

Department of Mathematics

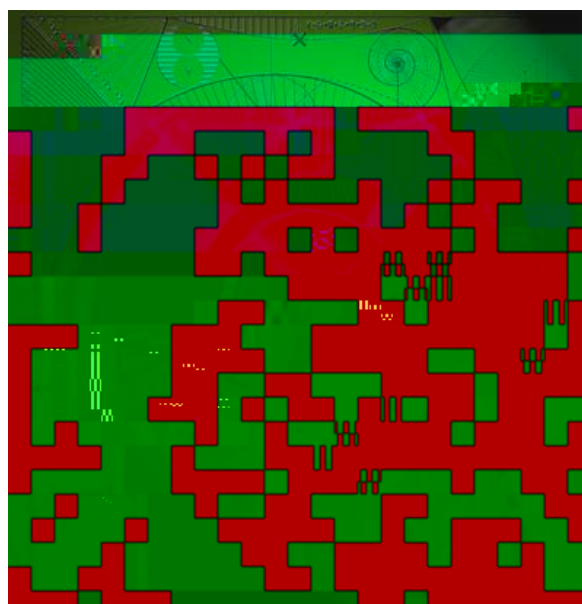
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A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

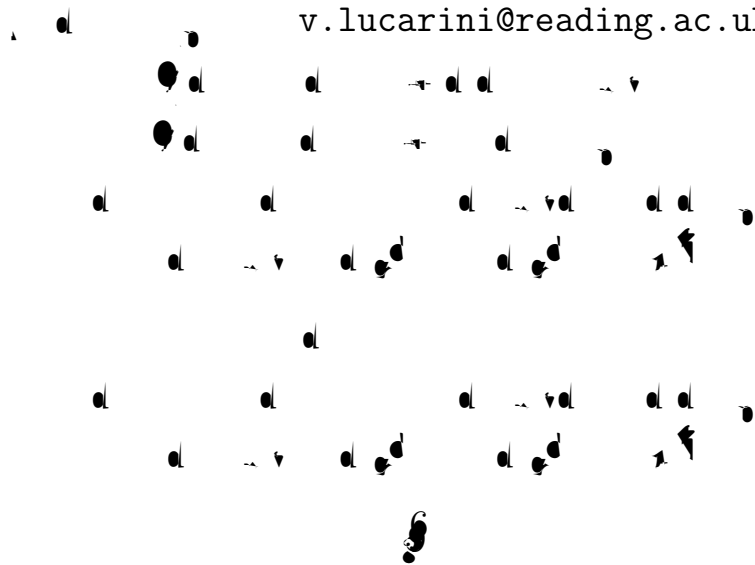
by

Valerio Lucarini and Stefania Sarno



A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

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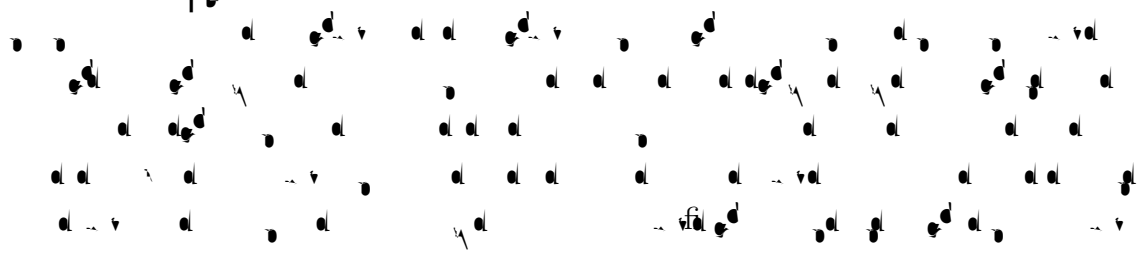
Abstract

The climate belongs to the class of non-equilibrium forced and dissipative systems, for which most results of quasi-equilibrium statistical mechanics, including the fluctuation-dissipation theorem, do not apply. In this paper we show for the first time how the Ruelle linear response theory, developed for studying rigorously the impact of perturbations on general observables of non-equilibrium statistical mechanical systems, can be applied with great success to analyze the climatic response to general forcings. The crucial value of the Ruelle theory lies in the fact that it allows to compute the response of the system in terms of expectation values of explicit and computable functions of the phase space averaged over the invariant measure of the unperturbed state. We choose as test bed a classical version of the Lorenz 96 model, which, in spite of its simplicity, has a well-recognized prototypical value as it is a spatially extended one-dimensional

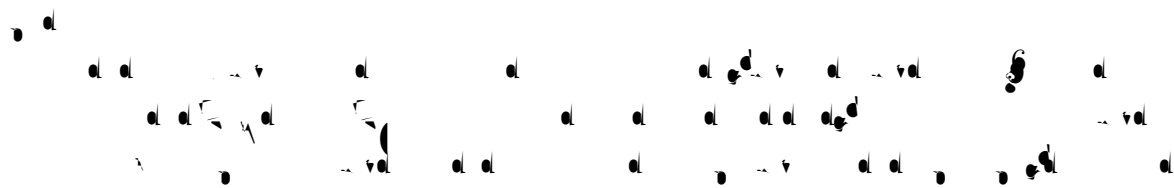
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1 Introduction







fi

66

fi

CO₂

ff

fi

$$\langle \Phi \rangle^{(1)} = \int_{-\infty}^{+\infty} \chi_{\Phi}^{(1)}(f) \times \delta(f - \omega) \chi_{\Phi}^{(1)}(f) ;$$

$$\chi_{\Phi}^{(1)} = \int_{-\infty}^{+\infty} t G_{\Phi}^{(1)}(t) dt :$$

$$\begin{aligned} & \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)}(t) f_{\alpha}(t) \\ & \chi_{\Phi}^{(1)} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} \\ & f_{\beta} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} \langle \Phi_{\beta} \rangle^{(1)} \\ & G_{\Phi}^{(1)}(t) = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} f_{\alpha}(t) \delta(t) \\ & \langle \Phi \rangle^{(1)} = \sum_{\alpha} \chi_{\Phi_{\alpha}}^{(1)} G_{\Phi}^{(1)}(t) \end{aligned}$$

