1] it follows from [19, Theorem 2.1] that the essential spectrum of A_c^+, i.e. the set of 2 C for which $A_c^+ \quad I^+$ is not Fredholm (here I^+ is the identity operator on $\ell^2(\mathbf{N})$), is the set S . Thus and by Theorem 27,

$$(f : j j 1g n H) \quad S \quad \mathrm{Spec}(A_c^+):$$

It remains to show that H $\mathrm{Spec}(A_c^+)$. But, applying [19, Theorem 2.4] (note that the set $E (U; W)$ in the notation of [19, Theorem 2.4] is precisely i.e. the set of of the set H for this operator), it follows that, for 2 H , either A_c^+ ((I(0er

If $\lambda = x + iy$ is an eigenvalue of $A_{c,\sigma}^{(N,\mathrm{per})}$ then $1 - \sigma \le |\lambda| \le 1 + \sigma$ and $|x| + |y| \le \sqrt{2(1+\sigma^2)}$, while if λ is an eigenvalue of $A_c^{(N)}$ then $|x| + |y| \le 2\sqrt{\sigma}$.

Proof The first of these statements is clear from the definition of $\Sigma_{N,\sigma}$, (40), and Proposition 1 which gives that $\Sigma_{1,\sigma} \subset S_\sigma$. The second of these statements is shown for $\sigma = 1$ in [6, Theorem 4.1]. The second statement follows for $0 < \sigma < 1$ by the observation that, where $d \in \{\pm 1\}^N$, $c = \sigma d \in \{\pm\sigma\}^N$, and D_N is the diagonal matrix with leading diagonal $(1, \sigma^{1/2}, \dots, \sigma^{(N-1)/2})$, it holds that $D_N^{-1} A_c^{(N)} D_N = \sqrt{\sigma} A_d^{(N)}$. The last sentence then follows from Lemma 26. □

Note that in the last sentence of the above theorem the condition $|x| + |y| \le 2\sqrt{\sigma}$ implies both that $|\lambda| \le 1 + \sigma$ and that $|x| + |y| \le \sqrt{2(1+\sigma^2)}$.

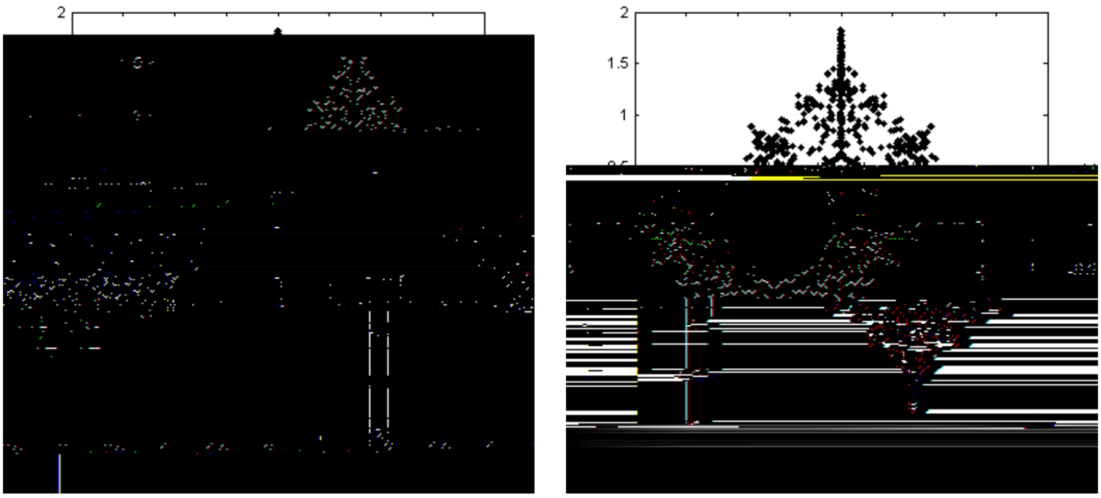

Figure 3: Plots of Spec($A_c^{(N)}$) (left) and Spec($A_{c,\sigma}^{(N,\mathrm{per})}$) (right) for a case when $N = 2000$, $\sigma = 0.9025$, $\tau = 1$, and the entries of the vector $c = (c_1, \dots, c_N)$ are independent and identically distributed with $\Pr(c_j = \pm\sigma) = 0.5$ for each j (the same vector c is used in the two plots).

In Figure 3 we plot the spectra of Spec$A_c^{(N)}$ and Spec$A_{c,\sigma}^{(N,\mathrm{per})}$ for $N = 2000$ and $\tau = 1$ for a typical realisation with the entries $c \in \{\pm\sigma\}^N$ randomly chosen with the c_j independently and identically distributed with $\Pr(c_j = \pm\sigma) = 0.5$ and $\sigma = 0.9025$ so that $\sqrt{\sigma} = 0.95$ (the several other realisations we have computed are very close in appearance to these plots). Theorem 36 tells us that Spec($A_c^{(N)}$) $\subset 0.95\,\Sigma_1 \subset 0.95S$ and that Spec($A_{c,\sigma}^{(N,\mathrm{per})}$) $\subset S_{0.9025}$, and that if $\lambda = x + iy$ is an eigenvalue of $A_c^{(N)}$ then $|x| + |y| \le 1.9$, while if λ is an eigenvalue of $A_{c,\sigma}^{(N,\mathrm{per})}$ then $0.075 \le |\lambda| \le 1.9025$ and $|x| + |y| \le \sqrt{2(1+\sigma^2)} \approx 1.905$.

It is clear from Figure 3 that Theorem 36 is only the beginning of the story. We observe in the gure a hole in the spectrum of $A_{c;}^{(N;\mathrm{per})}$, but it is a hole of radius approximately 0.6, not 0.075, with a large proportion of the eigenvalues positioned on the boundary of this hole, while outside the hole the spectra of $A_{c;}^{(N;\mathrm{per})}$ and $A_c^{(N)}$ appear near identical. The same qualitative behaviour is visible in Figure 4, which is a similar plot except that is reduced to 0.5 and we change the probability distribution, making it twice as likely that the entries of the vector c are

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