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by

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Abstract

In this paper we derive novel approximations to trapped waves in a two-dimensional Dirichlet acoustic waveguide whose walls vary slowly along the guide. The guide contains a single smoothly bulging region, but is otherwise straight, and

where $=\ _0$ is the dimensionless wavenumber.

We first investigate what type of modes exist as solutions to the boundary-value problem consisting of just (2.5) and (2.6), the *quasi-modes*, and consider the reflection of one of these quasimodes at a taper in a duct.

2.2. Quasi-modes

We use the WKBJ-type ansatz

$$= A \ ^{\mathsf{P}} \qquad (2.8)$$

where

$$A = A_0(\ , y) + \epsilon A_1(\ , y) + \epsilon^2 A_2(\ , y) + \qquad (2.9)$$

and

$$= \epsilon^{-1} \ _{-1}(\ , y) + \epsilon \ _1(\ , y) + \epsilon^2 \ _2(\ , y) + \qquad (2.10)$$

An (ϵ^0) term is not included in the expansion for since it can be subsumed into A_0. Note that is allowed to be complex-valued to ensure that (2.8) can include both propagating (imaginary) and evanescent ($\mathsf{Re}(\ ) \neq 0$) modes. The replacement of a single unknown, , by two unknowns, A and , clearly leads to an under-determined scheme since we now have twice as many unknowns as equations. However, it turns out that the first few terms in the series (2.9) and (2.10) can be calculated before this inconsistency halts further progress.

The expressions (2.8), (2.9) and (2.10) are substituted into (2.5), and terms at each order are equated. At (ϵ^{-2}), we have

$$A_0 \ _{-1\mathsf{y}}^2 = 0$$

whose solution we write as

$$_{-1} = \ (\ ). \qquad (2.11)$$

where is to be determined. The (ϵ^{-1}) equation is then trivially satisfied, and at (ϵ^0) we have

$$A_{0\mathsf{yy}} + (\ ^2 + \ '^2)A_0 = 0 \qquad (2.12)$$

The appropriate solution of this equation is

$$A_0 = \ _0(\ ) \ (\ , y), \qquad = (2 \ )^{1=2}\,\mathsf{sin}[\ (y + \ _-)]. \qquad (2.13)$$

where

$$(\ ) = \ _+(\ ) + \ _-(\ ) \qquad (2.14)$$

is the duct width,

$$^2(\ ) = \ ^2 + \ '^2(\ ). \qquad (2.15)$$

and we must choose

$$(\ ) \qquad _{\mathsf{n}}(\ ) = \frac{n}{(\ )} \qquad (2.16)$$

for $n \in \mathbb{N}$ to ensure that the boundary conditions (2.6) are satisfied.

To solve (2.15) for , we must be careful to distinguish cases for which $\ _{\mathsf{n}}^2(\ ) \quad ^2 \quad 0$ and for which $\ _{\mathsf{n}}^2(\ ) \quad ^2 \quad 0$. Thus

$$(\ ) \qquad _{\mathsf{n}}(\ ) = \begin{cases} i\int^{\mathsf{x}} (\ ^2 \quad _{\mathsf{n}}^2(\ _0))^{1=2}\,\mathsf{d} \ _0, & _{\mathsf{n}}(\ ). \\ \int^{\mathsf{x}} (\ _{\mathsf{n}}^2(\ _0) \quad ^2)^{1=2}\,\mathsf{d} \ _0, & [illegible] \end{cases}$$

which is used to determine ϕ_0. Because $A_0 = 0$ on $y = \ \ _{\pm}$, this condition reduces to

$$0 = \int_{-h_-}^{h_+} \frac{d}{d} \left(\ '_n A_0^2 \right) dy = \frac{d}{d} \left(\int_{-h_-}^{h_+} \ '_n A_0^2 \, dy \right) = \frac{d}{d} \left(\ '_n \phi_0^2 \right). \tag{2.20}$$