2.2 Kramer-Kronig relation and Dm rDle

$$\begin{aligned} & G_{\Phi}^{(1)}(t) \in L^2 \\ & \chi_{\Phi}^{(1)} = \chi \end{aligned}$$

$$G_{\Phi}^{(1)}(t) = \int_0^t \Theta(t-s) \left(P \frac{d}{ds} - \pi \delta \right) \chi_{\Phi}^{(1)}(s) ds$$

$$\int_{-\infty}^{+\infty} t \Theta(t-k) \, i! \, t^{-i} = -i \frac{k^{i-k}}{(i-k)!} \left(P \frac{d}{dt} - \pi \delta \right) \approx k \frac{(k+1)}{(k+1)!}$$

$$\lim_{t \rightarrow +\infty} \frac{\chi_{\Phi}^{(1)}(t)}{t^{\beta}} = \alpha$$

$$G_{\Phi}^{(1)}(t) \approx \alpha \Theta(t) t^{\beta} \quad o(t^{\beta})$$

$$\lim_{t \rightarrow \infty} \frac{\chi_{\Phi}^{(1)}(t)}{t^{\beta}} = \alpha$$

$$\chi_{\Phi}^{(1)}(t) \approx \alpha t^{-\beta-1} \quad o(t^{-\beta-1})$$

$$\alpha = \alpha i^{(\beta+1)} \beta \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta}$$

$$\chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta} \quad \chi_{\Phi}^{(1)}(t) = \alpha \Theta(t) t^{\beta}$$

$$-\frac{\pi}{2} i^{2p-1} \chi_{\Phi}^{(1)}(t) = P \int_0^{\infty} v \frac{v^{2p} \chi_{\Phi}^{(1)}(v)}{v^2 - t^2} dv$$

$$\frac{\pi}{2} i^{2p} \chi_{\Phi}^{(1)}(t) = P \int_0^{\infty} v \frac{v^{2p+1} \chi_{\Phi}^{(1)}(v)}{v^2 - t^2} dv$$

$$P \int_0^{\infty} v \frac{v^{2p} \chi_{\Phi}^{(1)}(v)}{v^2 - t^2} dv = \beta$$

p ;:::; $\beta=$ β

fi

fi

p

$!$

$$\chi^{(1)} \left(\frac{-P}{\pi} \int \epsilon v \frac{\chi^{(1)} v}{v} \right);$$

$\chi^{(1)}$

fi

$!$

$!$

$c_1 \quad c_2 !.^2 \quad o !^3$

c_1

$!\rightarrow \infty$

2.3 A practical formula for the linear dependence and consistency relation between dependence of different observable

$$\begin{aligned} & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t_0) + \int_{t_0}^t f'(t) dt \\ & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t_0) + \int_{t_0}^t f'(t) dt \end{aligned}$$

$$\begin{aligned} & \delta \Phi_\epsilon(t; t_0; x_0) = \Phi_\epsilon(t; t_0; x_0) - \Phi_0(t; t_0; x_0) \\ & \delta \Phi_\epsilon(t; t_0; x_0) = O(\epsilon) \end{aligned}$$

$$\chi_\Phi^{(1)} \equiv \lim_{\epsilon \rightarrow 0, T \rightarrow \infty} \chi_\phi^{(1)}; x_0; \epsilon; T$$

$$\begin{aligned} & \chi_\Phi^{(1)}; x_0; \epsilon; T = \frac{1}{T} \int_0^T \delta \Phi_\epsilon(t; x_0) dt \\ & \delta \Phi_\epsilon(t; x_0) = \Phi_\epsilon(t; x_0) - \Phi_0(t; x_0) \end{aligned}$$

$$\begin{aligned} & \chi_\Phi^{(1)} - \chi_\Phi^{(1)} : \\ & \chi_\Phi^{(1)} = \Phi \end{aligned}$$

$$\begin{aligned} & \Phi(x) = \Gamma(x); \\ & \Gamma = F \cdot \nabla \Phi \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + v \cdot \nabla_x \right) F(x) - X(x) f(t) = F(x) \epsilon, \quad !t X(x) \\ & \Phi(x) \end{aligned}$$

$$i = \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 / \quad \chi_\Phi^{(1)} \approx$$

$$G_\Phi^{(1)}(t) = \Theta(t) \langle \Xi \rangle_0 + t^0;$$

3 Application of the Response Theory to the Lorenz-96 Model

3.1 Statistical properties of the Driven Lorenz-96 Model

$$\frac{dx_i}{dt} = x_{i-1}x_{i+1} - x_{i-2} - x_i + F$$

$$i = 1, \dots, N \quad x_{i+N} = x_{i-N} = x_i$$

$$E = \sum_i x_i^2$$

E —

$$\begin{aligned} & \gamma \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right) \\ & \in \mathbb{R}^N \quad \text{if } t \neq 0 \end{aligned}$$

3.2 Asymptotic properties of the linear Dependability

$$\epsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\frac{X_i}{t} = \frac{x_{i-1} - x_{i+1} - x_{i+2} - \dots - x_i}{F} \in \mathbb{R}^N \quad \text{if } t \neq 0$$

$$\gamma \left(\frac{X_i}{t} \right) = \frac{X_i}{t} \quad \text{if } t \neq 0$$

$$\begin{aligned}
 & \text{"} = x_i^2 \\
 & \text{fi} \\
 & \text{!} \rightarrow \infty
 \end{aligned}$$

$$\begin{aligned}
 & \chi_{\varepsilon, N}^{(1)} \left(i \frac{m}{!} - \frac{F - m}{!^2} \right) o !^{-2} : \\
 & \pi = \\
 & \chi
 \end{aligned}$$

