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An Extension of Chao's Estimator of

An Extension of Chao's Estimator of Population Size Based on the First Three Capture Frequency Counts

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Abstract

A new estimator for estimating the size of an elusive target population is presented using frequency counts from capture-recapture sampling. The proposed estimator is developed by extending the idea of Chao's estimator using monotonicity of ratios of neighbouring frequency counts under a specific Poisson mixture sampling framework, the Poisson-Gamma mixture or negative binomial. The new estimator is achieved using a simple linear model on the basis of the log-ratio of neighbouring frequency counts as dependent variable which is valid under the Poisson-Gamma mixture. A simulation study is provided to study the performance of the proposed estimator under a variety of heterogeneous Poisson capture probabilities. Confidence interval estimation is done by means of an approximating normal approach and a modified bootstrap method, and was found to perform well. A variety of real data sets were also examined in order to illustrate the use of the proposed method.

Keywords:

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lation, usually defined by a specific disease experiencing potential severe undercount (e.g. Böhning et al., 2004; Corrao et al., 2000; Gallay et al., 2000; Hay et al., 2009; Hook and Regal, 1995; NaT/Fa3363.cmB2f394.1336632.575

it is more reasonable to assume that the actual target population may consist of a variety of subgroups. This leads to a Poisson mixture model of the form

$$p_j = \int_0^\infty \frac{e^{-\lambda}\lambda^j}{j!} f(\lambda) d\lambda, \tag{1}$$

where $f(\lambda)$ represents the heterogeneity distribution of the model parameter in the population. A prominent example for $f(\lambda)$ is the Gamma-distribution $f(\lambda) = \theta\lambda^{k-1}\exp(-\lambda/\theta)/\Gamma(k)$ with parameters $\theta, k > 0$, so that p_j is Poisson-Gamma mixture, or if the marginal is worked out, the *Negative Binomial* distribution. Let $r_j = \frac{jp_j}{p_{j-1}}$, where $p_j = \frac{\Gamma(k+j)}{\Gamma(j+1)\Gamma(k)}\theta^k(1-\theta)^j$ with $\theta = (1-\theta)/\theta$, then we achieve $r_j = (k+j-1)(1-\theta)$. This clearly implies that there is a linear relationship $r_j = (k-1)(1-\theta) + (1-\theta)\,j$ between *Bi-*

(which is not necessarily so when working on the r_j scale). Now, for $j = 2$ or $j = 3$ in (2) we get $\log(r_2) = \log(\frac{2f_2}{f_1}) = \alpha + 2\beta$ and $\log(r_3) = \log(\frac{3f_3}{f_2}) = \alpha + 3\beta$. Solving these equations in α and β can easily be achieved as $\hat{\alpha} = 3\log(\frac{2f_2}{f_1}) - 2\log(\frac{3f_3}{f_2})$ and $\hat{\beta} = \log(\frac{3f_3}{f_2}) - \log(\frac{2f_2}{f_1})$. Then, plugging $\hat{\alpha}$ and $\hat{\beta}$ into (2) and using $j = 1$, (2) provides $\log(r_1) = \log(\frac{f_1}{f_0}) = \alpha + \beta$, or

$$\log(\frac{f_1}{f_0}) = 3\log(\frac{2f_2}{f_1}) - 2\log(\frac{3f_3}{f_2}) + \log(\frac{3f_3}{f_2})$$ rg001RG001rg001RG1001-245.6641-6v9olS15y07.9831i8.

Proof.

The result in Theorem 3 indicates the large potential of reducing bias with the new estimator. To explore this a bit further we consider exponential mixing in (1).

Corollary 1. *Let the mixing density $f(\lambda)$ in (1) be the exponential, $k = 1$, so that the marginal (1) is the geometric. Then:*

$$\lim_{N\to\infty} \frac{E(\hat{N}_{New})}{N} = 1 - \frac{\theta}{4} \text{ and } \lim_{N\to\infty} \frac{E(\hat{N}_{Chao})}{N} = 1 - \frac{\theta}{2}.$$

The condition in corollary 1 might appear di□cult to be checked. However, exponential mixing means that the shape parameter k equals one which implies that the line in the ratio plot passes through the origin. This can be simply diagnosed and formally tested. An asymptotic unbiased Chao-type estimator for this case ($k = 1$) is provided as $n + f_1^2/f_2$ and an asymptotic unbiased estimator incorporating the first three capture frequency counts is also available as $n + f_1^3 f_3/f_2^3$.

