

Department of Mathematics

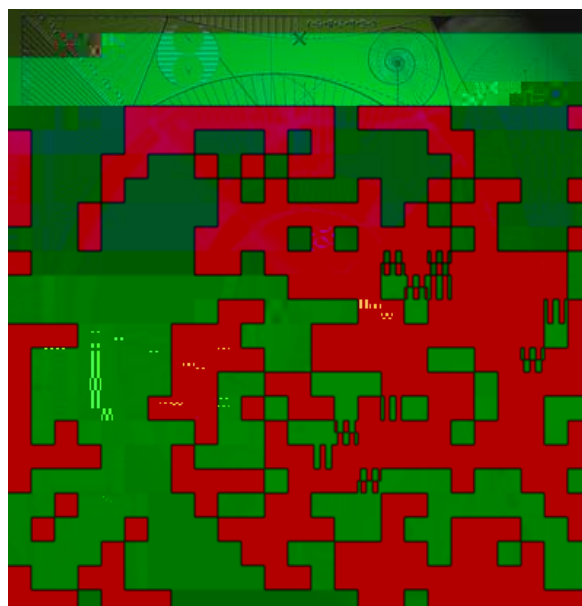
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Boundary value problems for the elliptic
sine-Gordon equation in a semi-strip

by

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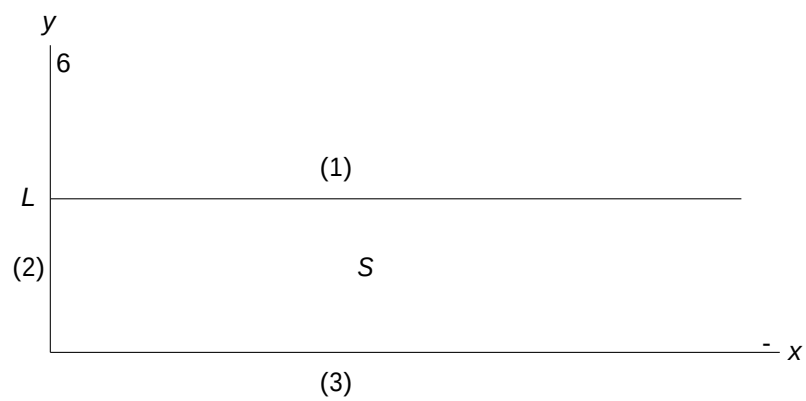


Figure 1: The semistrip S

and we will analyze boundary value problems posed in the semi-infinite strip

$$S = \{0 < x < 1; \quad 0 < y < L\};$$

where L is a positive finite constant. The sides

2 Spectral analysis under the assumption of existence

In what follows we assume that (1.1) is supplemented with appropriate boundary conditions on the boundary of the semistrip S so that the existence of a unique solution $q(x; y)$ can be assumed. Furthermore, we assume the following:

$$\begin{aligned} q(x; L); q_y(x; L); q(x; 0); q_y(x; 0) &\in L^1(\mathbb{R}^+); \\ xq(x; L); xq_y(x; L); xq(x; 0); xq_y(x; 0) &\in L^1(\mathbb{R}^+); \\ q(0; y); q_x(0; y); yq(0; y); yq_x(0; y) &\in L^1([0; L]); \end{aligned} \quad (2.1)$$

The sine-Gordon equation is the compatibility condition of the following Lax pair for the 2×2 matrix-valued function $(x; y; \lambda)$, $\lambda \in \mathbb{C}$:

$$x + \frac{(\lambda)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q(x; y; \lambda); \quad (2.2)$$

$$y + \frac{!(\lambda)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = iQ(x; y; \lambda); \quad (2.3)$$

where

$$(\lambda) = \frac{1}{2i} \left(\frac{1}{\lambda} - \lambda \right); \quad !(\lambda) = \frac{1}{2} \left(\frac{1}{\lambda} + \lambda \right); \quad (2.4)$$

$$Q(x; y; \lambda) = \frac{i}{4} \begin{pmatrix} 0 & 1(1 - \cos q) \\ q_x & iq_y + \frac{i \sin q}{2} \end{pmatrix} A; \quad q = q(x; y); \quad (2.5)$$

Equations (2.2) and (2.3) can be written as the single equation

$$d e^{(\lambda)x + !(\lambda)y} \frac{\partial}{\partial \lambda} (x; y; \lambda) = W(x; y; \lambda); \quad (2.6)$$

where the differential form W is given by

$$W(x; y; \lambda) = e^{(\lambda)x + !(\lambda)y} \frac{\partial}{\partial \lambda} (Q(x; y; \lambda) (x; y; \lambda) dx + iQ(x; y; \lambda) (x; y; \lambda) dy); \quad (2.7)$$

and ∂_λ acts on a 2×2 matrix A by

$$\partial_\lambda A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A;$$

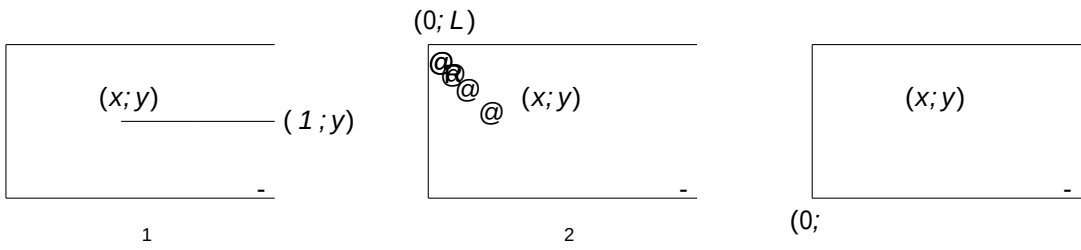
Remark 2.1 Note that

$$\overline{(\lambda)} = (\lambda) = \left(\frac{1}{\lambda} \right); \quad \overline{!(\lambda)} = !(\lambda) = !\left(\frac{1}{\lambda} \right);$$