$$\chi_{E,N}^{(1)} = \frac{F}{-}$$

$$\int_0^\infty \chi_{E,1}^{(1)} \frac{\pi}{l} ;$$

$$\chi_{M,\epsilon,1}^{(1)}$$

$$\chi_{M,1}^{(1)} \frac{\pi}{l^2} ;$$

$$\int_0^\infty \chi_{M,1}^{(1)} \frac{\pi}{l} ;$$

$$E_j \quad x_j$$

$$G_{E_j,1}^{(1)}(t) = \int \rho_0(x) \Theta(t) @_j \left[-x_1^2 - (x_j - x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F) t - o(t) \right] \\ \Theta(t) \left[\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0 t - o(t) \right];$$

$$\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0$$

$$\rho_0$$

$$\chi_{E_j,1}^{(1)} = i \frac{m}{l} - \frac{m}{l^2} - o(l^{-2}) ;$$

$$E_j \quad \chi_{E,1}^{(1)} = \sum_k \chi_{E_k,1}^{(1)} \\ \chi_{E_j,1}^{(1)} \approx \chi_{E,1}^{(1)}$$

$$\begin{aligned}
& \int_0^\infty \chi_{E_j,1}^{(1)} \left(\frac{\pi}{m} \right) \\
& \propto l^{-2} \\
& \chi_{E,1}^{(1)} \chi_{E_j,1}^{(1)} \\
& x_j \\
& = E_{j+1} E_{j+2} E_{j-1}
\end{aligned}$$

$$\chi_{\psi,1}^{(1)} \left(-\frac{\langle x_{j-1} x_{j+1} - x_{j-2} \rangle_0}{l^2} - \frac{F-m}{l^2} \right) o l^{-2} :$$

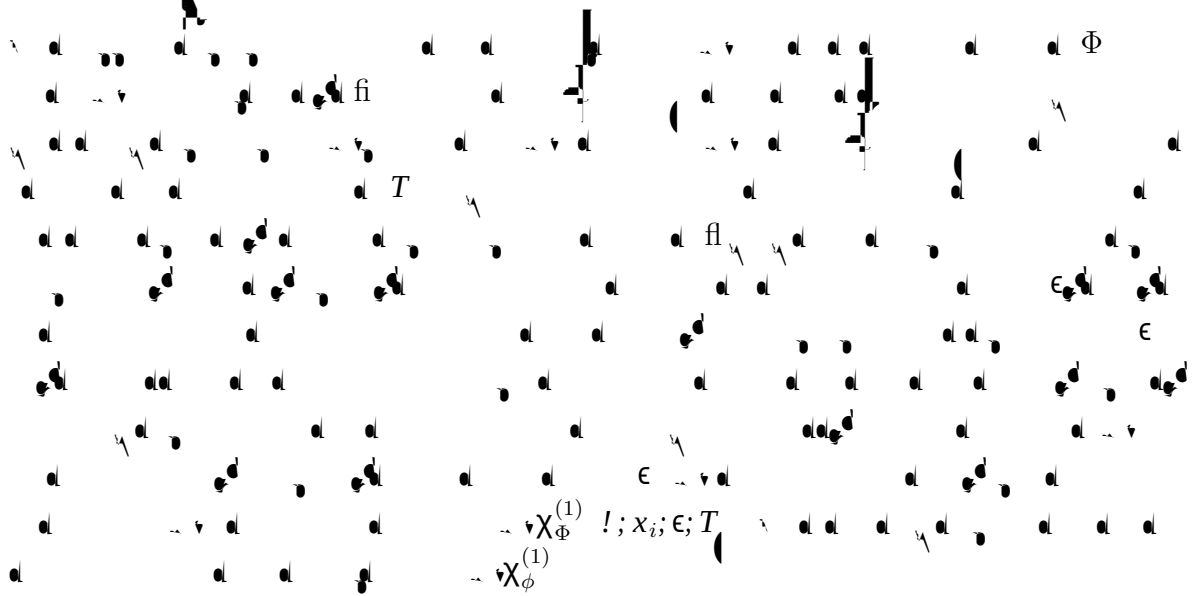
$$\begin{aligned}
& \chi_{\psi,1}^{(1)} \chi_{E_j,1}^{(1)} \chi_{E,1}^{(1)} \\
& \chi_{E,1}^{(1)} \chi_{E_{loc},1}^{(1)} E_{loc} E_j = x_j^2 = x_{j+1}^2 = x_{j+2}^2 = x_{j-1}^2 \\
& x_j
\end{aligned}$$

$$\chi_{x_j,1}^{(1)} \left(\frac{1}{l} - \frac{1}{l^2} \right)$$

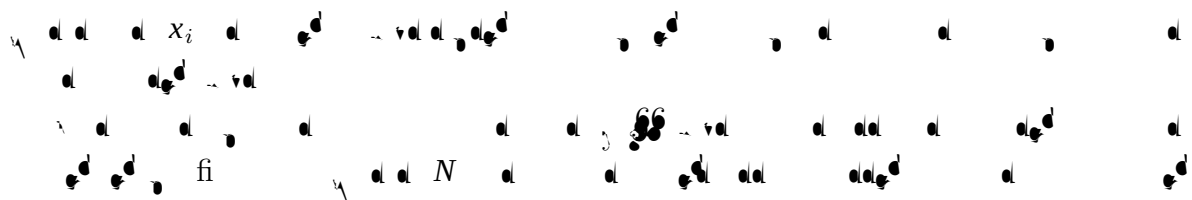


4 Re DI

4.1 Simulation and Data Processing



$$\chi_{\Phi}^{(1)} \sim_{K \rightarrow \infty} \frac{1}{K} \sum_{i=1}^K \chi_{\Phi}^{(1)} ; x_i ; \epsilon ; T ;$$



