School of Mathematics, Meteorology
and Physics

Department of Mathematics

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A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

by

Valerio Lucarini and Stefania Sarno

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v.lucarini@reading.ac.uk

Abstract

The climate belongs to the class of non-equilibrium forced and dissipative systems, for which most results of quasi-equilibrium statistical mechanics, including the fluctuation-dissipation theorem, do not apply. In this paper we show for the first time how the Ruelle linear response theory, developed for studying rigorously the impact of perturbations on general observables of non-equilibrium statistical mechanical systems, can be applied with great success to analyze the climatic response to general forcings. The crucial value of the Ruelle theory lies in the fact that it allows to compute the response of the system in terms of expectation values of explicit and computable functions of the phase space averaged over the invariant measure of the unperturbed state. We choose as test bed a classical version of the Lorenz 96 model, which, in spite of its simplicity, has a well-recognized prototypical value as it is a spatially extended one-dimensional

Con en

1 Introduction

[illegible] fi [illegible] fi [illegible] fi [illegible] CO_2 [illegible] ffi [illegible] fl [illegible]

$$\langle\Phi\rangle^{(1)}(\omega) = \int_{-\infty}^{+\infty} d\omega_1 \chi_\Phi^{(1)}(\omega_1) f(\omega_1) \times \delta(\omega - \omega_1) = \chi_\Phi^{(1)}(\omega) f(\omega);$$

$$\chi_\Phi^{(1)}(\omega) = \int_{-\infty}^{+\infty} dt G_\Phi^{(1)}(t) \exp(i\omega t).$$

$f_\alpha(t)$ $X(x)$ $\langle\Phi_\alpha\rangle^{(1)}(t)$ $\chi_\Phi^{(1)}(\omega)$ $f_\beta(t)$ $f_\beta(\omega)$ $f_\beta(\omega)$ $\chi_{\bar\Phi}^{(1)}(\omega)$ $\langle\Phi_\beta\rangle^{(1)}(\omega)$ Φ $X(x)$ $\langle\Phi_\beta\rangle^{(1)}(t)$ $G_\Phi^{(1)}(t)$ $f(t) = \delta(t)$ $\langle\Phi\rangle^{(1)}(t) = G_\Phi^{(1)}(t)$

2.2 Kramer -Kronig rela ion and

$G_\Phi^{(1)}(t) \in L^2$ $\chi_\Phi^{(1)}(\omega)$ ω $\chi(\omega)$

$$\int_{-\infty}^{+\infty} dt\,\Theta(t)\, t^k e^{i\omega t} = (-i)^k \frac{d^k}{d\omega^k}\left(P\frac{1}{\omega} - i\pi\delta(\omega)\right) \approx k!\, \frac{i^{(k+1)}}{\omega^{(k+1)}}$$

$t \to 0^+$

$$G^{(1)}_\Phi(t) \approx \bar{\alpha}\,\Theta(t)\, t^\beta + o(t^\beta)$$

$\omega \to \infty$

$$\chi^{(1)}_\Phi(\omega) \approx \alpha\, \omega^{-\beta-1} + o(\omega^{-\beta-1})$$

$\alpha = \bar{\alpha} i^{(\beta+1)} \beta!$ β $\bar{\alpha}$ Φ

ω $\chi^{(1)}_\Phi(\omega) = \chi^{(1)}_\Phi(-\omega)^*$ χ χ ω $\alpha = \alpha_R$ β $\alpha = \alpha_I$ β

$\chi^{(1)}_\Phi(\omega) = \chi^{(1)}_\Phi(-\omega)^*$

$$-\frac{\pi}{2}\omega^{2p-1}\,\Im\chi^{(1)}_\Phi(\omega) = P\int_0^\infty d\nu\, \frac{\nu^{2p}\,\Re\chi^{(1)}_\Phi(\nu)}{\nu^2 - \omega^2}$$

$$\frac{\pi}{2}\omega^{2p}\,\Re\chi^{(1)}_\Phi(\omega) = P\int_0^\infty d\nu\, \frac{\nu^{2p+1}\,\Im\chi^{(1)}_\Phi(\nu)}{\nu^2 - \omega^2}$$