2.1 Bounded and analytic eigenfunctions

We define three solutions $j(x; y; \lambda)$ $j = 1; 2; 3$; of (2.6) by

$$j(x; y; \lambda) = I + \int_{(x_j; y_j)}^{(x; y)} e^{-(\lambda)x + !(\lambda)y} \frac{\partial}{\partial \lambda} W(\lambda; \lambda); \quad (2.8)$$



2.2 Spectral functions

Any two solutions ψ , $\tilde{\psi}$ of (2.6) are related by an equation of the form

$$\psi(x; y; \lambda) = \tilde{\psi}(x; y; \lambda) e^{-\left(\frac{\omega(\lambda)}{2}x + \frac{\omega(\lambda)}{2}y\right)} C(\lambda): \quad (2.12)$$

We introduce the notations

$$S_1(\lambda) = \psi_1(0; L; \lambda); \quad S_2(\lambda) = \psi_2(0; 0; \lambda); \quad S_3(\lambda) = \psi_1(0; 0; \lambda): \quad (2.13)$$

Then equation (2.12) implies the following equations:

$$\psi_1(x; y; \lambda) = \psi_2(x; y; \lambda) e^{-\left(\frac{\omega(\lambda)}{2}x + \frac{\omega(\lambda)}{2}y\right)} e^{\frac{\omega(\lambda)}{2}L} S_1(\lambda)$$

$a_1($

where the matrices $J_{\pm}$ and J are defined as follows:

$$\begin{aligned} J_+ &= \begin{pmatrix} (12) & 1 \\ 1 & a_3(\lambda) \end{pmatrix} \begin{pmatrix} (1) \\ 3 \end{pmatrix}; \quad \arg(\lambda) \in [0; \frac{\pi}{2}]; \\ J_- &= \begin{pmatrix} (12) & 1 \\ 1 & a_1(\lambda) \end{pmatrix} \begin{pmatrix} (2) \\ 2 \end{pmatrix}; \quad \arg(\lambda) \in [\frac{\pi}{2}; \pi]; \\ J_+ &= \begin{pmatrix} 1 & (3) \\ a_3(\lambda) & 1 \end{pmatrix} \begin{pmatrix} (34) \\ 1 \end{pmatrix}; \quad \arg(\lambda) \in [\pi; \frac{3\pi}{2}]; \\ J_- &= \begin{pmatrix} 1 & (4) \\ a_1(\lambda) & 1 \end{pmatrix} \begin{pmatrix} (34) \\ 2 \end{pmatrix}; \quad \arg(\lambda) \in [\frac{3\pi}{2}; 2\pi]; \end{aligned}$$

$$J(x; y; \lambda) = J_-(x; y; \lambda); \quad \text{if } \arg(\lambda) \in [\pi; 2\pi]; \quad \lambda = 0; \frac{\pi}{2}; \frac{3\pi}{2}; \quad (2.24)$$

where, using the global relation, we find

$$\begin{aligned} J^0 &= \begin{pmatrix} 0 & \frac{a_2(\lambda)}{a_1(\lambda)a_3(\lambda)} & \frac{b_3(\lambda)}{a_3(\lambda)} e^{-i(x; y; \lambda)} \\ \frac{e^{i\omega(\lambda)L} b_1(\lambda)}{a_1(\lambda)} e^{i(x; y; \lambda)} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ C \\ A \end{pmatrix}; \\ J^{-2} &= \begin{pmatrix} 0 & 1 & \frac{b_2(\lambda)}{a_1(\lambda)a_3(\lambda)} e^{-i(x; y; \lambda)} \\ 1 & A & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad J^{3-2} = \begin{pmatrix} 0 & 1 & \frac{b_2(\lambda)}{a_1(\lambda)a_3(\lambda)} e^{i(x; y; \lambda)} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ A \\ 1 \end{pmatrix} \end{aligned}$$

and

$$J = J^{3-2}(J^0)^{-1}J^{-2}; \quad (2.25)$$

where

$$(x; y; \lambda) = (\lambda)x + i(\lambda)y; \quad (2.26)$$

All the matrices J have unit determinant: for J^{-2} and J^{3-2} this is immediate, whereas for J^0 we find

$$\det(J^0) = \frac{a_2(\lambda) + e^{-i(\lambda)L} b_1(\lambda) b_3(\lambda)}{a_1(\lambda) a_3(\lambda)} = \frac{a_1(\lambda) a_3(\lambda)}{a_1(\lambda) a_3(\lambda)} = 1;$$

where we have used the equation

$$a_2(\lambda) = a_1(\lambda) a_3(\lambda) - b_3(\lambda) b_1(\lambda) e^{-i(\lambda)L}; \quad \lambda \in \mathbb{R}; \quad (2.27)$$

Equation (2.27) is a consequence of equations (2.21) and (2.22) (see also equation (4.19) below).