F Φ K ff $!$ $!$ $!_l$ $:\pi$ $!_h$ π $:\pi$ t t T $\pi=!$

$\epsilon \approx$ ff ff ϵ $:$ K ϵ K

f t $(($ K T

$$\chi_{\varepsilon,N}^{(1)} = N \chi_{E,N}^{(1)} \quad E=N$$

Diagram illustrating a process involving fermions (solid lines) and photons (wavy lines). The diagram shows a central vertical line with a thick black bar in the middle, representing a photon. Two horizontal lines extend from the top and bottom of this central line, each ending in a loop of fermions (represented by solid lines). These loops are connected to external fermion lines (solid lines) that enter and exit the diagram. The diagram is labeled with various mathematical symbols, including $m=!$ and $!$.

$$\sim -F - m_{\pi}^2$$

$$\chi_{\mu,N}^{(1)} = N \chi_{M,N}^{(1)} \quad \mu \leq M \leq N$$

The diagram illustrates the relationship between various sets and functions. At the top, a set $X_{\mu, N}^{(1)}$ is shown with a mapping to a set of functions $\mathcal{F}$. Below this, a set of functions $\mathcal{F}$ is mapped to a set of functions $\mathcal{F}_\varepsilon$. The diagram also includes a set of functions $\mathcal{F}_\varepsilon$ and a set of functions $\mathcal{F}_\varepsilon^{(1)}$. The diagram is composed of several interconnected parts, including a central set of functions $\mathcal{F}_\varepsilon$ and a set of functions $\mathcal{F}_\varepsilon^{(1)}$.

$$\chi_{\varepsilon,N}^{(1)} \in \begin{cases} \frac{\omega}{\omega_l} \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \frac{m}{\omega}; & l \geq l; \end{cases}$$

$$! \rightarrow \chi_{\varepsilon, N}^{(1)}$$

$$\chi_{\varepsilon,N}^{(1)} = \begin{cases} \chi_{\varepsilon,N}^{(1)} & ; \quad \leq l \leq l; \\ \chi_{\varepsilon,N}^{(1)} & ; \quad l \leq l \leq h; \\ -\frac{F-2m}{\omega^2} & ; \quad l \geq h; \end{cases}$$

$\chi_{\mu,N}^{(1)}$

Diagram illustrating the process $e^+ e^- \rightarrow \pi^0 \gamma \gamma \gamma$. The diagram shows the annihilation of an electron and a positron into a virtual photon, which then splits into a π^0 and a photon. The π^0 subsequently decays into two photons. The final state consists of a π^0 , a photon, and two more photons from the π^0 decay.

$$\frac{m}{F} \ll \lambda \gamma F^{\gamma-1};$$

$$\lambda \frac{e F}{F} \gamma - m F \gamma =:$$

$$X_i x$$

$$\begin{aligned}
& \text{fi} \\
& \chi_{\mu,N}^{(1)} \quad E_j \quad \text{fi} \quad \chi_{x_j,1}^{(1)} \quad \chi_{M,1}^{(1)} \\
& M \quad \chi_j \\
& \text{fi} \\
& \chi_{M,1}^{(1)} \quad \chi_{\mu,N}^{(1)} \quad \chi_{x_j,1}^{(1)} \quad \chi_j
\end{aligned}$$

4.4 Further implication of Kramer-Kronig relation and Dyadic

$$\begin{aligned}
& \text{fi} \\
& \text{th}
\end{aligned}$$

$\chi_{x_j,1}^{(1)}$ $\chi_{M,1}^{(1)}$ $\chi_{E_j,1}^{(1)}$ $\chi_{E,1}^{(1)}$ E_{loc}