Note that (3) is only well-defined as long as f_2 is positive. Therefore, we suggest to use a modification of (3) which allows $f_2 = 0$, as follows

$$\hat{N}_{New_{Mo}} = n + \frac{3}{4}\frac{f_1(f_1-1)(f_1-2)f_3}{(f_2+1)(f_2+2)(f_2+3)}. \tag{4}$$

In addition, we consider the following truncated version of $\hat{N}_{New}$ to improve its variance. It can be seen from Theorem 3 (and by replacing f_j by their theoretical value p_j) that the expected value of $\hat{\gamma} = \frac{3f_1f_3}{2f_2^2}$ approaches

$$\frac{3\Gamma(k+1)\Gamma(k+3)\Gamma(3)^2\Gamma(k)^3}{2\Gamma(2)\Gamma(k)\Gamma(4)\Gamma(k)\Gamma(k+2)^2} = \frac{k+2}{k+1}$$

for N becoming large assuming the negative binomial distribution for the count probabilities p_j, $j = 0, \ldots, m$. Note that

$$1 \leq \frac{k+2}{k+1} \leq 2$$

for $0 \leq k \leq \infty$. Hence, truncation at the upper and lower bound of the asymptotically expected value of the multiplier $\hat{y}$ appears reasonably and leads to an adjusted form $\hat{N}_{New}$ as follows:

$$\hat{N}_{NewAdj} = \begin{cases} n + \frac{f_1^2}{2f_2}, & \text{if } \frac{3f_1 f_3}{2f_2^2} \leq 1 \\ n + \frac{f_1^2}{[illegible]}, & \text{if } [illegible] \end{cases}$$

where E_n and Var_n refer to the marginal distribution of n and $\hat{\lambda}_0 = \frac{3}{4}\frac{f_1^3 f_3}{f_2^3}$. Assuming that $E_{\hat{\lambda}_0|n}(n + \hat{\lambda}_0)$ in the second term [2] of (6) can be estimated by $n + \hat{\lambda}_0$ we have that

$$Var_n\{E_{\hat{\lambda}_0|n}(n + \hat{\lambda}_0)\} = \widehat{Var_n}\{n + \hat{\lambda}_0\} = Var_n\{n\} = Np_0(1 - p_0). \qquad (7)$$

Since $\hat{p_0} = \frac{\hat{f}_0}{n+\hat{f}_0}$ and $N\widehat{(1 - p_0)} = n$, (7) can be estimated by

$$\widehat{Var_n}\{E_{\hat{\lambda}_0|n}(n + \hat{\lambda}_0)\} = \frac{\frac{3n}{4} f_1^3 f_3}{n f_2^3 + \frac{3}{4} f_1^3 f_3}. \qquad (8)$$

Now, consider the first term in (6), $E_n\{Var_{\hat{\lambda}_0|n}(n + \hat{\lambda}_0)\}$, and assume again that $E_n\{Var_{\hat{\lambda}_0|n}(n+\hat{\lambda}_0)\}$ can be estimated by $Var_{\hat{\lambda}_0|n}(n+\hat{\lambda}_0) = Var_{\hat{\lambda}_0|n}(\frac{3}{4}\frac{f_1^3 f_3}{f_2^3})$. Using the multivariate delta-method (see Bishop et al., 1975) we are able to achieve that

$$Var_{\hat{\lambda}_0|n} = \nabla g\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}^T Cov\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \nabla g\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \qquad (9)$$

where $g(f_1, f_2, f_3) = \frac{3}{4}\frac{f_1^3 f_3}{f_2^3}$ and $\nabla_i g(f_1, f_2, f_3) = \frac{\partial}{\partial f_i} g(f_1, f_2, f_3)$. It is easy to see that

$$\nabla g\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \frac{9}{4}\frac{f_1^2 f_3}{f_2^3} & -\frac{9}{4}\frac{f_1^3 f_3}{f_2^4} & \frac{3}{4}\frac{f_1^3}{f_2^3} \end{pmatrix}^T. \qquad (10)$$

Recall that the covariance matrix of the multinomial vector $(f_1, f_2, f_3)^T$ is estimated by

$$\hat{Cov}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_1(1 - \frac{f_1}{n}) & -\frac{f_1 f_2}{n} & -\frac{f_1 f_2}{} \end{pmatrix}$$

Hence (9) becomes ultimately

$$Var_{\hat{\lambda}_0|n}(\frac{3}{4}\frac{f_1^3 f_3}{f_2^3}) = (\frac{9}{4})^2 \frac{f_1^5 f_3^2}{f_2^6}\{\frac{f_1}{f_2} + 1\} + (\frac{3}{4})^2 \frac{f_1^6 f_3}{f_2^6}\{1 - \frac{f_3}{n}\}. \qquad (12)$$

Substituting (8) and (12) into (6), we finally have that

$$Var_{\hat{\lambda}_0|n}(n + \frac{3}{4}\frac{f_1^3 f_3}{f_2^3}) = (\frac{9}{4})^2 \frac{f_1^5 f_3^2}{f_2^6}\{ f$$ [illegible]

estimator under investigation. We do not only bootstrap from the observed sample of size n because the variance of estimating N arises from two sources, the random variation due to drawing n individuals from the target population of size N and the random variation from estimating the parameter of interest from the observed n units, as just mentioned in subsection 4.1 (see, for a review, van der Heijden et al., 2003a; Böhning, 2008).

2) Then, the resampled zero counts of individuals never identified are omitted. Then, using only the new sample of size n a new estimate $\hat{N}$ is calculated.

3) Step 1) and 2) are repeated B times. This provides $\hat{N}_1, \hat{N}_2, \hat{N}_3, \ldots, \hat{N}_B$.