The function $(x; y; \lambda)$

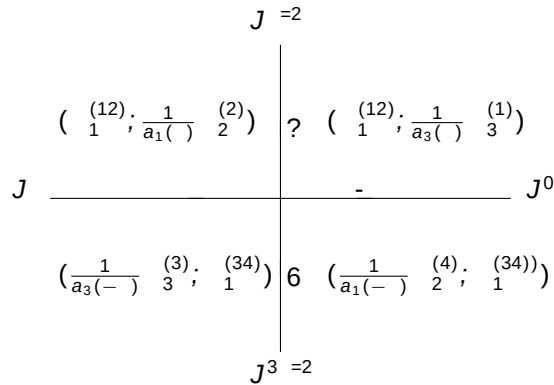


Figure 3: Bounded eigenfunctions and the Riemann-Hilbert problem

The possible zeros of a_1 in the region $\arg(z) \in [\frac{\pi}{2}; \frac{3\pi}{2}]$ are simple; these zeros are denoted $z_j, j = 1; \dots; N_1$

(2.28)

The possible zeros of a_3 in the region $\arg(z) \in [0; \frac{\pi}{2}]$ are simple; these zeros are denoted $z_j, j = 1; \dots; N_3$

The residues of the function at the corresponding poles can be computed using equations (2.14)-(2.16). Indeed, equation (2.16) yields

$$b_1^{(12)} = a_3^{(3)} + b_3 e^{(x;y;z)} b_3^{(1)};$$

hence

$$\text{Res}_{z_j} \frac{b_1^{(1)}}{a_3} = \frac{b_1^{(1)}(z_j)}{a_3'(z_j)} = \frac{b_1^{(12)}(z_j)}{a_3'(z_j) b_3(z_j)} e^{-(x;y;z_j)}; \quad (2.29)$$

where a_3' denotes the derivative of a_3 with respect to z . Similarly, using (2.14),

$$\text{Res}_{z_j} \frac{b_2^{(2)}}{a_1} = \frac{b_2^{(2)}(z_j)}{a_1'(z_j)} = \frac{b_1^{(12)}(z_j)}{a_1'(z_j) b_1(z_j) e^{-!(z_j)L}} e^{-(x;y;z_j)}; \quad (2.30)$$

Using the notation $[]_1$ for the first column, $[]_2$ for the second column for the solution of the RH problem (2.23), at equations (2.29) and (2.30) imply the following residue conditions:

$$\text{Res}_{z_j} [(x;y;z_j)]_2 = e^{-(x;y;z_j)} \text{at } z_j$$

The inverse problem

Rewriting the jump condition, we obtain

$$u_+ - u_- = u_+ J = (I - J) u_- = -J u_-; \quad (2.32)$$

where $J = I - J$: The asymptotic conditions (2.10)-(2.11) imply

$$u(x; y; \lambda) = I + \frac{u^*(x; y)}{\lambda} + O\left(\frac{1}{\lambda^2}\right); \quad |y| \rightarrow \infty; \quad (2.33)$$

Equations (2.32) and (2.33) define a Riemann-Hilbert problem.

The solution of this RH problem is given by

$$u(x; y; \lambda) = I + \frac{1}{2\pi i} \int_{\Gamma} \frac{u^*(x; y; \lambda') J(x; y; \lambda')}{\lambda - \lambda'} d\lambda'; \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \quad (2.34)$$

where

$$\Gamma = \mathbb{R} \cup i\mathbb{R}.$$

Equations (2.33) and (2.34) imply

$$u^* = \frac{1}{2\pi i} \int_{\Gamma} u(x; y; \lambda') J(x; y; \lambda') d\lambda'; \quad (2.35)$$

Using (2.33) in the first ODE in the Lax pair (2.2), we find

$$\frac{i}{4} [\sigma_3; u^*] = i \frac{q_x}{4} \frac{iq_y}{\lambda} + q_x - iq_y = 2(u^*)_{21} = 2 \lim_{\lambda \rightarrow \infty} (u^*)_{21}; \quad (2.36)$$

(σ_1, σ_3 denote the usual Pauli matrices).

In order to obtain an expression in terms of q rather than its derivatives, we consider the coefficient of the term λ^{-1} . The (1,1) element of this coefficient yields

$$\cos q(x; y) = 1 + 4i(u^*)_{11} - 2(u^*)_{21}^2. \quad (2.37)$$

3 Spectral theory assuming the validity of the global relation

3.1 The spectral functions

The above analysis motivates the following definitions for the spectral functions.

The spectral functions at the $y = 0$ and $y = L$ boundaries

Definition 3.1 Given the functions $q(x; L)$, $q_y(x; L)$ satisfying conditions (2.1), define the map

$$S_1 : f q(x; L); q_y(x; L) g \mapsto f a_1(\lambda); b_1(\lambda) g$$

by

$$\begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = [u_1(0; L)]_1; \quad \lambda \in \mathbb{C}^+;$$

where $[u_1(x; L)]_1$ denotes the first column vector of the unique solution $u_1(x; L)$ of the Volterra linear integral equation

$$u_1(x; \lambda) = I \int_x^\infty e^{-(\lambda - x)\frac{\alpha_0}{2}} Q(\lambda; L; \eta) u_1(\eta; \lambda) d\eta; \quad (3.1)$$

and $Q(x; L; \lambda)$ is given in terms of $q(x; L)$ and $q_y(x; L)$ by equation (2.5).

Proposition 3.1 The spectral functions $a_1(\lambda)$, $b_1(\lambda)$ have the following properties.