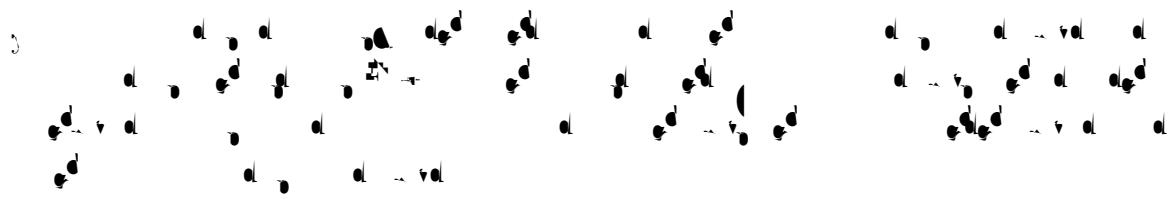
61

५५

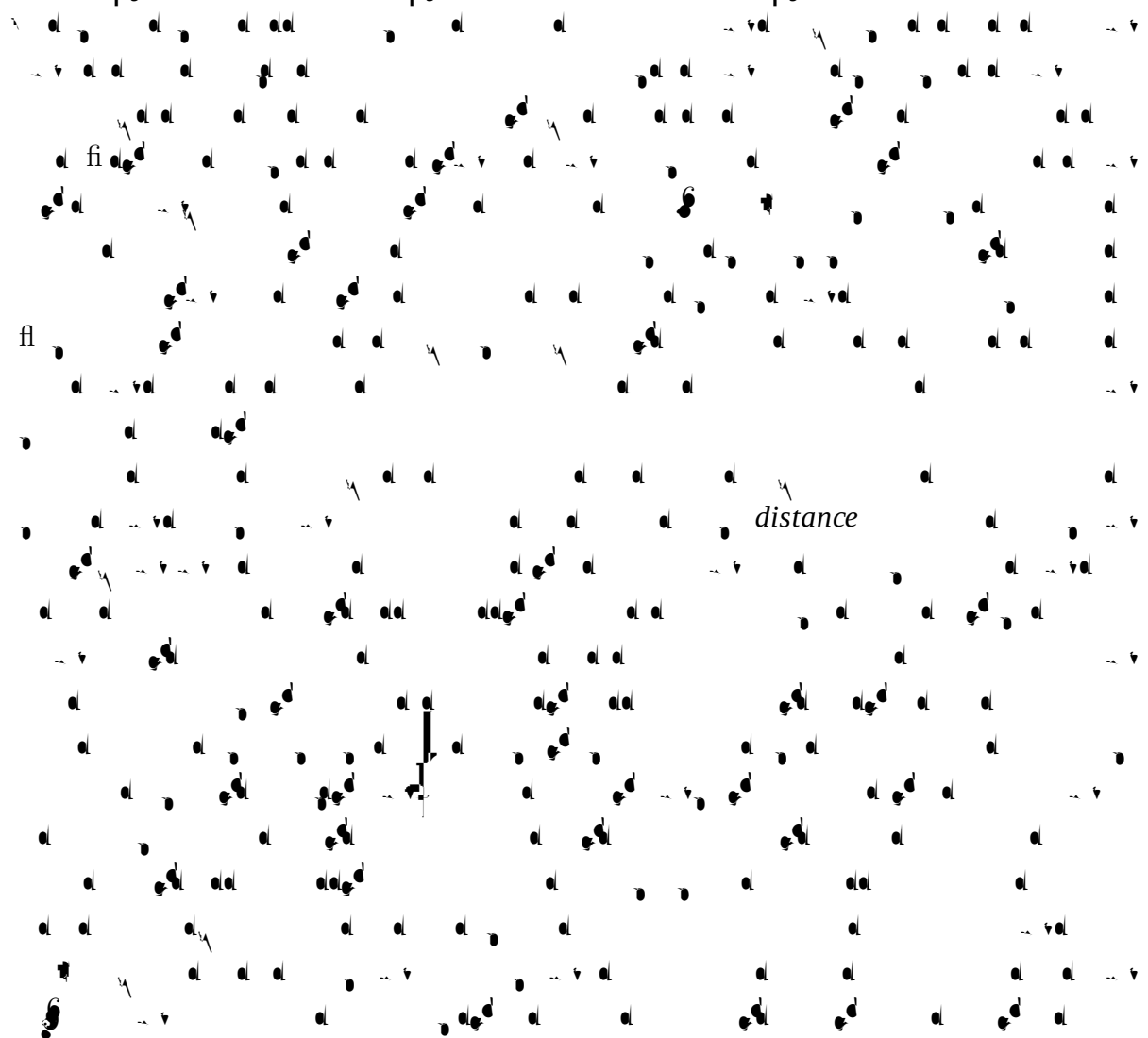
$$G_{\varepsilon, N}^{(1)}$$

CC

f



6 Summary, Discussion and Conclusion



Hasselmann programme 6 *closure theory*

Handwritten musical notation on a single page. The notation consists of numerous small, dark, irregular marks and symbols scattered across the page, resembling a dense, abstract pattern. Some faint, larger symbols are visible, including what appears to be a stylized 'ff' (fortissimo) and a '66'.

The diagram is a complex, abstract representation of the relationship between climate system components and spectroscopy. It features a dense network of black dots and lines, creating a web-like structure. Key labels are scattered throughout the diagram, including "climate system" on the left, "spectroscopy of the" on the right, "CO₂" in the center-right, "delta t" at the bottom right, "F" and "N" in the upper middle, "fi" in several locations, and "66" in the upper left. The overall layout suggests a flow or interaction between these elements, with the "climate system" and "spectroscopy of the" labels acting as primary nodes in the network.

Handwritten musical score on ten staves. The notation includes various musical symbols such as notes, rests, and dynamic markings. The text "ff" appears on the second staff, and "fi" appears on the fourth, sixth, and eighth staves. The text "CO₂" is written on the eighth staff, and "66" appears on the ninth staff. The score is written in a cursive, handwritten style.

- <http://www-pcmdi.llnl.gov/>
climateprediction.net <http://cmip-pcmdi.llnl.gov/cmip5/>

VL acknowledges the financial support of the EU-ERC research grant NAMASTE and of the Walker Institute Research Development Fund.

Reference

66

80

86

90

ff

29

This image shows a page of musical notation, likely a score for a piece. The notation is written on multiple staves, with various musical symbols including notes, rests, and bar lines. The page is numbered with several measure numbers: 36, 14, 6, 66, 461, 61, 136, 66, 22, 131, 12, 57, and 6. The notation is dense and complex, suggesting a high level of musical skill. The page is oriented vertically, with the music written from top to bottom. The background is white, and the notation is in black ink.

Handwritten musical notation on a page. The notation includes various notes, rests, and dynamic markings. The number 155 is written in the center, and the number 275 is written below it. The notation is written in a cursive, handwritten style.