4) The lower and upper bound of the 95% confidence interval are calculated from $P_{2.5}$ and $P_{97.5}$, the 2.5^{th} and 97.5^{th} percentile of the data set obtained in 3), respectively.

5) The standard error of estimate is now found from the sample $\hat{N}_1, \hat{N}_2, \hat{N}_3, \ldots, \hat{N}_B$.

5. Real Data Examples and Empirical Applications

There exist a variety of published studies applying the ideas of capture-recapture to the estimation of the total number (adjusted for hidden cases) of patients with infectious diseases such as tuberculosis, HIV/AIDS, legionnaires disease, or malaria (e.g. Gallay et al., 2000; Nardone et al., 2003; van Hest et al., 2008). However, most studies use frequency data from multiple sources with problems of matching and potentially different target areas. Here, we illustrate the use of our proposed estimator in particular data sets with repeated identifications from only one source which is the underlying assumption to apply the model in (1). To achieve a better judgment of the proposed estimator we include the following estimators in the comparison:

- Chao: $\hat{N}_{Chao} = n + f_1^2/(2f_2)$,
  $\widehat{Var(\hat{N}_{Chao})} = \frac{1}{4}\frac{f_1^4}{f_2^3} + \frac{f_1^3}{f_2^2} + \frac{1}{2}\frac{f_1^2}{f_2} - \frac{1}{4}\frac{f_1^4}{nf_2^2} - \frac{1}{2}\frac{f_1^4}{f_2(2nf_2+f_1^2)}$
- MLE: $\hat{N}_{MLE} = \frac{n}{1-\exp(-\hat{\lambda})}$ where $\hat{\lambda}$ is

the needle exchange from which we obtained these data; however

We use this frequency table as basis for all estimators considered as provided in Table 4.

Please insert Table 3-4 here

As can be seen in Table 4, the estimated number of heroin users from our method is between the estimators obtained using Chao's and Zelterman's methods. However, similar to the previous application, the proposed estimator shows larger variation.

5.2. Butterfly Data

The Malayan butterfly data go back to Fisher *et al.* (1943) (see Chao and Bunge, 2002) and have been frequently serving as test data for estimators under development. The frequency count of identifying distinct species is shown in

6. Simulation Study

6.1. Simulation Scenarios

A simulation experiment is undertaken to study the performance of the proposed estimator and some competitors such as maximum likelihood, Zelterman's and Chao's estimator. The scope of study covers a variety of situations of heterogeneity in the capture probabilities. Counts have been sampled from the following distributions: Negative Binomial(k, θ); $k = 2, 3, \theta = 0.4, 0.5, 0.6$ and $k = 4, 5, \theta$

of the above three scenarios, for example, for $N = 1,000$ and $NB(4, 0.7)$, we got $f_0 = 240$, $f_1 = 288$, $f_2 = 216$, $f_3 = 130$, $f_4 = 68$, $f_5 = 33$, $f_6 = 15$, $f_7 = 6$, $f_8 = 3$ and $f_9 = 1$. Then, f_0 was omitted and only the remaining zero-truncated frequencies $f_1, f_2, ..., f_m$ with $n = \sum_{j=1}^{m} f_j$ were used to estimate f_0 and N. The results for all estimators are shown in Table 7. It is clear that if heterogeneity becomes more pronounced our proposed estimator noticeably provides the most accurate results. However, these are the results from only one simulated sample. We now undertake a more profound simulation investigation.

Please insert Table 7 here

6.2.1. Heterogeneity in Identification

As a summary result it can be said that under a Negative-Binomial the MLE and Chao's estimator show a clear underestimation of population size, whereas Zelterman, the new estimator and the adjusted form tend to overestimate for a small population size, see Table 8. It also can be seen from Table 8– 9 that the proposed estimator and it's adjusted form perform similarly in cases of large population size. In addition, the adjusted form also significantly reduces the variance if compared with it's original form, in particular for small size. Furthermore, the proposed estimators show a good performance for estimating population size as does Chao's and Zelterman's estimator, in particular they provide smallest *RBias* and *RMSE* for the larger sample size. In summary, it is reasonable to state that the proposed estimators (in particular the adjusted versions) are suitable under the Negative Binomial distributional model.

Please insert Table 8-9 here

For the case that the identification probabilities arise from the Geometric distribution, the new estimator generally shows a good performance in terms of accuracy as it gives on average the smallest *RBias* in almost all cases, see Table 10. According to *RMSE*, although the new estimator seems to be of lack of precision for the small population size, it shows excellent performance against the other methods for larger size, see Table 11.

Please insert Table 10-11 here

standard error of the new estimator computing based 10,000 repeated simulation samples, whereas $Mean(\widehat{Se(\hat{N})})$ and $Mean(\widehat{Se(\hat{N})_{Bt}})$ represent an average estimated standard error from equation (13) and the bootstrap percentile method, respectively. It is seen from Table 14-16 that, $Se(\hat{N})$, $Mean(\widehat{Se(\hat{N})})$ and $Mean(\widehat{Se(\hat{N})_{Bt}})$ are quite similar in their values. $Mean(\widehat{Se(\hat{N})})$ is slightly smaller than $Se(\hat{N})$ and $Mean(\widehat{Se(\hat{N})_{Bt}})$. As a result, it is reasonable to state that the variance approximation of the new estimator in equation (13) can be utilized to represent the true variance.