- (i) $a_1(\lambda)$, $b_1(\lambda)$ are continuous and bounded for $\operatorname{Im}(\lambda) = 0$, and analytic for $\operatorname{Im}(\lambda) > 0$.
- (ii) $a_1(\lambda) = 1 + O(\lambda^{-1})$, $b_1(\lambda) = O(\lambda^{-1})$ as $|\lambda| \rightarrow \infty$; $\operatorname{Im}(\lambda) = 0$:
- (iii) $a_1(\lambda) = \cos \frac{q(x; L)}{2} + O(\lambda^{-1})$, $b_1(\lambda) = i \sin \frac{q(x; L)}{2} + O(\lambda^{-1})$ as $|\lambda| \rightarrow 0$; $\operatorname{Im}(\lambda) = 0$:
- (iv) $a_1(\lambda) \overline{a_1(\bar{\lambda})} = b_1(\lambda) \overline{b_1(\bar{\lambda})} = 1$, $\operatorname{Im}(\lambda) = 0$.
- (v) The map $Q_1 : f a_1; b_1 g \mapsto f q(x; L) q_y(x; L) g$, inverse to S_1 , is given by

$$\cos q(x; L) = 1 + 4i \lim_{\lambda \rightarrow \infty} (M_x)_{11} + 2 \lim_{\lambda \rightarrow \infty} (M)_{21};$$

$$q_y(x; L) = i q_x(x; L) + 2 \lim_{\lambda \rightarrow \infty} (M)_{21};$$

where M is the solution of the following Riemann-Hilbert problem:

* The function

$$M(x; \lambda) = \begin{cases} M_+(x; \lambda) & \lambda \in \mathbb{C}^+ \\ M_-(x; \lambda) & \lambda \in \mathbb{C}^- \end{cases}$$

is a sectionally meromorphic function of $\lambda \in \mathbb{C}$:

* $M = I + O(\lambda^{-1})$ as $|\lambda| \rightarrow \infty$, and

$$M_+(x; \lambda) = M_-(x; \lambda) J_1(x; \lambda); \quad \lambda \in \mathbb{R};$$

where

$$J_1(x; \lambda) = \begin{pmatrix} 1 & \frac{b_1(-\lambda)}{a_1(\lambda)} e^{-(\lambda)x} \\ \frac{b_1(\lambda)}{a_1(-\lambda)} e^{(\lambda)x} & \frac{1}{a_1(\lambda) a_1(-\lambda)} \end{pmatrix}; \quad \lambda \in \mathbb{R}; \quad (3.2)$$

* The function $a_1(\lambda)$ may have N_1 simple poles λ_j in $\mathbb{C}^+$.

* Let $[M]_i$ denote the i -th column vector of M , $i = 1, 2$. The possible poles of M_+ occur at λ_j , and the possible poles of M_- occur at λ_j in $\mathbb{C}^-$, and the associated residues are given by

$$\begin{aligned} \operatorname{Res}_{\lambda_j} [M(x; \lambda)]_2 &= \frac{e^{-(\lambda_j)x}}{a_1(\lambda_j) b_1(-\lambda_j)} [M(x; \lambda_j)]_1; \\ \operatorname{Res}_{\lambda_j} [M(x; \lambda)]_1 &= \frac{e^{(\lambda_j)x}}{a_1(-\lambda_j) b_1(\lambda_j)} [M(x; -\lambda_j)]_2; \end{aligned} \quad (3.3)$$

The spectral functions

(v) The map $Q_2 : f a_2; b_2 g \rightarrow f q(0; y) q_y(0; y) g$, inverse to S_2 , is given by

$$\cos q(0; y) = 1 + 4i \lim_{y \rightarrow \infty} (M_y)_{11} + 2 \lim_{y \rightarrow \infty} (M)_{21};$$

$$q_x(0; y) = i q_y(0; y) + 2 \lim_{y \rightarrow \infty} (M)_{21};$$

where M is the solution of the following Riemann-Hilbert problem:

* The function

$$M(y; \lambda) = \begin{pmatrix} M_+(y; \lambda) & \text{Re} & 0 \\ M_-(y; \lambda) & \text{Re} & 0 \end{pmatrix}$$

is a sectionally meromorphic function of $\lambda \in \mathbb{C}$:

* $M = I + O(\lambda^{-1})$ as $|\lambda| \rightarrow \infty$, and

$$M_+(y; \lambda) = M_-(y; \lambda) J_2(y; \lambda); \quad \lambda \in i\mathbb{R};$$

where

$$J_2(y; \lambda) = \begin{pmatrix} 1 & \frac{b_2(-\lambda)}{a_2(-\lambda)} e^{-i(\lambda)y} \\ \frac{b_2(\lambda)}{a_2(\lambda)} e^{i(\lambda)y} & \frac{1}{a_2(\lambda)a_2(-\lambda)} \end{pmatrix}; \quad \lambda \in i\mathbb{R};$$

* M satisfies appropriate residue conditions at the zeros of $a_2(\lambda)$.