206

81

59

131

234

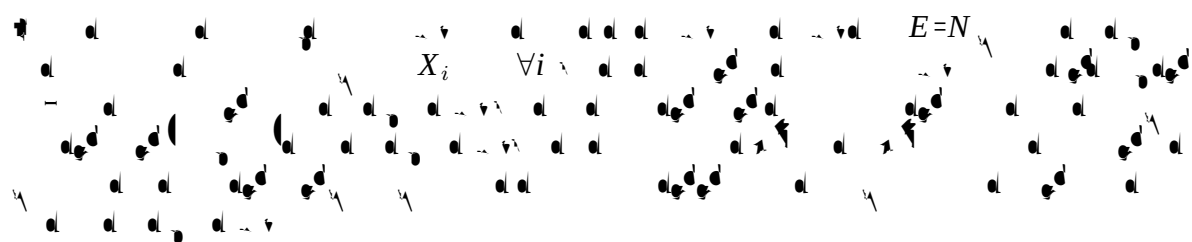
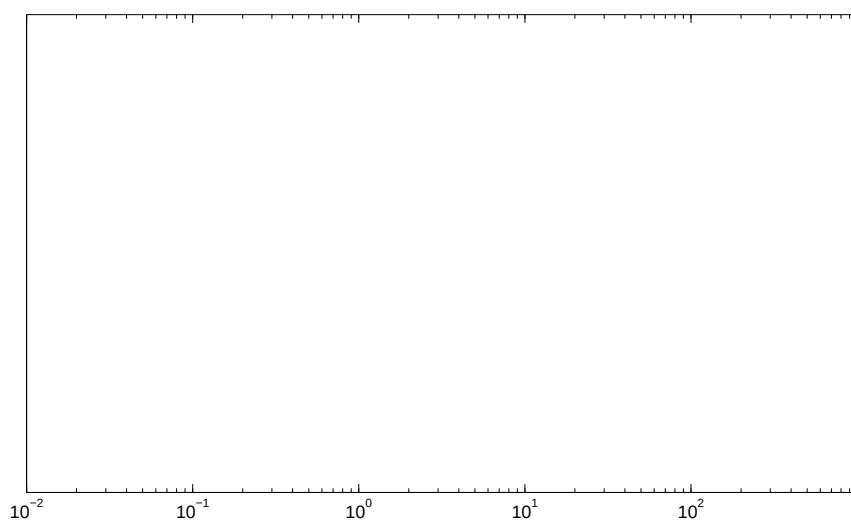
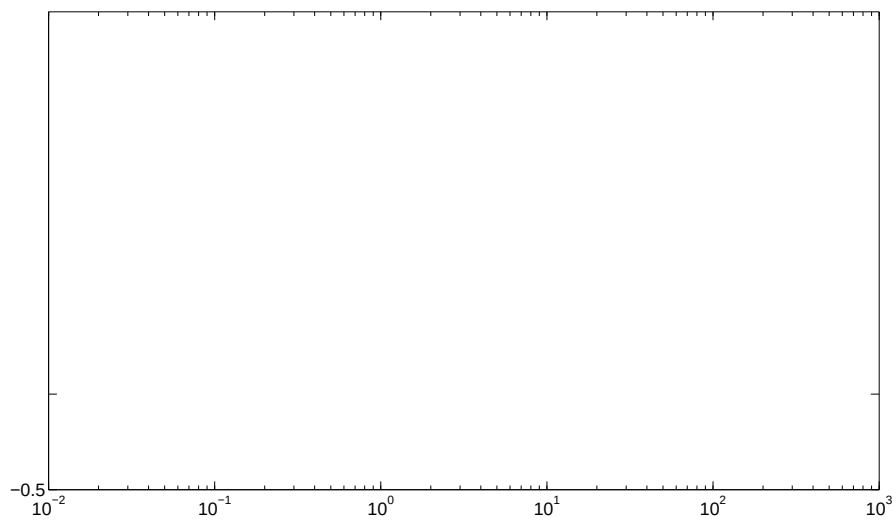
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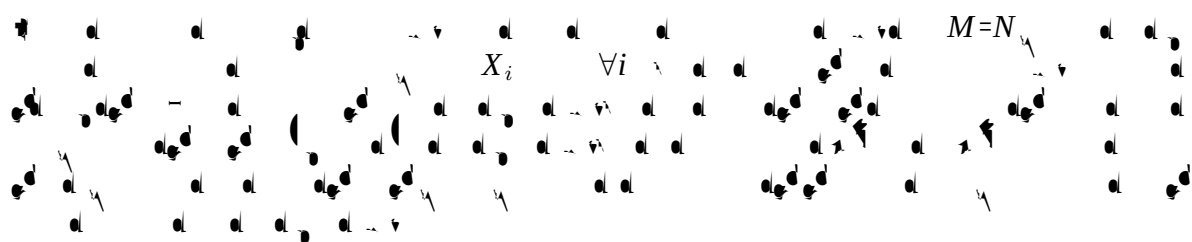
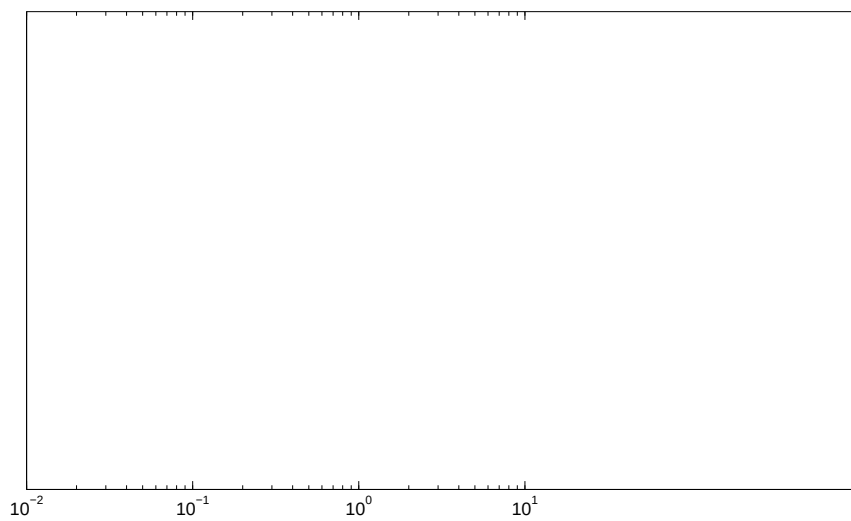
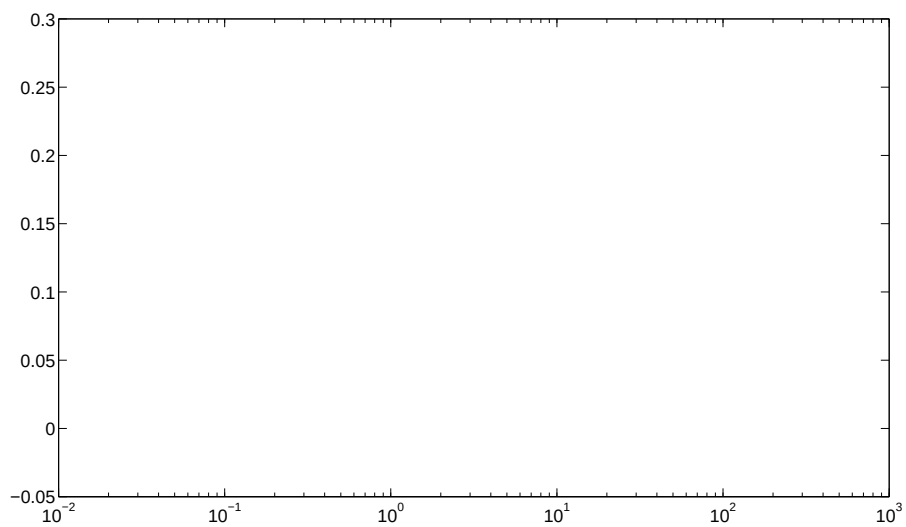
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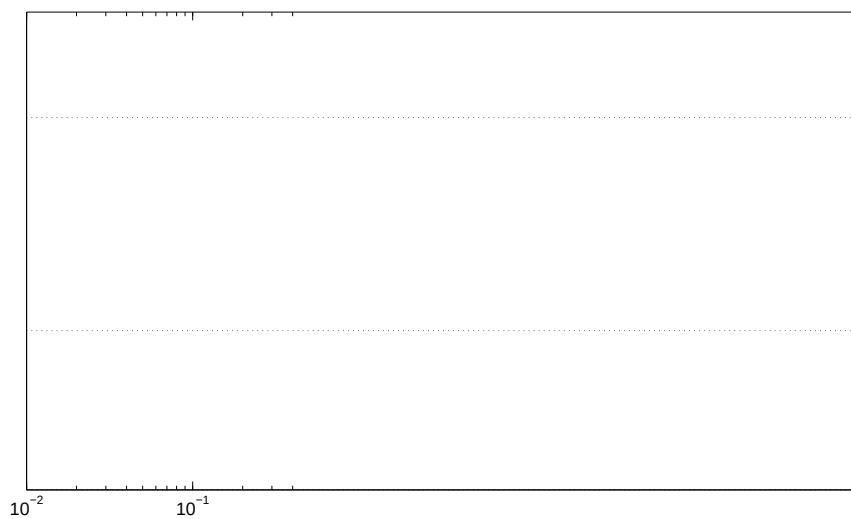
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58