from which

$$\phi_0 = c | \ ' |^{-1/2} \tag{2.21}$$

for constant c. This gives the quasi-modes

$$_n = c_n | \ '_n |^{-1/2} \ \ ^{-1} f_n \ \ _n(\ , y)$$

so that there is a nonuniformity when $(\quad _* \quad)^{3=2} = \quad (\epsilon)$ i.e., when $(\quad _* \quad) = \quad (\epsilon^{2=3})$. A similar analysis can be carried out for the evanescent exponential term in (2.26), and we see that in the limit $\quad ! \quad ^+_*$,

$$\exp\left\{ \quad \epsilon^{-1} \int_{x_*}^{x} \left(\quad ^2(\quad _0) \quad \quad ^2\right)^{1=2} d \quad _0 \right\} \quad \exp\left\{ \quad \frac{2}{3}\epsilon^{-1} \quad ^{1=2}(\quad \quad _*)^{3=2} \right\} \tag{2.28}$$

Having determined the size of the nonuniformity, we now return to the governing equation (2.5), and write $(\quad , y) = \quad (\quad) \quad (\quad , y)$ where the stretched variable $\quad = \epsilon^{-2=3}(\quad \quad _*)$ is $\quad (1)$ in the region of interest. This substitution yields the equation

$$\epsilon^{2=3}(\quad '' \quad \quad) \quad = \quad \epsilon^{4=3}(\quad '' + 2$$

2.3.2. A uniform

which is solved by

$$C_1 = c_1(\;)\; (\;, y) \tag{2.43}$$

The $(\epsilon^{2=3})$ coefficient of **Ai** is then the equation

$$_{1yy} + {}^2 \; _1 = \; g_0'(g_0' g_1 + 2g_0 g_1') \; _0 \tag{2.44}$$

Multiplying (2.44) by $_0$ and integrating from $y = \; _-$ to $y = \; _+$ shows that the solvability condition $g_0' g_1 + 2g_0 g_1' = 0$ must be satisfied, from which $g_1 = \; _1 g_0^{-1=2}$ for constant $_1$; the solution of (2.44) is then $_1 = \; _1(\;)\; (\;, y)$.

Finally, the $(\epsilon^{4=3})$ coefficient of **Ai**$'$ is

$$C_{2yy} + {}^2 C_2 = \; 2g_0' \; _{0x} \; g_0'' \; _0 \tag{2.45}$$

Again, multiplying (2.45) by $_0$ and integrating from $y = \; _-$ to $y = \; _+$ shows that we require

$$\int_{-h_-}^{h_+} (2g_0' \; _{0x} + g_0'' \; _0) \; _0 \, dy = 0,$$

which reduces to

$$\frac{d}{d} \left({}^2_0 g_0' \right) = 0$$

This has solution

$$_0 = \; |g_0'|^{-1=2} \tag{2.46}$$

for constant , and where from (2.41),

$$|g_0'| = | \; {}^2 \; {}^2|^{1=2} |g_0|^{-1=2}$$

To leading order, the uniform approximation to the n-th quasi-mode reflection/transmission problem is thus

$$(\;, y) \; \frac{|g_0(\;)|^{1=4} \text{Ai}(\epsilon^{-2=3} g_0(\;))\; (\;, y)}{| \; {}^2 \; {}^2(\;)|^{1=4}} \tag{2.47}$$

where g_0 is given from (2.42). The form of this expression bears an obvious similarity to the corresponding ODE expansion derived via the Langer transformation (see [11]).

It only remains to determine the constant , which we do by ensuring that as $! \; 1$, this representation of agrees with (2.25), where is now again regarded as unknown. Now, as $! \; 1$, $> \; _*(\;)$ and $g_0(\;) \; 0$, so that

$$\text{Ai}(\epsilon^{-2=3} g_0(\;)) \quad {}^{-1=2} \epsilon^{1=6} (\; g_0(\;))^{-1=4} \sin \left(\frac{}{4} + \epsilon^{-1} \int_x^{x_*} \left(\; {}^2 \; {}^2(\; _0) \right) d \; _0 \right)$$

3. Trapping of waves in a symmetric duct

In this section, we revive the subscript notation denoting dependence on n **2** $\mathbb{N}$, writing $_{\mathbf{n}}$**( )** for **( )**, $_{\mathbf{n}}$**(** $, y$**)** for **(** $, y$**)**, and so on.