Proof of propositions (3.1)-(3.3)

The proof of properties (i)-(iv) follows from the discussion in Section 2.2. In particular, property (iii) follows from the asymptotic behaviour at $\lambda \rightarrow 0$, which is

Regarding the rigorous derivation of the above results, we note the following: If $f q(x; L); q_y(x; L)g$, $f q(x; 0); q_y(x; 0)g$ and $f q(y; 0); q_x(y; 0)g$ are in L^1 , then the Volterra integral equations (3.1), (3.4) and (3.5) respectively, have a unique solution, and hence the spectral functions $f a_j; b_j g$, $j = 1; \dots; 3$, are well defined. Moreover, under the assumption (2.1) the spectral functions belong to $H^1(\mathbb{R})$, hence the Riemann-Hilbert problems that determine the inverse maps can be characterized through the solutions of a Fredholm integral equation, see [10, 49].
QED

3.2 The Riemann-Hilbert problem

Theorem 3.1 Suppose that a subset of the boundary values $f q(x; L); q_y(x; L)g$; $f q(x; 0); q_y(x; 0)g$, $0 < x < 1$, and $f q(y; 0); q_x(y; 0)g$, $0 < y < L$, satisfying (2.1), are prescribed as boundary conditions. Suppose that these prescribed boundary conditions are such that the global relations (2.21) and (2.22) can be used to characterize the remaining boundary values. Define the spectral functions $f a_j; b_j g$, $j = 1; \dots; 3$, by definitions (3.1)-(3.3). Assume that the possible zeros $f_j g_{j=1}^{N_1}$ of $a_1(\cdot)$ and $f_j g_{j=1}^{N_2}$ of $a_3(\cdot)$ are as in assumption 2.28. Define $M(x; y; \cdot)$ as the solution of the following 2×2 matrix Riemann-Hilbert problem:

- * The function $M(x; y; \cdot)$ is a sectionally meromorphic function of λ away from $\mathbb{R} \cup i\mathbb{R}$.
- * The possible poles of the second column of M occur at $\lambda = \lambda_j$, $j = 1; \dots; N_2$, in the first quadrant and at $\lambda = \lambda_j$, $j = 1; \dots; N_1$, in the second quadrant of the complex plane. The possible poles of the first column of M occur at $\lambda = \lambda_j$ ($j = 1; \dots; N_1$) and $\lambda = \lambda_j$ ($j = 1; \dots; N_2$).
- The associated residue conditions satisfy the relations (2.31).
- * $M = I + O(\lambda^{-1})$ as $|\lambda| \rightarrow \infty$, and

$$M_+(x; y; \cdot) = M_-(x; y; \cdot)J(x; y; \cdot); \quad \lambda \in \mathbb{R} \cup i\mathbb{R};$$

where $M = M_+$ for λ in the first or third quadrant, and $M = M_-$ for λ in the second or fourth quadrant of the complex plane, and J is defined in terms of $f a_j; b_j g$ by equations (2.25).

Then M exists and is unique, provided that the H^1 norm of the spectral functions is sufficiently small.

Define $q(x; y)$ in terms of $M(x; y; \cdot)$ by

$$q_x - i q_y = 2 \lim_{\lambda \rightarrow \infty} (\lambda M)_{21}; \quad (3.7)$$

$$\cos q(x; y) = 1 + 4i \left(\lim_{\lambda \rightarrow \infty} (\lambda M_x)_{11} \right) - 2 \left(\lim_{\lambda \rightarrow \infty} (\lambda M)_{21} \right)^2; \quad (3.8)$$

Then $q(x; y)$ solves (1.1). Furthermore, $q(x; y)$ evaluated at the boundary, yields the functions used for the computation of the spectral functions.

Proof: Under the assumptions (2.1), the spectral functions are in H^1 .

In the case when $a_1(\cdot)$ and $a_3(\cdot)$ have no zeros, the Riemann-Hilbert problem is regular and it is equivalent to a Fredholm integral equation. However, we have *not* been able to establish a vanishing lemma, hence we require a small norm assumption for solvability. If $a_1(\cdot)$ and $a_3(\cdot)$ have zeros, the singular RH problem can be mapped to a regular one coupled with a system of algebraic equations [22]. Moreover, it follows from standard arguments, using the dressing method [47, 48], that if M solves the above RH problem and $q(x; y)$ is defined by (3.7)-(3.8), then $q(x; y)$ solves equation (1.1). The proof that q evaluated at the boundary yields the functions used for the computation of the spectral functions follows arguments similar to the ones used in [24].

QED

4 Linearizable boundary conditions

We now concentrate on the particular boundary conditions (1.2).

In this case, equations (2.17)-(2.19) simplify as follows:

$$\begin{aligned} \begin{pmatrix} A_1(x; \cdot) \\ B_1(x; \cdot) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4} \int_x^\infty e^{i(\cdot)(x-\cdot)} \begin{pmatrix} q_y(\cdot; L) A_1(\cdot; \cdot) \\ q_y(\cdot; L) B_1(\cdot; \cdot) \end{pmatrix} d\cdot; \\ 0 < x < 1; \quad \operatorname{Im}(\cdot) &= 0; \end{aligned} \quad (4.1)$$

$$\begin{aligned} \begin{pmatrix} A_2(y; \cdot) \\ B_2(y; \cdot) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4} \int_y^L e^{i(\cdot)(y-\cdot)} \begin{pmatrix} \frac{(1-\cos d)}{h} A_2(\cdot; \cdot) + [q_x(0; y) - i \frac{\sin d}{h}] B_2(\cdot; \cdot) \\ q_x(0; y) + i \frac{\sin d}{h} A_2(\cdot; \cdot) + \frac{(1-\cos d)}{h} B_2(\cdot; \cdot) \end{pmatrix} d\cdot; \\ 0 < y < L; \quad 2 \in \mathbb{C}; \end{aligned} \quad (4.2)$$

$$\begin{aligned} \begin{pmatrix} A_3(x; \cdot) \\ B_3(x; \cdot) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4} \int_x^\infty e^{i(\cdot)(x-\cdot)} \begin{pmatrix} q_y(\cdot; 0) A_3(\cdot; \cdot) \\ q_y(\cdot; 0) B_3(\cdot; \cdot) \end{pmatrix} d\cdot; \\ 0 < x < 1; \quad \operatorname{Im}(\cdot) &= 0; \end{aligned} \quad (4.3)$$