P $p = 0, \ldots, \beta - 1$ β

p $;:::;\beta=$ β

p $! \rightarrow$

$$\chi^{(1)} \quad \frac{}{\pi}P\int d\nu \frac{\chi^{(1)} \; \nu}{\nu} ;$$

$\chi^{(1)}$

$\chi^{(1)}$

$! \rightarrow$

$!$

$! \sim$

c_1 $c_2 ! :^2$ o $!^3$ c_1 c_2

c_1

$! \rightarrow \infty$

2.3 A prac ical formDla for he linear D cep ibili and con- i enc rela ion be een D cep ibili ie of differen ob- er able

fi $f(t)$ fi $f(t)$ $f(t)$ ϵ ωt $\Phi(x)$ fi

$$\delta\Phi_\epsilon(t;t_0;x_0) \quad \Phi_\epsilon(t;t_0;x_0) - \Phi_0(t;t_0;x_0)$$

x_0 t_0 ϵ $\delta\Phi_\epsilon(t;t_0;x_0)$ $O(\epsilon)$

$$\chi_\Phi^{(1)}(\omega) \equiv {}_{\epsilon\to 0\; T\to\infty} \chi_\phi^{(1)}(\omega;x_0;\epsilon;T)$$

$$\chi_\Phi^{(1)}(\omega;x_0;\epsilon;T) \quad \frac{}{T}\frac{}{\epsilon}\int_0^T dt\,\delta\Phi_\epsilon(t;x_0) \quad \omega t$$

ffi fi fi fi

$\delta\Phi_\epsilon(t;x_0)\; dt \quad \delta\, d\Phi_\epsilon(t;x_0)\; dt$

$$\chi_{\dot\Phi}^{(1)}(\omega) \quad -\omega\,\chi_\Phi^{(1)}(\omega) :$$

fi ffi $\chi_{\dot\Phi}^{(1)}(\omega)$ Φ

$$\Phi(x) \quad \Gamma(x) ;$$

$\Gamma \quad F\cdot\nabla\Phi$ Φ

$x \quad F \quad x \quad X \quad x \; f \; t \quad F \quad x \quad \epsilon \quad ! \; t \; X \quad x$

$\Phi \; x$

3 Applica ion of he Re pon e Theor o he Loren 96 Model

3.1 S a i ical proper ie of he Dnper Drbed Loren 96 Model

ab-initio

$$\frac{d x_i}{d t} = x_{i-1}\,(x_{i+1} - x_{i-2}) - x_i + F$$

i ; ; :::::; N i x_{i+N} x_{i-N} x_i

$i \to i$

F N

N F F

N F

6

fi

$E = \sum_i x_i^2$ E

E —

3.2 A mp o ic proper ie of he linear D cep ibili

$$\frac{d x_i}{d t} \quad x_{i-1} \; x_{i+1} - x_{i+2} \; - x_i \quad F \quad \epsilon \quad ! t \; X_i$$

$X_i \quad X_i \; x_1; : : : ; x_N \quad x_i$

3.2.1 Global perturbation

X_i $\forall i$ E $G^{(1)}_{E,a}(t)$

$$
\begin{aligned}
G^{(1)}_{E,N}(t) &= \int \rho_0(x)\,\Theta(t)\,\Lambda\Pi_0(t)\,E(x) \\
&= \int \rho_0(x)\,\Theta(t)\,\tilde{\ }\cdot\nabla E(x(t)) \\
&= \int \rho_0(x)\,\Theta(t)\sum_i \partial_i E(x(t)))
\end{aligned}
$$

$x(t)$ $E(x(t))$ t t ρ_0 E

$$
\begin{aligned}
G^{(1)}_{E,N}(t) &= \int \rho_0(x)\,\Theta(t)\sum_i \partial_i\Big(E|_{t=0} + tE|_{t=0} + o(t)\Big) \\
&= \int \rho_0(x)\,\Theta(t)\Big[\sum_i x_i - \Big(\sum_i x_i - NF\Big)t + o(t)\Big].
\end{aligned}
$$

$$
\begin{aligned}
\chi^{(1)}_{E,N}(\omega) &= i\Big(\sum_i \langle x_i\rangle_0\Big)/\omega - \Big(\sum_i \langle x_i\rangle_0 - NF\Big)/\omega^2 + o(\omega^{-2}) \\
&= iN\frac{m}{\omega} - N\frac{F-m}{\omega^2} + o(\omega^{-2})
\end{aligned}
$$

i

$'' = x_i^2$ x_i

fi $\chi^{(1)}_{\varepsilon,N} = N\chi^{(1)}_{E,N}$ $\chi^{(1)}_{\varepsilon,N}$

$! \to \infty$

$$\chi^{(1)}_{\varepsilon,N}\ ! \quad i\,\frac{m}{!} - \frac{F - \ m}{!^{2}} \quad o\ !^{-2}:$$