Please insert Table 14-16 here

7. Conclusions and Discussion

A diversity of estimators in the capture-recapture field exists such as the estimators of Chao (1987) and Zelterman (1988), being widely applied in many areas of interest, especially in public health and social sciences. Here, we have introduced a new method of estimating the population size under a specific form of heterogeneity for the identification probability of distinct individuals. We have also been able to see how accurate and precise the method is performing when it is compared to other frequently used estimators. Overall, the proposed estimator is more accurate as well as providing small bias in the homogeneous Poisson case which asymptotically disappears. It is also found that the new estimator compares well with Chao's estimator since it's other methods under consideration provided the known underestimation phenomenon in almost all scenarios of heterogeneous identification probabilities. However, although the proposed estimator showed superior performance in terms of accuracy, it evidently gave also the largest variation. Hence, the new method has lack of precision; nonetheless, the variation of the new estimators considerably decreased for large population size (1,000 and more) which is typically the case for situations of interest in surveillance and public health. In addition, the adjusted forms of the new estimator can be utilized for sample sizes below $1,000$ which significantly reduces the variance. We also provided a formula of variance approximation of the new estimator. This variance formula is not only useful to determine the efficiency of estimating, but it can be also used to construct confidence intervals. In short, the new estimator can be an alternative form of population size estimation especially for large populations and heterogeneous capturing probabilities.

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Table 1:

Table 2: Estimated total number of Scottish drug injectors in 1997

Method	$\hat{N}$	$\widehat{Se(\hat{N})}$	95% CI (Approximate Normal)	$Se(\hat{N})_{BT}$	95% CI (Bootstrap Percentile)
MLE	648	0.67	645 - 649	1.00	646 - 649
Chao	828	34.85	759 - 896	36.91	763 - 907
New	975	137.99	704 - 1,245	150.94	788 - 1,379
NewAdj	975	-	-	103.76	779 - 1,169
NewMo	948	-	-	145.78	774 - 1,326
Zel	1,042	85.25	874 - 1,209	87.44	909 - 1,246

Table 3: The frequency count of times that heroin users contacted health treatment centers in Bangkok, Thailand in 2002; $n = 9,302$

j	1	2	3	4	5	6	7	8	9	10	11
f_j	2,176	1,600	1,278	976	748	570	455	368	281	254	188
j	12	13	14	15	16	17	18	19	20	21	
f_j	138	99	67	44	34	17	3	3	2	1	

Table 4: Estimated total number of heroin users in Bangkok, Thailand 2000

Method	$\hat{N}$	$\widehat{Se(\hat{N})}$	95% CI (Approximate Normal)	$Se(\hat{N})_{BT}$	95% CI (Bootstrap Percentile)
MLE	9,454	12.84	9,430 - 9,479	13.40	9,518 - 9,573
Chao	10,782	80.21	10,625 - 10,940	85.71	10,625 - 10,945
New	11,714	250.16	11,224 - 12,205	265.07	11,256 - 12,279
NewAdj	11,714	-	-	249.39	11,257 - 12,241
NewMo	11,701	-	-	255.71	11,250 - 12,216
Zel	12,078	184.54	11,717 - 12,440	188.45	11,728 - 12,476

Table 5: Malayan butterfly data (Fisher *et al.*, 1943)

j	1	2	3	4	5	6	7	8	9	10	11	12	13
f_j	118	74	44	24	29	22	20	19	20	15	12	14	6
j	14	15	16	17	18	19	20	21	22	23	24	24+	n
f_j	12	6	9	9	6	10	10	11	5	3	3	119	620

Table 6: Estimated total number of Malayan butterfly species

Method	$\hat{N}$	$\widehat{Se(\hat{N})}$	95% CI (Approximate Normal)	$Se(\hat{N})_{BT}$	95% CI (Bootstrap Percentile)
MLE	624	1.80	604 - 645	2.23	616 - 624
Chao	715	22.07	671 - 756	24.78	672 - 766
New	754	63.49	630 - 879	85.33	659 - 1,017
NewAdj	754	-	-	66.17	

Table 7: Estimated population size based upon one sample and four different estimators, true $N = 1,000$

Capture Probability(p_j)	MLE	Chao	New	NewAdj	NewMo	Zel
$NB(4, 0.6)$	921	973	994	994	989	1,005
$NB(4, 0.7)$	892	953	993	993	984	980
$NB(4, 0.8)$	860	919	987	987	970	935
$Geo(0.3)$	732	850	926	926	914	930
$Geo(0.4)$	675	800	899	899	883	859
$Geo(0.5)$	635	750	878	878	857	791
$0.5Poi(0.5) + 0.5Poi(1.0)$	923	948	993	993	967	953
$0.5Poi(0.5) + 0.5Poi(2.0)$	796	869	963	963	948	900
$0.5Poi(0.5) + 0.5Poi(3.0)$	743	842	974	974	959	914