In equations (4.1) and (4.3), the only dependence on $\cdot$ is through $(\cdot)$. Thus, since $(\frac{1}{\cdot}) = (\cdot)$, it follows that the vector functions $(A_1; B_1)$ and $(A_3; B_3)$ satisfy the same symmetry properties. Hence,

$$a_j(\frac{1}{\cdot}) = a_j(\cdot); \quad b_j(\frac{1}{\cdot}) = b_j(\cdot); \quad j = 1, 3; \quad \operatorname{Im}(\cdot) = 0; \quad (4.4)$$

It turns out that the vector function $(A_2; B_2)$ also satisfies a certain symmetry condition, as stated in the following proposition.

Proposition 4.1 *Let $q_x(0; y)$ be a sufficiently smooth function. Then the vector solution of the linear Volterra integral equation (4.2) satisfies the following symmetry conditions (where we do not indicate the explicit dependence of A_2, B_2 on y):*

$$\begin{aligned} A_2(\frac{1}{\cdot}) &= \frac{1}{1 - F(\cdot)^2} [A_2(\cdot) - F(\cdot) B_2(\cdot) + F(\cdot) e^{i(\cdot)(y-L)} B_2(\cdot) - F(\cdot)^2 e^{i(\cdot)(y-L)} A_2(\cdot)]; \\ B_2(\frac{1}{\cdot}) &= \frac{1}{1 - F(\cdot)^2} [B_2(\cdot) - F(\cdot) A_2(\cdot) + F(\cdot) e^{i(\cdot)(y-L)} A_2(\cdot) - F(\cdot)^2 e^{i(\cdot)(y-L)} B_2(\cdot)]; \\ 0 < y < L; \quad 2 \in \mathbb{C}; \end{aligned} \quad (4.5)$$

where the function $F(\lambda)$ is given by

$$F(\lambda) = i \frac{1}{1 + \frac{\lambda^2}{2}} \tan \frac{d}{2}. \quad (4.6)$$

Proof: Let the 2×2 matrix valued function $U_2(y; \lambda)$ be defined by

$$U_2(y; \lambda) = \begin{pmatrix} A_2(y; \lambda) & B_2(y; \lambda) \\ B_2(y; \lambda) & A_2(y; \lambda) \end{pmatrix}; \quad 0 < y < L; \quad \lambda \in \mathbb{C}; \quad (4.7)$$

Then U_2 satisfies the ODE

$$\begin{aligned} (U_2)_y + \frac{i(\lambda)}{2} [U_3; U_2] &= iQ(0; y; \lambda) U_2; \quad 0 < y < L; \\ U_2(L; \lambda) &= I; \end{aligned} \quad (4.8)$$

where $Q(x; y; \lambda)$ is defined in (2.5), and $q(0; y) = d$.

Letting

$$U_2(y; \lambda) = U_2(y; \lambda) e^{\frac{i\omega(\lambda)}{2} U_3(y-L)} \quad (4.9)$$

it follows that U_2 satisfies the ODE

$$\begin{aligned} (U_2)_y &= V U_2; \\ U_2(L; \lambda) &= I; \quad 0 < y < L; \quad \operatorname{Re}(\lambda) \geq 0; \end{aligned} \quad (4.10)$$

where

$$V(y; \lambda) = \frac{1}{4} \begin{pmatrix} (\lambda + \frac{\cos d}{i \sin d}) & q_x(0; y) + \frac{i \sin d}{\lambda + \frac{\cos d}{i \sin d}} \\ q_x(0; y) & \lambda + \frac{\cos d}{i \sin d} \end{pmatrix}; \quad (4.11)$$

We seek a non singular matrix $R(\lambda)$, independent of y , such that

$$V(y; \frac{1}{\lambda}) = R(\lambda) V(y; \lambda) R(\lambda)^{-1}. \quad (4.12)$$

It can be verified that such a matrix is given by

$$R(\lambda) = \begin{pmatrix} 1 & F(\lambda) \\ F(\lambda) & 1 \end{pmatrix}; \quad (4.13)$$

where F is defined by (4.6).

Replacing in equation (4.10) by $\frac{1}{\lambda}$, and using (4.12), we need the following equation: $U_2(L; \frac{1}{\lambda}) = R(\lambda) U_2(L; \lambda) R(\lambda)^{-1}$

This equation and equation (4.9) imply

$$z(y; \frac{1}{-}) = R(\lambda) z(y; \lambda) e^{i(\lambda - \frac{\alpha_0}{2})(y-L)} R(\lambda)^{-1} : \quad (4.14)$$