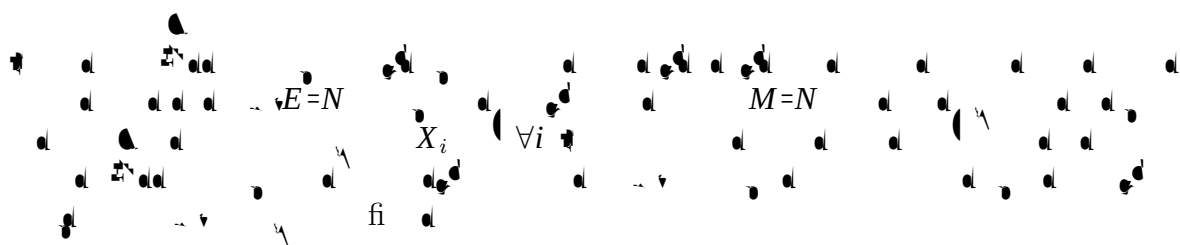
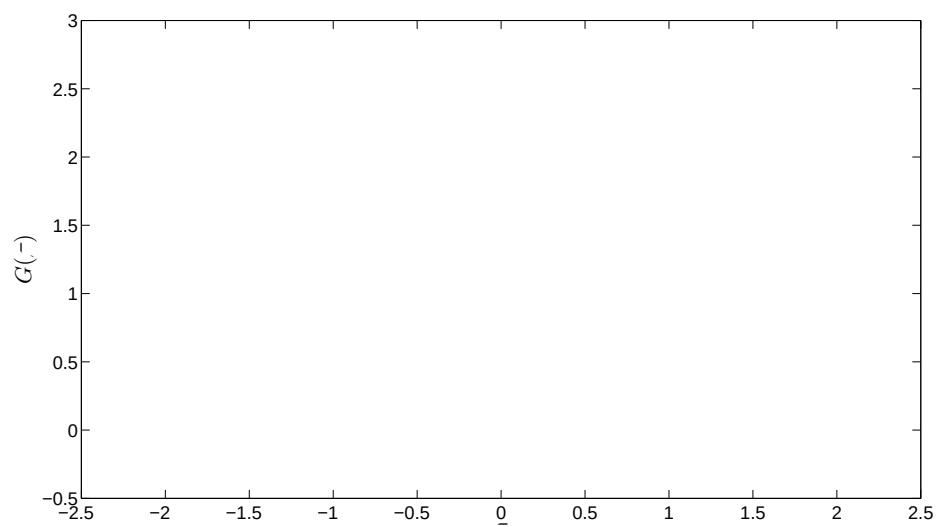
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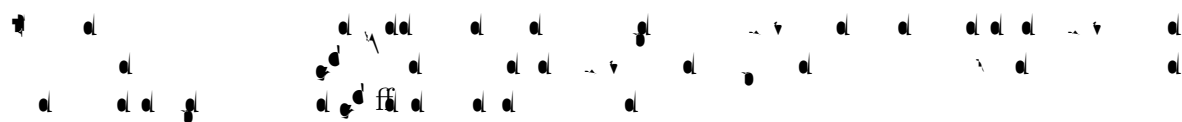
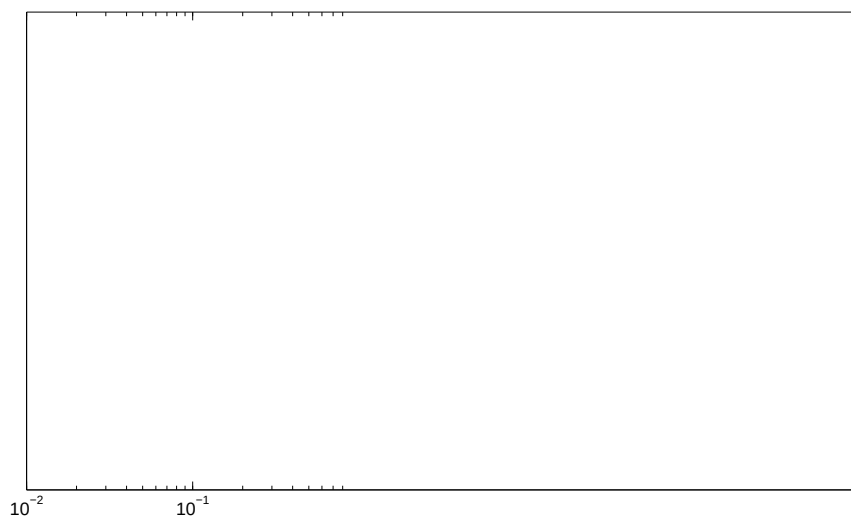