Table 8: *RBias* of population size estimators for counts drawn from $NB(k, \theta)$

p_j	MLE	Chao	New	NewAdj	NewMo	Zel
			$N = 100$			
$NB(2, 0.4)$	-0.1261	-0.0349	0.1023	0.0324	-0.0113	0.0459
$NB(2, 0.5)$	-0.1685	-0.0591	0.1328	0.0433	-0.0167	0.0053
$NB(2, 0.6)$	-0.2058	-0.0892	0.1886	0.0560	-0.0344	-0.0417
$NB(3, 0.4)$	-0.0510	-0.0014	0.0817	0.0317	0.0031	0.0758
$NB(3, 0.5)$	-0.0851	-0.0154	0.0823	0.0389	-0.0027	0.0468
$NB(3, 0.6)$	-0.1203	-0.0335	0.1073	0.0559	-0.0084	0.0161
$NB(4, 0.6)$	-0.0716	-0.0092	0.0814	0.0463	0.0020	0.0396
$NB(4, 0.7)$	-0.1013	-0.0270	0.1157	0.0696	-0.0061	0.0094
$NB(4, 0.8)$	-0.1220	-0.0445	0.2471	0.1232	-0.0233	-0.0219
$NB(5, 0.6)$	-0.0426	0.0014	0.0612	0.0372	0.0034	0.0479
$NB(5, 0.7)$	-0.0700	-0.0108	0.0837	0.0569	-0.0009	0.0256
$NB(5, 0.8)$	-0.0917	-0.0219	0.1871	0.1132	0.0018	0.0034
			$N = 1,000$			
$NB(2, 0.4)$	-0.1321	-0.0519	-0.0106	-0.0106	-0.0180	0.0093
$NB(2, 0.5)$	-0.1776	-0.0815	-0.0167	-0.0167	-0.0268	-0.0310
$NB(2, 0.6)$	-0.2164	-0.1173	-0.0249	-0.0248	-0.0398	-0.0806
$NB(3, 0.4)$	-0.0554	-0.0147	0.0009	0.0013	-0.0034	0.0335
$NB(3, 0.5)$	-0.0911	-0.0301	-0.0026	-0.0020	-0.0084	0.0144
$NB(3, 0.6)$	-0.1269	-0.0515	-0.0032	-0.0024	-0.0117	-0.0136
$NB(4, 0.6)$	-0.0779	-0.0241	0.0014	0.0024	-0.0042	0.0100
$NB(4, 0.7)$	-0.1093	-0.0455	-0.0001	0.0019	-0.0091	-0.0184
$NB(4, 0.8)$	-0.1407	-0.0790	0.0014	0.0066	-0.0172	-0.0625
$NB(5, 0.6)$	-0.0486	-0.0116	0.0023	0.0033	-0.0017	0.0173
$NB(5, 0.7)$	-0.0770	-0.0264	0.0020	0.0038	-0.0045	-0.0006
$NB(5, 0.8)$	-0.1068	-0.0522	0.0037	0.0083	-0.0093	-0.0348

Table 8(con.): *RBias* of population size estimators for counts drawn from $NB(k, \theta)$

p_j	MLE	Chao	New	NewAdj	NewMo	Zel
			$N = 10,000$			
$NB(2, 0.4)$	-0.1327	-0.0531	-0.0169	-0.0169	-0.0177	0.0068
$NB(2, 0.5)$	-0.1777	-0.0832	-0.0269	-0.0269	-0.0278	-0.0343
$NB(2, 0.6)$	-0.2176	-0.1197	-0.0386	-0.0386	-0.0401	-0.0838
$NB(3, 0.4)$	-0.0559	-0.0159	-0.0035	-0.0035	-0.0039	0.0299
$NB(3, 0.5)$	-0.0915	-0.0310	-0.0070	-0.0070	-0.0076	0.0124
$NB(3, 0.6)$	-0.1279	-0.0537	-0.0125	-0.0125	-0.0133	-0.0173
$NB(4, 0.6)$	-0.0785	-0.0258	-0.0047	-0.0047	-0.0052	0.0069
$NB(4, 0.7)$	-0.1105	-0.0477	-0.0087	-0.0087	-0.0096	-0.0214
$NB(4, 0.8)$	-0.1420	-0.0815	-0.0147	-0.0147	-0.0165	-0.0656
$NB(5, 0.6)$	-0.0491	-0.0128	-0.0018	-0.0017	-0.0021	0.0145
$NB(5, 0.7)$	-0.0777	-0.0278	-0.0040	-0.0040	-0.0046	-0.0030
$NB(5, 0.8)$	-0.1080	-0.0544	-0.0081	-0.0081	-0.0094	-0.0377
			$N = 100,000$			
$NB(2, 0.4)$	-0.1327	-0.0533	-0.0177	-0.0177	-0.0178	0.0064
$NB(2, 0.5)$	-0.1778	-0.0833	-0.0277	-0.0277	-0.0278	-0.0346
$NB(2, 0.6)$	-0.2177	-0.1200	-0.0398	-0.0398	-0.0399	-0.0841
$NB(3, 0.4)$	-0.0560	-0.0160	-0.0040	-0.0040	-0.0040	0.0294
$NB(3, 0.5)$	-0.0915	-0.0312	-0.0077	-0.0077	-0.0077	0.0120
$NB(3, 0.6)$	-0.1280	-0.0540	-0.0134	-0.0134	-0.0135	-0.0176
$NB(4, 0.6)$	-0.0786	-0.0259	-0.0051	-0.0051	-0.0052	0.0067
$NB(4, 0.7)$	-0.1107	-0.0480	-0.0095	-0.0095	-0.0096	-0.0218
$NB(4, 0.8)$	-0.1423	-0.0818	-0.0162	-0.0162	-0.0163	-0.0659
$NB(5, 0.6)$	-0.0492	-0.0129	-0.0021	-0.0021	-0.0021	0.0143
$NB(5, 0.7)$	-0.0777	-0.0280	-0.0046	-0.0046	-0.0046	-0.0033
$NB(5, 0.8)$	-0.1080	-0.0546	-0.0090	-0.009:		