The first column vector of this equation implies (4.5).
QED

Remark 4.1 *Recalling that $a_2(\lambda) = A_2(0; \lambda)$, and $b_2(\lambda) = B_2(0; \lambda)$, equations (4.5) immediately imply the following important relations:*

$$\begin{aligned} a_2(\frac{1}{-}) &= \frac{1}{1 - F(\lambda)^2} [a_2(\lambda) - F(\lambda) b_2(\lambda) + F(\lambda) e^{-i(\lambda - \frac{\alpha_0}{2})L} b_2(\lambda) - F(\lambda)^2 e^{-i(\lambda - \frac{\alpha_0}{2})L} a_2(\lambda)]; \\ b_2(\frac{1}{-}) &= \frac{1}{1 - F(\lambda)^2} [b_2(\lambda) - F(\lambda) a_2(\lambda) + F(\lambda) e^{-i(\lambda - \frac{\alpha_0}{2})L} a_2(\lambda) - F(\lambda)^2 e^{-i(\lambda - \frac{\alpha_0}{2})L} b_2(\lambda)]; \\ Im(\lambda) &= 0: \end{aligned} \quad (4.15)$$

In summary, the basic equations characterizing the spectral functions are:

- (a) the symmetry relations (4.4) and (4.15);
- (b) the global relations (2.21) and (2.22);
- (c) the conditions of unit determinant.

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Proof: for simplicity, we will use the notations

$$f = f(\cdot); \quad \hat{f} = f(\cdot): \quad (4.22)$$

Replacing $\cdot$ with $\cdot$ in (2.21) and solving the resulting equation and equation (2.22) for a_2

The left hand side of (4.29), using (4.16) with $j = 1$, simplifies as follows:

$$\frac{a_1}{b_1} + \hat{a}_1 b_1 - \frac{a_1}{b_1} (1 + b_1 \hat{b}_1) = \hat{a}_1 b_1 - a_1 \hat{b}_1.$$

Thus, equation (4.29) becomes

$$b_1 \hat{a}_1 - \hat{b}_1 a_1 = e^{-\ell(\cdot)L} (b_3 \hat{a}_3 - \hat{b}_3 a_3) + e^{-\ell(\cdot)L} F - F. \quad (4.30)$$

Replacing in this equation $\hat{a}_1$ by $\hat{a}_1$ yields

$$(b_1 \hat{a}_1 - \hat{b}_1 a_1) = e^{\ell(\cdot)L} (b_3 \hat{a}_3 - \hat{b}_3 a_3) + e^{\ell(\cdot)L} F - F. \quad (4.31)$$

Indeed, equation (4.17) yields

$$\hat{b}_1 = \frac{1}{a_1}(G + \hat{a}_1 b_1):$$

Replacing $\hat{b}_1$ in equation (4.16) with $j = 1$ by the above expression, and making use of (4.34), we find the first of equations (4.36). The second of equations (4.36) can be obtained in a similar way by eliminating $\hat{a}_1$ instead of $\hat{b}_1$.

Remark 4.5 The equations satisfied by a_3 and b_3 can be obtained from equations (4.36) by replacing $G(\cdot)$ by $G(\cdot)$. Hence

$$a_3(\cdot) = \frac{1}{h(\cdot)}(a_3(\cdot) - G(\cdot)b_3(\cdot)); \quad b_3(\cdot) = \frac{1}{h(\cdot)}(b_3(\cdot) - G(\cdot)a_3(\cdot)); \quad 2 \text{ R}; \quad (4.37)$$

where G is given by (4.21) and $h(\cdot)$ is given by (4.35).

Remark 4.6 The function G is an entire function, thus each of equations (4.36) defines the jump condition of a scalar RH problem. However, it will be shown in section 5 that equations (4.36) and (4.37) are sufficient to determine the jump matrix (2.25).

Remark 4.7 Equations (4.17)-(4.20) imply the following identity:

$$e^{i(\cdot)L}[a_1(\cdot)^2 - b_1(\cdot)^2] + e^{-i(\cdot)L}[a_1(\cdot)^2 - b_1(\cdot)^2] = (e^{i(\cdot)L} + e^{-i(\cdot)L})(1 - F^2) + 2F^2; \quad 2 \text{ R}; \quad (4.38)$$

Indeed, equations (4.19)-(4.20) imply

$$e^{i(\cdot)L}(a_2^2 - b_2^2) = e^{i(\cdot)L}\hat{a}_1^2(a_3^2 - b_3^2) - e^{-i(\cdot)L}b_1^2(\hat{a}_3^2 - \hat{b}_3^2) - 2\hat{a}_1 b_1(a_3 \hat{b}_3 - \hat{a}_3 b_3) \quad (4.39)$$

Replacing in this equation by (4.37), adding the resulting equation to equation (4.39) and using equation (4.30) we find

$$e^{i(\cdot)L}(a_2^2 - b_2^2) + e^{-i(\cdot)L}(\hat{a}_2^2 - \hat{b}_2^2) = (e^{i(\cdot)L} + e^{-i(\cdot)L})(a_1^2 - b_1^2)(\hat{a}_1^2 - \hat{b}_1^2) + 2(a_1 \hat{b}_1 - \hat{a}_1 b_1)(a_3 \hat{b}_3 - \hat{a}_3 b_3); \quad (4.40)$$

Using (4.32), the right hand side of (4.40) equals the following expression:

$$(e^{i(\cdot)L} + e^{-i(\cdot)L}) \left(a_1 \hat{b}_1 - \hat{a}_1 b_1 \right) \left(e^{i(\cdot)L} + e^{-i(\cdot)L} \right) (a_1 \hat{b}_1 - \hat{a}_1 b_1) - 2(a_3 \hat{b}_3 - \hat{a}_3 b_3) :$$

Using equations (4.17) and (4.18) the last expression becomes the right hand side of (4.38).