π=

χ

$$\chi^{(1)}_{E,N} \;! \qquad \frac{F}{-\;!}$$

$$\int_0^\infty \chi^{(1)}_{E,1}(\omega)\, \omega \; \frac{\pi}{\;}.$$

$\chi^{(1)}_{M,\epsilon,1}$

$$\chi^{(1)}_{M,1}(\omega) \quad \frac{\;}{\omega} \quad \frac{\;}{\omega^2};$$

$$\int_0^\infty \chi^{(1)}_{M,1}(\omega)\, \omega \; \frac{\pi}{\;};$$

E_j x_j

$$G^{(1)}_{E_j,1}(t) \quad \int \rho_0\, x\, \Theta(t)\, \partial_j \left[-x_1^2 \quad \left(x_j \;\; x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F \right) t \quad o(t) \right]$$
$$\Theta(t) \left[\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F \rangle_0\, t \quad o(t) \right].$$

$$\langle x_j \rangle_0 \quad \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F \rangle_0$$

ρ_0

$$\chi^{(1)}_{E_j,1} \quad i \frac{m}{\omega} \quad \frac{m}{\omega^2} \quad o(\omega^{-2}).$$

$\chi^{(1)}_{E,1}$ $\sum_k \chi^{(1)}_{E_k,1}$

E_j E

$\chi^{(1)}_{E_j,1} \approx \chi^{(1)}_{E,1}$

$$\int_0^\infty \chi^{(1)}_{E_j,1}(\omega)\, \omega \quad \frac{\pi}{m}$$

$\propto \omega^{-2}$

ω^{-2}

$\chi^{(1)}_{E,1}$ $\chi^{(1)}_{E_j,1}$

x_j

$= E_{j+1} \quad E_{j+2} \quad E_{j-1}$

$$\chi^{(1)}_{\psi,1}(\omega) \quad -\frac{\langle x_{j-1} \; x_{j+1} - x_{j-2} \rangle_0}{\omega^2} \quad -\frac{F - m}{\omega^2} \quad o(\omega^{-2}):$$

$\chi^{(1)}_{\psi,1}$ $\chi^{(1)}_{E_j,1}$

$\chi^{(1)}_{E,1}$

$\chi^{(1)}_{E,1}$ $\chi^{(1)}_{E_{loc},1}$ E_{loc} E_j $= x_j^2$ $= x_{j+1}^2$ $= x_{j+2}^2$ $= x_{j-1}^2$

x_j

$$\chi^{(1)}_{x_j,1}(\omega) \quad \overline{\omega} \quad \overline{\omega^2}$$

4 Re Dl

4.1 SimDla ion and Da a Proce ing

$$\chi^{(1)}_{\Phi}\;! \sim_{K\to\infty} \overline{K} \sum_{i=1}^{K} \chi^{(1)}_{\Phi}\;!; x_i; \epsilon; T \;;$$

F [illegible]

[illegible] Φ [illegible] $!$ [illegible]

[illegible] K [illegible]

$!$ [illegible] $!_l$: π $!_h$ π [illegible] : π [illegible]

[illegible] $!$ [illegible] t [illegible] t T [illegible] $\pi = !$

[illegible]

$\epsilon \approx$ [illegible]

[illegible]

[illegible] ϵ : [illegible] K [illegible]

[illegible] ϵ [illegible] K

[illegible] f t [illegible]

[illegible]

K [illegible] T [illegible]