Table 9: *RMSE* of population size estimators for counts drawn from $NB(k, \theta)$

p_j	MLE	Chao	New	NewAdj	NewMo	Zel
			$N = 100$			
$NB(2, 0.4)$	0.017436	0.007103	0.322176	0.023931	0.055283	0.026119
$NB(2, 0.5)$	0.031113	0.014611	0.444714	0.049028	0.103335	0.027807
$NB(2, 0.6)$	0.047743	0.027054	0.738501	0.088966	0.169868	0.036023
$NB(3, 0.4)$	0.003222	0.002308	0.220680	0.009114	0.017575	0.026815
$NB(3, 0.5)$	0.008521	0.004579	0.152294	0.018016	0.031767	0.019818
$NB(3, 0.6)$	0.017133	0.010355	0.187029	0.041730	0.058681	0.021858
$NB(4, 0.6)$	0.006593	0.004972	0.131836	0.020873	0.032070	0.017851
$NB(4, 0.7)$	0.013899	0.011858	0.239724	0.052637	0.076174	0.022364
$NB(4, 0.8)$	0.026576	0.032748	1.084269	0.157403	0.242154	0.044130
$NB(5, 0.6)$	0.002662	0.002600	0.052894	0.010521	0.012551	0.014369
$NB(5, 0.7)$	0.007154	0.006949	0.108087	0.030457	0.037630	0.017049
$NB(5, 0.8)$	0.015763	0.021379	0.540063	0.107888	0.156400	0.032862
			$N = 1,000$			
$NB(2, 0.4)$	0.017599	0.003176	0.002812	0.002621	0.002759	0.001757
$NB(2, 0.5)$	0.031791	0.007479	0.005924	0.005529	0.005857	0.002936
$NB(2, 0.6)$	0.047332	0.015324	0.012544	0.011473	0.012379	0.009179
$NB(3, 0.4)$	0.003133	0.000381	0.000728	0.000671	0.000650	0.002206
$NB(3, 0.5)$	0.008418	0.001267	0.001744	0.001646	0.001649	0.001445
$NB(3, 0.6)$	0.016367	0.003402	0.004228	0.003998	0.004013	0.001853
$NB(4, 0.6)$	0.006209	0.000977	0.001817	0.001702	0.001683	0.001245
$NB(4, 0.7)$	0.012299	0.003003	0.005032	0.004666	0.004726	0.002101
$NB(4, 0.8)$	0.020837	0.008663	0.017714	0.015629	0.016264	0.007297
$NB(5, 0.6)$	0.002441	0.000346	0.000809	0.000753	0.000733	0.001098
$NB(5, 0.7)$	0.006148	0.001275	0.002629	0.002418	0.002449	0.001296
$NB(5, 0.8)$	0.012102	0.004395	0.010177	0.009095	0.009400	0.003764

Table9(con.): $RMSE$ of population size estimators for counts drawn from $NB(k, \theta)$