5 Spectral theory in the linearisable case

In the case of the linearisable boundary conditions (1.2), it is possible to express $q(x; y)$ in terms of the solution of a RH problem whose jump matrices are computed explicitly in terms of the given constant d . Indeed, recall that the jump matrices of the basic RH problem of section 2.4 are defined as follows:

$$J^{-2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-i(x; y; \cdot)} A; \quad J^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i(x; y; \cdot)} A; \quad I(\cdot) = \frac{b_2(\cdot)}{a_1(\cdot)a_3(\cdot)};$$

$$J^0 = \begin{pmatrix} 0 & R(\lambda) \\ \frac{b_3(-)}{a_3(-)} e^{-\omega(\lambda)L} e^{(x;y; \lambda)} & 1 \end{pmatrix} \begin{matrix} 1 \\ C \\ A \end{matrix}; \quad R(\lambda) = \frac{a_2(\lambda)}{a_1(\lambda)a_3(\lambda)};$$

and

$$J = J^3 = {}^{=2}(J^0)^{-1} J^{{}^{=2}}; \quad (5.1)$$

Equation (4.19) and (4.20) imply that

$$R(\lambda) = 1 - e^{-l(\lambda)L} \frac{\hat{b}_3}{a_3} \frac{b_1}{\hat{a}_1}; \quad l(\lambda) = \frac{\hat{b}_3}{a_3} e^{l(\lambda)L} \frac{\hat{b}_1}{a_1}; \quad (5.2)$$

Thus in the linearisable case, the jump matrices involve only the ratios $\frac{\hat{b}_3}{a_3}$ and $\frac{\hat{b}_1}{a_1}$, evaluated at λ and at $-\lambda$. Equations (4.35) and (4.34) imply that these ratios are given by

$$\frac{\hat{b}_3}{a_3} = \frac{G}{h} + \frac{b_3}{a_3 h}; \quad \frac{\hat{b}_1}{a_1} = \frac{G}{h} + \frac{b_1}{a_1 h}; \quad (5.3)$$

Hence the jump matrices depend on the known function $\frac{G}{h}$ as well as on the unknown functions $\frac{b_1}{a_1 h}$ and $\frac{b_3}{a_3 h}$. Using the fact that these unknown functions are bounded and analytic in $\mathbb{C}^+$, it is possible to formulate a RH problem, equivalent to the basic one defined by (5.1), in terms of the known function $\frac{G}{h}$ only. This new RH problem is therefore defined by the following jump matrices:

$$J^{{}^{=2}} = \begin{pmatrix} 0 & 1 \\ 1 & r(\lambda) e^{-(x;y; \lambda)} \end{pmatrix} \begin{matrix} 1 \\ A \end{matrix}; \quad J^3 = {}^{=2} \begin{pmatrix} 0 & 1 \\ r(\lambda) e^{(x;y; \lambda)} & 1 \end{pmatrix} \begin{matrix} 1 \\ A \end{matrix}; \quad r(\lambda) = \frac{G}{h} (1 + e^{-l(\lambda)L});$$

$$J^0 = \begin{pmatrix} 0 & R(\lambda) \\ \frac{e^{-\omega(\lambda)L}}{h} G e^{(x;y; \lambda)} & 1 \end{pmatrix} \begin{matrix} 1 \\ C \\ A \end{matrix}; \quad R(\lambda) = 1 - \frac{G^2}{h\hat{h}} e^{-l(\lambda)L};$$

and

$$J = J^3 = {}^{=2}(J^0)^{-1} J^{{}^{=2}}; \quad (5.4)$$

Theorem 5.1 *Let $q(x; y)$ satisfy equation (1.1) and the boundary conditions (1.2). Then $q(x; y)$ is given by equations (2.36)-(2.37) with $\tilde{}$ replaced by $\sim$, where $\sim$ is the solution of the Riemann-Hilbert problem (2.23) with the jump matrix J replaced by the matrix J defined as follows:*

$$J(x; y; \lambda) = J^{\sim}(x; y; \lambda); \quad \text{if } \arg(\lambda) = \begin{cases} 0; & \frac{3}{2}; \end{cases} \quad (5.5)$$

where

$$J^0 = \begin{pmatrix} 0 & 1 \\ e^{-l(\lambda)L} \frac{G(\lambda)}{h(-)} e^{(x;y; \lambda)} & 1 \end{pmatrix} \begin{matrix} 1 \\ C \\ A \end{matrix};$$

$$J=2$$

Thus the above Riemann-Hilbert problem (5.7) is regular.

We now prove that the Riemann-Hilbert problem defined by (5.7) is uniquely solvable. It can be verified that when $d \in \mathbb{R}$, then $h(\lambda) = \overline{h(\overline{\lambda})}$. In this case, the jump matrices $J(\lambda)$ satisfy the following conditions: the matrices are Schwarz invariant on the imaginary axis and have zero real part on the real axis of the complex plane. Under these assumptions, it follows from general results (see e.g. [10, 29, 49]) that the so-called "vanishing lemma" holds. This guarantees the existence of a unique solution.

QED

6 Conclusions

We have studied boundary value problems for the elliptic sine-Gordon posed on a semistrip. In particular we have shown that if the prescribed boundary conditions are zero along the unbounded sides of the semistrip and constant on the bounded side, then it is possible to obtain the solution in terms of a Riemann-Hilbert problem which is uniquely defined in

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