4.2 Global Per Drba ion

fi $\chi^{(1)}_{\varepsilon,N}$ $=N\chi^{(1)}_{E,N}$ " $E=N$

$! \approx$ $! \approx$

$!$

$\sim m = !$

$! \approx$

$\sim - F - m = !^2$

$\chi^{(1)}_{\mu,N}$ $=N\chi^{(1)}_{M,N}$ μ $M=N$

$! \approx$

$\chi^{(1)}_{\mu,N}$ … $\lesssim$ … ω_l

… $\chi^{(1)}_{\varepsilon,N}$

$$\chi^{(1)}_{\varepsilon,N}(\omega) = \begin{cases} \frac{\omega}{\omega_l}\,\chi^{(1)}_{\varepsilon,N}(\omega_l); & \le \omega \le \omega_l; \\ \chi^{(1)}_{\varepsilon,N}(\omega); & \omega_l \le \omega \le \omega_h; \\ \frac{m}{\omega}; & \omega \ge \omega_h; \end{cases}$$

… $\omega \to$ … $\chi^{(1)}_{\varepsilon,N}$ …

$$\chi^{(1)}_{\varepsilon,N}(\omega) = \begin{cases} \chi^{(1)}_{\varepsilon,N}(\omega_l); & \le \omega \le \omega_l; \\ \chi^{(1)}_{\varepsilon,N}(\omega); & \omega_l \le \omega \le \omega_h; \\ -\frac{F-2m}{\omega^2}; & \omega \ge \omega_h; \end{cases}$$

… ω … $\chi^{(1)}_{\mu,N}(\omega)$ …

$$\frac{[illegible]\, m\, F}{[illegible]\, F} \quad \lambda\gamma F^{\gamma-1}; \qquad [illegible]$$

$$\frac{[illegible]\, e\, F}{[illegible]\, F} \quad \lambda \frac{\gamma}{[illegible]} F^{\gamma} \quad m\, F \frac{\gamma}{[illegible]}: \qquad [illegible]$$

4.4 F r her implica ion of Kramer -Kronig rela ion and m r le

ff

ff

ff

fi

$\mathsf{X}^{(1)}_{x_j,1}$ $\mathsf{X}^{(1)}_{M,1}$

$\mathsf{X}^{(1)}_{E_j,1}$ $\mathsf{X}^{(1)}_{E,1}$

E_{loc}

5 Prac ical Implica ion for Clima e Change S Dd- ie

$$\langle T_S \rangle^{(1)} \; t \quad \int_{-\infty}^{+\infty} \sigma_1 G_{T_S}^{(1)} \; \sigma_1 \; f \;\; t - \sigma_1 \; :$$

t

6 Summary, Discussion and Conclusion

distance

Hasselmann programme

closure theory

[illegible]

spectroscopy of the climate system

CO_2

CO_2

δt

[illegible]

[illegible] CO_2 [illegible]

http://www-pcmdi.llnl.gov/
climateprediction.net http://cmip-pcmdi.llnl.gov/cmip5/