Now, the uniformly-valid approximation to the reflection/transmission process in a tapering duct, equation (2.47), behaves as a decaying wave in $>$ $_*^{(\mathbf{n})} > \mathbf{0}$, and as a propagating wave in **0** $_*^{(\mathbf{n})}$, where $_*^{(\mathbf{n})}$ is the root of $_{\mathbf{n}}$**(** $_*^{(\mathbf{n})}$**) =** . In particular, if the duct narrows monotonically from its maximum width at **= 0**, and lies in **(** $_{\mathbf{n}}$**(0)**$, n$. **2)** (where n . **2 = lim**$_{\mathbf{x}\to\infty}$ $_{\mathbf{n}}$**( )**), then the n-th quasi-mode (2.47) will behave as a propagating wave for **2 [0**, $_*^{(\mathbf{n})}$**)**, and as a decaying wave in **(** $_*^{(\mathbf{n})}$. **1)**. Furthermore, if we now choose so that in addition $_{\mathbf{n}}$**(0**$, y$**) = 0** for y **2 (** **(** **(0**

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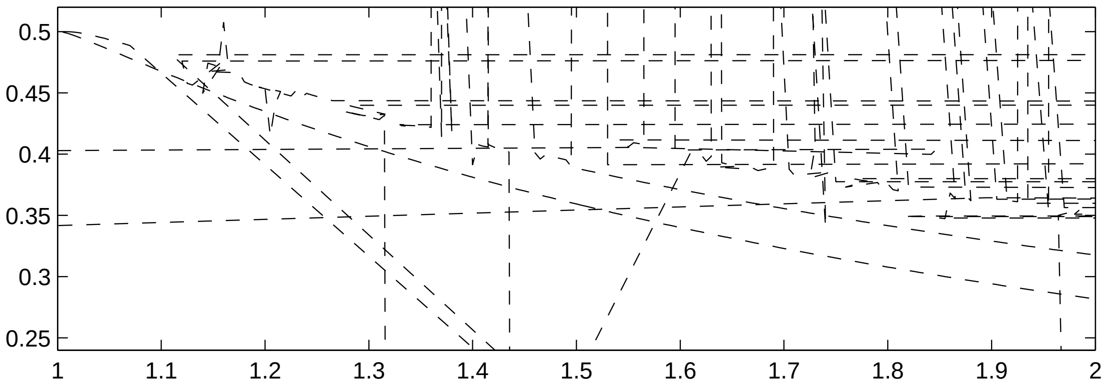
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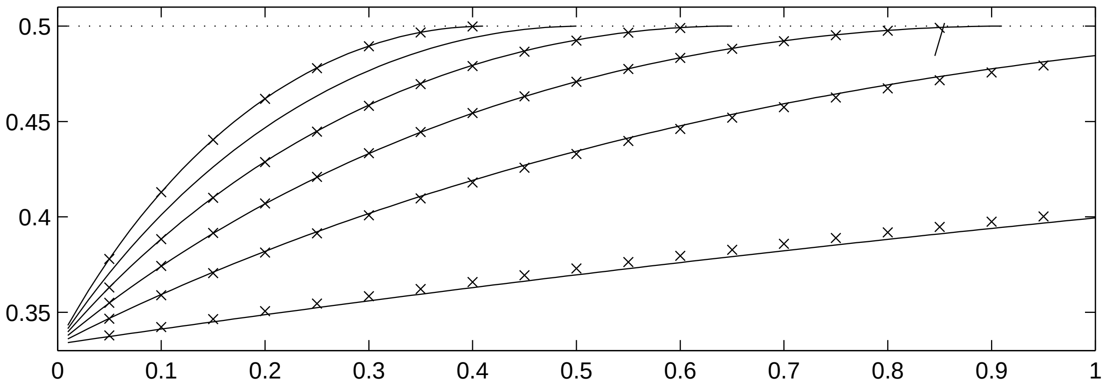
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