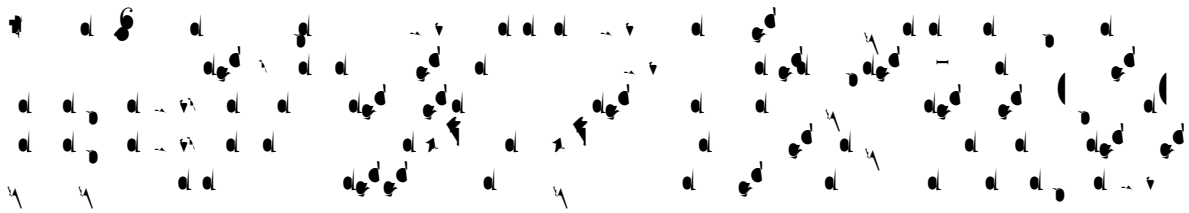
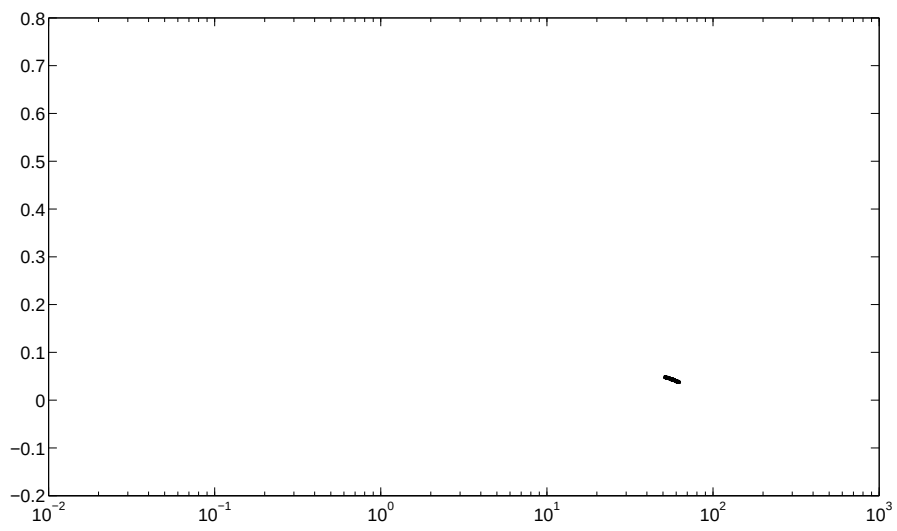


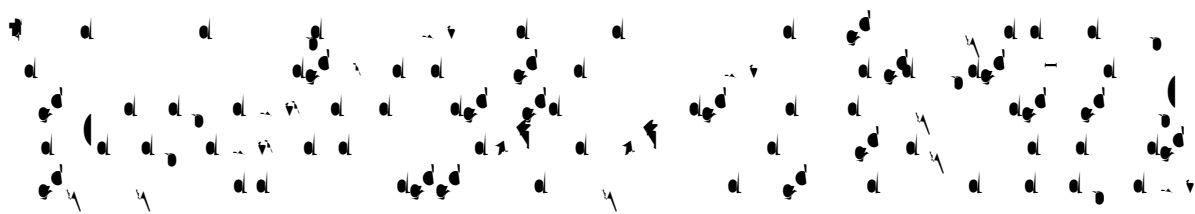
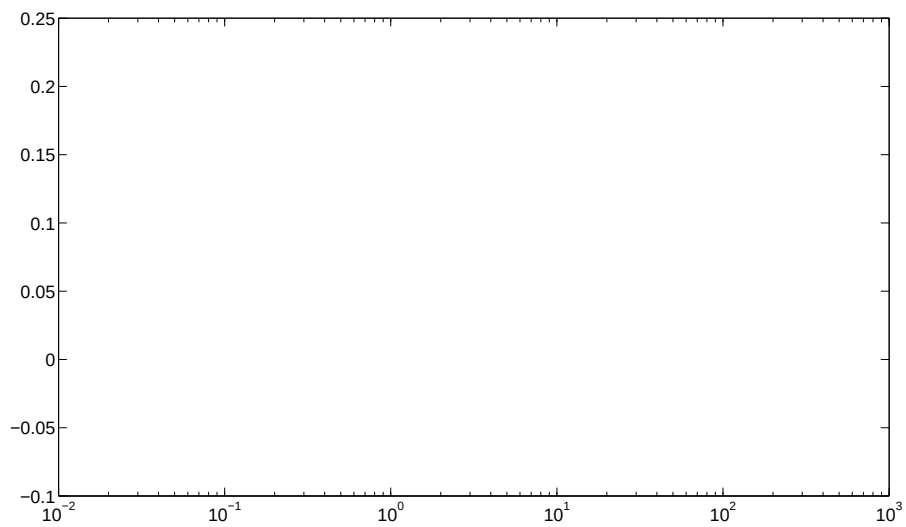


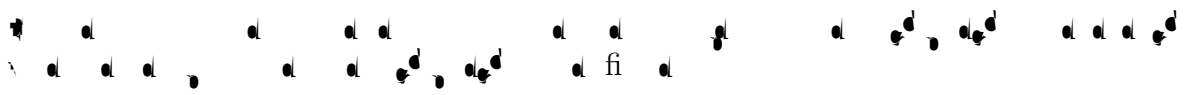
$$E=N \quad \quad \quad X_i \quad \quad \quad M=N \quad \quad \quad \forall i \quad \quad \quad \text{fi}$$











$X_{\varepsilon, N}^{(1)}$