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Reference

ffi 80

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fl 29

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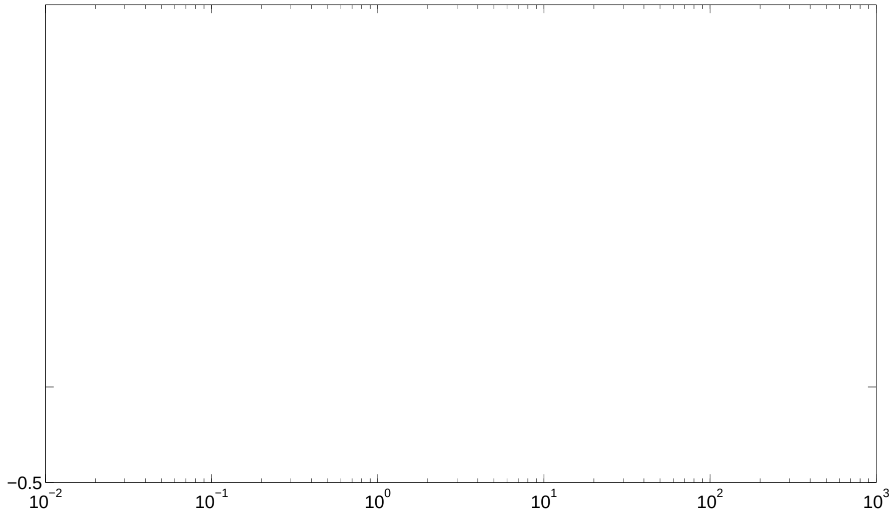
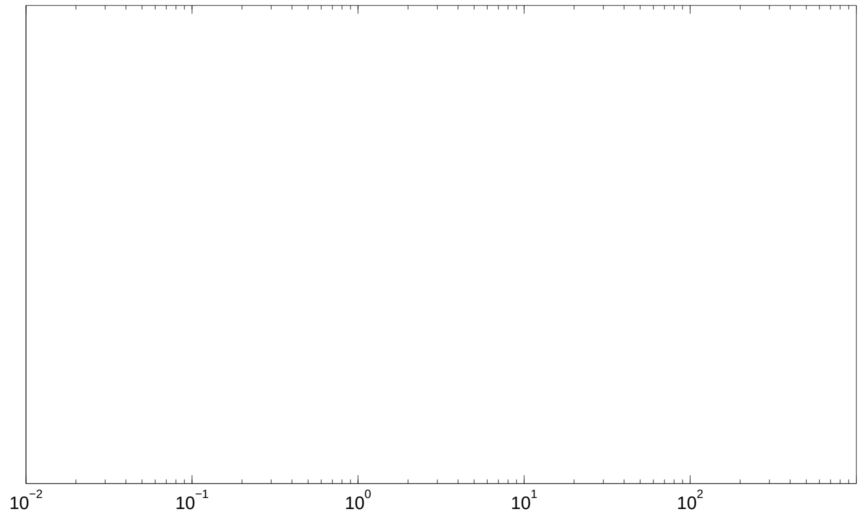
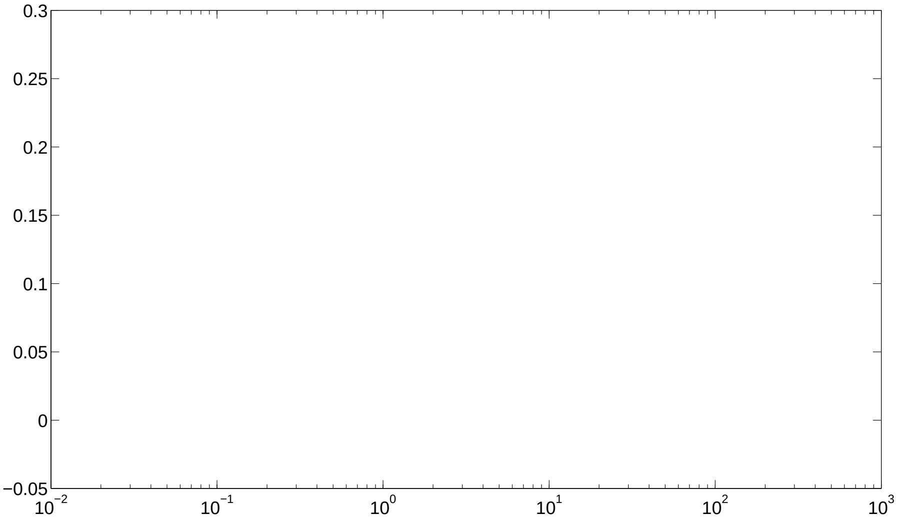
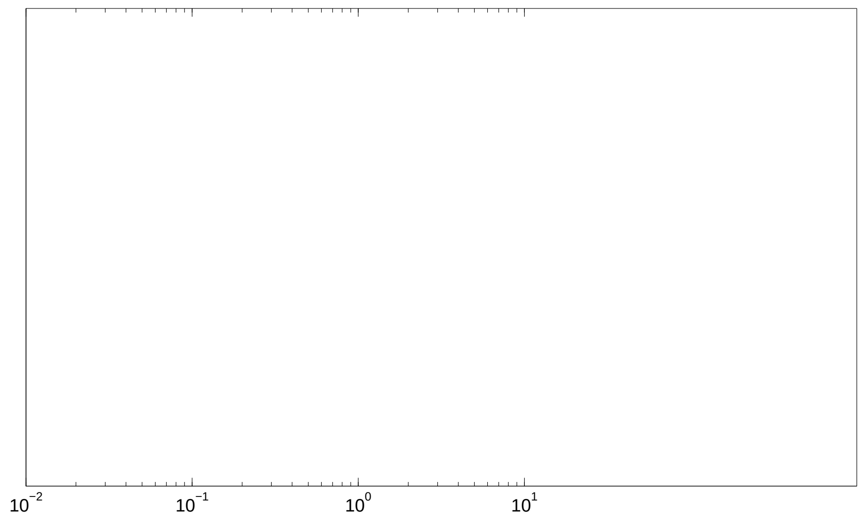