p_j	MLE	Chao	New	NewAdj	NewMo	Zel
			$N = 10,000$			
$NB(2, 0.4)$	0.017620	0.002869	0.000518	0.000518	0.000541	0.000205
$NB(2, 0.5)$	0.031610	0.006997	0.001197	0.001197	0.001246	0.001365
$NB(2, 0.6)$	0.047396	0.014489	0.002517	0.002517	0.002622	0.007285
$NB(3, 0.4)$	0.003133	0.000268	0.000073	0.000073	0.000076	0.000994
$NB(3, 0.5)$	0.008385	0.000999	0.000207	0.000207	0.000213	0.000278
$NB(3, 0.6)$	0.016390	0.002965	0.000543	0.000543	0.000561	0.000467
$NB(4, 0.6)$	0.006177	0.000703	0.000181	0.000181	0.000185	0.000160
$NB(4, 0.7)$	0.012236	0.002363	0.000524	0.000524	0.000537	0.000631
$NB(4, 0.8)$	0.020280	0.006889	0.001799	0.001795	0.001840	0.004636
$NB(5, 0.6)$	0.002419	0.000185	0.000075	0.000075	0.000076	0.000289
$NB(5, 0.7)$	0.006060	0.000830	0.000250	0.000249	0.000253	0.000133
$NB(5, 0.8)$	0.011729	0.003126	0.000952	0.000949	0.000967	0.001665
			$N = 100,000$			
$NB(2, 0.4)$	0.017610	0.002846	0.000336	0.000336	0.000338	0.000057
$NB(2, 0.5)$	0.031612	0.006950	0.000813	0.000813	0.000818	0.001213
$NB(2, 0.6)$	0.047401	0.014404	0.001682	0.001682	0.001694	0.007096
$NB(3, 0.4)$	0.003133	0.000268	0.000073	0.000073	0.000076	0.000994
$NB(3, 0.5)$	0.008385	0.000999	0.000207	0.000207	0.000213	0.000278
$NB(3, 0.6)$	0.016390	0.002965	0.000543	0.000543	0.000561	0.000467
$NB(4, 0.6)$	0.006176	0.000674	0.000042	0.000042	0.000042	0.000055
$NB(4, 0.7)$	0.012264	0.002312	0.000135	0.000135	0.000137	0.000493
$NB(4, 0.8)$	0.020251	0.006720	0.000414	0.000414	0.000420	0.004376
$NB(5, 0.6)$	0.002424	0.000170	0.000012	0.000012	0.000012	0.000212
$NB(5, 0.7)$	0.006047	0.000788	0.000045	0.000045	0.000045	0.000023
$NB(5, 0.8)$	0.011672	0.002996	0.000169	0.000169	0.000171	0.001459

Table 10: *RBias* of population size estimators for counts drawn from $Geo(\theta)$

p_j	MLE	Chao	New	NewAdj	NewMo	Zel
			$N = 100$			
$Geo(0.3)$	-0.2622	-0.1244	0.1640	-0.0172	-0.0677	-0.0231
$Geo(0.4)$	-0.3172	-0.1704	0.1695	-0.0331	-0.1032	-0.0970
$Geo(0.5)$	-0.3581	-0.2112	0.3865	-0.0365	-0.1242	-0.1575
$Geo(0.6)$	-0.3811	-0.2395	0.7796	-0.0182	-0.1585	-0.2017
			$N = 1,000$			
$Geo(0.3)$	-0.2692	-0.1475	-0.0610	-0.0642	-0.0739	-0.0663
$Geo(0.4)$	-0.3271	-0.1980	-0.0844	-0.0889	-0.1013	-0.1382
$Geo(0.5)$	-0.3710	-0.2467	-0.1021	-0.1096	-0.1256	-0.2047
$Geo(0.6)$	-0.4060	-0.2947	-0.1117	-0.1272	-0.1479	-0.2672
			$N = 10,000$			
$Geo(0.3)$	-0.2701	-0.1497	-0.0733	-0.0733	-0.0745	-0.0703
$Geo(0.4)$	-0.3277	-0.1996	-0.0984	-0.0984	-0.1000	-0.1409
$Geo(0.5)$	-0.3724	-0.2497	-0.1229	-0.1229	-0.1251	-0.2086
$Geo(0.6)$	-0.4078	-0.2993	-0.1457	-0.1457	-0.1491	-0.2728
			$N = 100,000$			
$Geo(0.3)$	-0.2702	-0.1500	-0.0749	-0.0749	-0.0750	-0.0709
$Geo(0.4)$	-0.3278	-0.2000	-0.1000	-0.1000	-0.1001	-0.1414
$Geo(0.5)$	-0.3725	-0.2500	-0.1249	-0.1249	-0.1251	-0.2090
$Geo(0.6)$	-0.4081	-0.3000	-0.1496	-0.1496	-0.1500	-0.2736

Table 12: $RBias$ of population size estimators for p_p

Table12(con.): *RBias* of population size estimators for

$p_j \quad 0.5Poi(\lambda) + 0.5Poi(\mu)$

λ	μ	MLE	Chao	New	NewAdj	NewMo	Zel
				$N = 10,000$			
0.5	2.0	-0.2068	-0.1329	-0.0400	-0.0400	-0.0415	-0.1021
0.5	3.0	-0.2580	-0.1572	-0.0213	-0.0217	-0.0229	-0.0846
0.5	4.0	-0.2810	-0.1529	0.0687	0.0066	0.0661	-0.0075
0.5	5.0	-0.2921	-0.1289	0.2346	0.0488	0.2296	0.1318
1.0	2.0	-0.0620	-0.0272	0.0004	0.0005	-0.0005	-0.0140
1.0	3.0	-0.1171	-0.0448	0.0099	0.0099	0.0090	-0.0028
1.0	4.0	-0.1477	-0.0457	0.0370	0.0370	0.0359	0.0396
1.0	5.0	-0.1642	-0.0368	0.0680	0.0680	0.0665	0.1030
				$N = 100,000$			
0.5	2.0	-0.2069	-0.1332	-0.0416	-0.0416	-0.0417	-0.1025
0.5	3.0	-0.2579	-0.1573	-0.0227	-0.0227	-0.0228	-0.0848
0.5	4.0	-0.2810	-0.1530	0.0667	0.0064	0.0664	-0.0078
0.5	5.0	-0.2921	-0.1293	0.2291	0.0480	0.2286	0.1306
1.0	2.0	-0.0621	-0.0273	-0.0002	-0.0002	-0.0003	-0.0142
1.0	3.0	-0.1172	-0.0449	0.0094	0.0094	0.0093	-0.0029
1.0	4.0	-0.1478	-0.0459	0.0359	0.0359	0.0358	0.0392
1.0	5.0	-0.1642	-0.0370	0.0668	0.0668	0.0666	0.1027