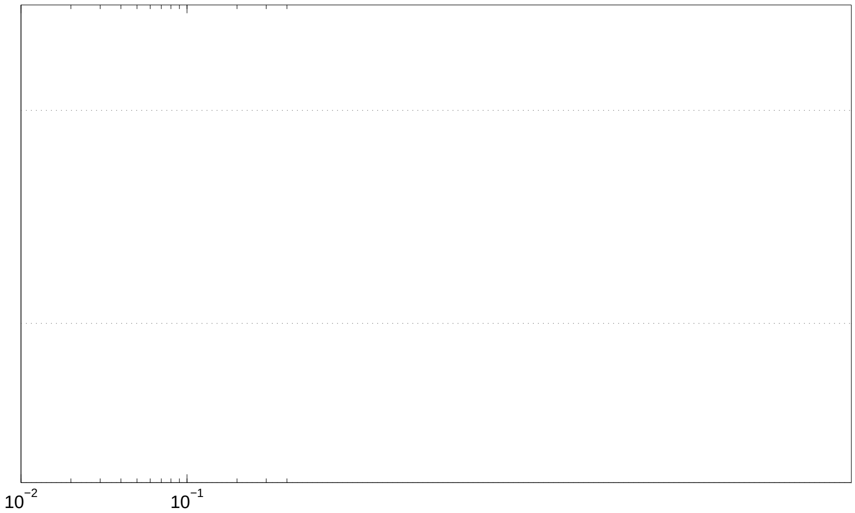


$E = N$ X_i $\forall i$



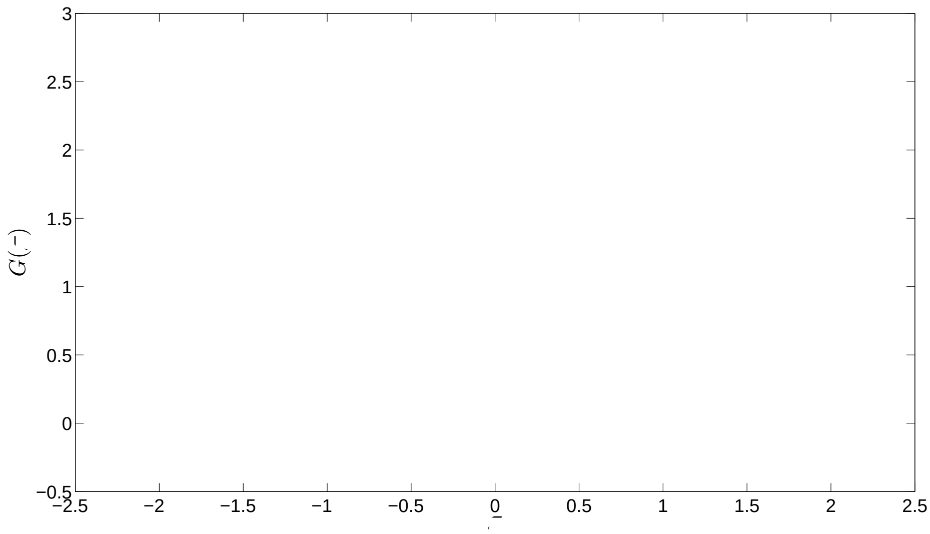


$M = N$ X_i $\forall i$



$E = N$ $M = N$

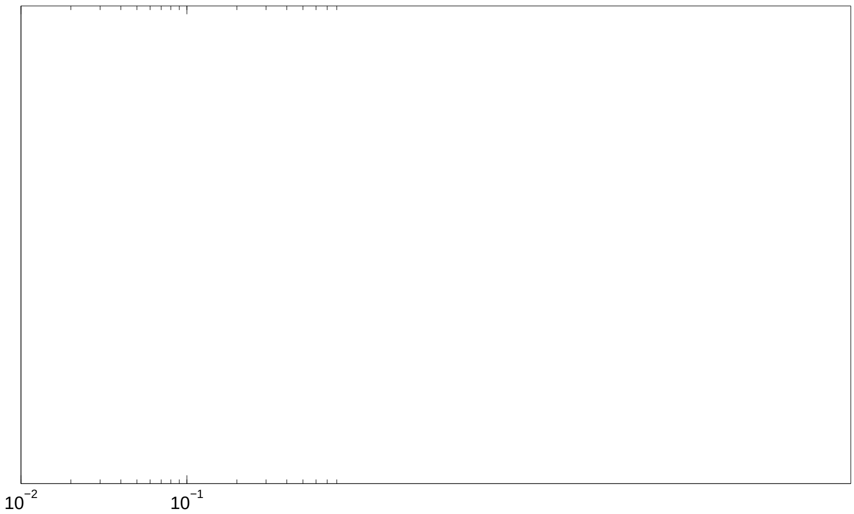
X_i $\forall i$ fi

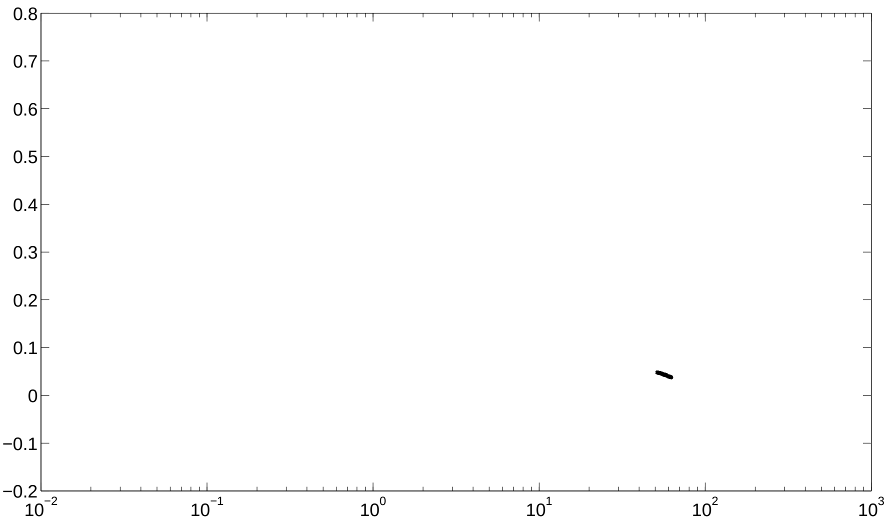


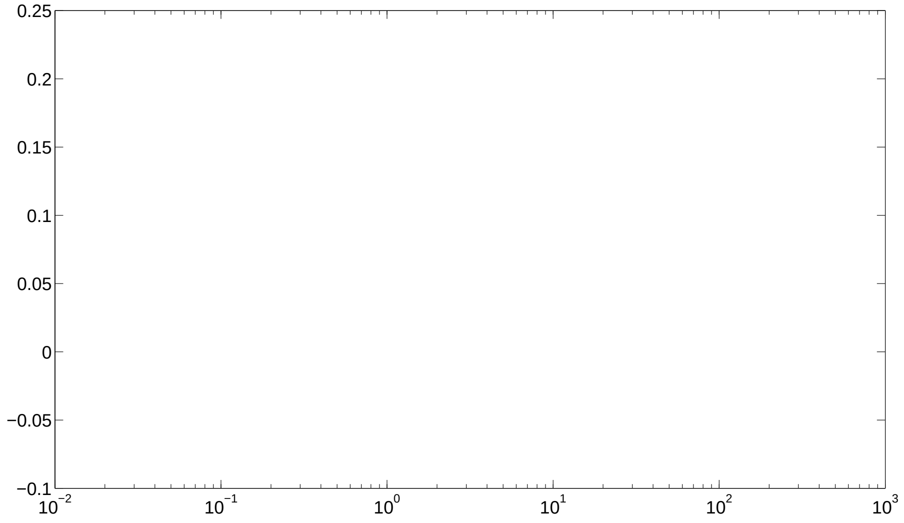
$E = N$ $M = N$

X_i $\forall i$

fi


10^{-2}
10^{-1}


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}


0.25
0.2
0.15
0.1
0.05
0
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−0.1
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}