Table 13: *RMSE* of population size estimators for $p_j \quad 0.5Poi(\lambda) + 0.5Poi(\mu)$

λ	μ	MLE	Chao	New	NewAdj	NewMo	Zel
				$N = 100$			
0.5	2.0	0.0436	0.0294	1.2641	0.0848	0.2344	0.0364
0.5	3.0	0.0656	0.0310	1.4570	0.0573	0.2156	0.0397
0.5	4.0	0.0774	0.0302	9.0444	0.0656	0.4079	0.0806
0.5	5.0	0.0839	0.0374	375.7771	0.1352	2.1824	0.3013
1.0	2.0	0.0077	0.0136	0.2933	0.0367	0.1752	0.0957
1.0	3.0	0.0143	0.0100	0.3123	0.0233	0.1694	0.0829
1.0	4.0	0.0217	0.0095	0.6376	0.0222	0.2736	0.1088
1.0	5.0	0.0266	0.0115	1.8505	0.0297	0.5621	0.1796
				$N = 1,000$			
0.5	2.0	0.0429	0.0186	0.0135	0.0121	0.0134	0.0124
0.5	3.0	0.0665	0.0250	0.0123	0.0073	0.0114	0.0090
0.5	4.0	0.0788	0.0235	0.0362	0.0046	0.0275	0.0045
0.5	5.0	0.0850	0.0171	0.1565	0.0086	0.1134	0.0299
1.0	2.0	0.0042	0.0017	0.0057	0.0045	0.0055	0.0052
1.0	3.0	0.0138	0.0026	0.0054	0.0032	0.0051	0.0047
1.0	4.0	0.0219	0.0026	0.0088	0.0026	0.0082	0.0072
1.0	5.0	0.0269	0.0020	0.0165	0.0034	0.0150	0.0129

Table13(con.): *RMSE* of population size estimators for

$p_j \quad 0.5Poi(\lambda) + 0.5Poi(\mu)$

λ	μ	MLE	Chao	New	NewAdj	NewMo	Zel
				$N = 10,000$			
0.5	2.0	0.0428	0.0178	0.0027	0.0027	0.0028	0.0107
0.5	3.0	0.0666	0.0248	0.0015	0.0015	0.0016	0.0074
0.5	4.0	0.0790	0.0235	0.0068	0.0004	0.0064	0.0005
0.5	5.0	0.0853	0.0168	0.0604	0.0029	0.0579	0.0183
1.0	2.0	0.0039	0.0009	0.0005	0.0005	0.0005	0.0005
1.0	3.0	0.0137	0.0021	0.0005	0.0005	0.0005	0.0005
1.0	4.0	0.0218	0.0022	0.0019	0.0009	0.0019	0.0018
1.0	5.0	0.0270	0.0014	0.0055	0.0016	0.0054	0.0052
				$N = 100,000$			
0.5	2.0	0.0428	0.0177	0.0018	0.0018	0.0019	0.0105
0.5	3.0	0.0665	0.0248	0.0006	0.0006	0.0006	0.0072
0.5	4.0	0.0790	0.0234	0.0047	0.0001	0.0046	0.0001
0.5	5.0	0.0853	0.0167	0.0530	0.0024	0.0528	0.0172
1.0	2.0	0.0039	0.0008	0.0001	0.0001	0.0001	0.0001
1.0	3.0	0.0137	0.0020	0.0001	0.0001	0.0001	0.0001
1.0	4.0	0.0218	0.0021	0.0013	0.0008	0.0013	0.0013
1.0	5.0	0.0270	0.0014	0.0045	0.0015	0.0045	0.0045

Table 14: Estimated standard error of estimating population size of the proposed estimator for p_j $NB(k, \theta)$

Table 15: Estimated standard error of estimating population size of the proposed estimator for p_j $Geo(\theta)$

p_j	$Se(\hat{N})$	Mean $\widehat{Se(\hat{N})}$	Mean $\widehat{Se(\hat{N})}_{Bt}$	$Se(\hat{N})$	Mean $\widehat{Se(\hat{N})}$	Mean $\widehat{Se(\hat{N})}_{Bt}$
	$N = 1,000$			$N = 10,000$		
$Geo(0.3)$	85.50	82.66	91.82	250.67	243.54	248.26
$Geo(0.4)$	110.63	103.79	115.74	310.10	319.81	324.46
$Geo(0.5)$	154.84	141.98	160.69	420.39	422.33	431.44
$Geo(0.6)$	201.55	192.32	225.04	576.47	567.02	576.62