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A Nyström Method for a Boundary

and

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Abstract

This paper is concerned with solving numerically the Dirichlet boundary value problem for Laplace's equation in a non-locally perturbed half-plane. This problem arises in the simulation of classical unsteady water

present and analyse a numerical scheme for computing the Dirichlet-to-Neumann map. i.e.

for $\mathbf{x_1}$ $\mathbb{R}$

a , t at **f** an **v** $\mathbf{v_1, v_2}$ at f

$$\frac{\partial \varphi}{\partial t} \quad - \; \mathbf{v}^2 - g\mathbf{f},$$

$$\frac{\partial \mathbf{f}}{\partial t} \quad \mathbf{v_2} - \mathbf{v_1}\mathbf{f}'.$$

, for at on a o , parat , t , t r nat on of t , , o t pot nt a

at an n t , fro t , o t on of t , t o para t r t , o n

ar po t on an t , D r t o n ar ata parat on nat ra

na , t , t , to , o , , , p t t , t pp n n r a t

o , t ro o t t , at r a , t rat r , for , a p , n , tta an

A a Ba fort , , , , Ba r B a , B a , o

Z an

t p , o t at t , n , ontr t on ar , n t papa r A a n no , t

t at t pap r app ar to , t , r t p at on to ta , t , n r a

o t on of t , o n ar a , pro , n t , n ra a , of ar trar

o n , ont n o D r t ata φ_Γ t n , t r t , o n ar nor φ_Γ

a Γ()09(n)28.0111((t)0.4362e)-414.647(pr)-0.39949(n)-427.285(t)90.15979.403(R0.11102e3749(b)089(r 1159.6060.4320.512(l394542)0.44

1l(-)-0.43820.430062(i)-424.44@

for o p t n t , D r t to N , ɜnn ɜp to pro , ɜ too for t , ɜ n

o p tɜt onɜ pro , ɜt ɜ t , t p for pro , of ɜt on of non

p o ɜt ɜ , , not , o t ɜt o r t o o , ɜpp n t ,

p , ɜ ɜ , n t , rfɜ , p o An ɜttrɜ t on of o r n ɜ , ,

ɜn o r ɜnɜ n t ɜt ɜ , t ɜt t ɜr fro o r r , t t ɜt o r , ,

tɜ , ɜn on nt n for t r , p , t to t , p o S t ,

on t on n of t , n t , ɜn t , rror n t , n ɜ , ,

r , ɜ n o n , n t , t ɜ S $\to\infty$, ɜ o not , t ɜt t ro o t , tɜ ,

ɜr , to pro , tɜ t r , t ɜn rror o n t ɜt ɜr , n for t r , p , t

to t , rfɜ , pro , f , n ɜ tɜ n on trɜ n , t , n , t ,

r q r , nt ɜn o n on ɜt , of f f o r , o r ot ɜt on

, ɜ ɜ n t , ɜpp ɜt on to t , ɜt on of t , p n nt ɜt ɜ ,

r , f ɜr , n o , on trɜ n , t ɜ ɜ f n t on of t ,

, tr t r , of t , pɜp ɜ fo o , t on r , ɜ t , nt , rɜ q ɜ

t on for ɜt on fro r , ton t ɜt , r t , t , ɜ n n ,

r , t n t , t on ɜr , ɜpp n prop rt , of t , nt , rɜ op ɜtor r , ɜr t

r , t for t , o t on of t , o n ɜr nt , rɜ q ɜt on ɜn ɜn , p t r p

r , ntɜt on for ɜn ɜpp n prop rt , of t , D r t to N , ɜnn ɜp n

, t on , t rn to r t ɜt on ɜn n ɜ ɜnɜ , t on ɜnɜ ,

ɜ N tro t o for t , o n ɜr nt , rɜ q ɜt on ɜ , on r t ɜt on

of t , nt , rɜ op ɜtor pɜrɜ tr , o t ɜt t , nt , rɜt on on

t , r ɜ n , t , trɜp r , ɜnɜ , r , t fro , r

C ɜn r , , n , t on , ɜ r t , ɜp

pro ɜt on to t , ɜt , of ɜ ont n o nt ɜ , f n t on on t ,

r ɜ n , ɜ , on o ɜ ɜt on ɜn tr ono tr nt po ɜt on n , t on

, , t , t o ɜn r , t of , t on to for ɜt , ɜn ɜnɜ , ɜn ɜp

pro ɜt , N tro t o p ɜ rɜ ɜ on nt t ,

D r t ɜtɜ φr ɜn ɜr , oot n pɜrt ɜr f C^∞ ℝ t o ,

top n ɜ t rt n

not r q r , ɜ o , t , t o of , t on ɜ , to t , r n t

, , for t , ɔt ɾ ɔ , pro , n , t on , ɾ , ɔn ɔnɔ ,

ɔr t o for ɔppro ɔt n t , D r t to N , ɔnn ɔp r ɔn n ,

for ɔppro ɔt n t , rfɔ , , o t v ɩ nɔ n , t on , trɔt , t ,

t or t ɔ on ɾ n , r , t n ɾ ɔ , ɔ p ,

Notation. , o , t ɾ , ɔr o notɔt on , t ro o t n pɔrt ɔr , n t on of ɔr o f n t on pɔ , t ɔt ɔr , n , , ɔr for t , n ɾ ɔ ɔnɔ

n ɔn op n or o , t G $\mathbb{R}^m$ m or ɔn n $\mathbb{N}_0$ t BC^n G not , t , t of f n t on φ G $\mathbb{R}$ t ɔt ɔr , o n , ɔn ont n o ɔn ɔ , pɔrt ɔ ɾ ɔt , p to or ɾ n t ɔt ɔr , ɔ o n , ɔn ont n o BC^n G ɔ Bɔnɔ pɔ , n ɾ t , ɔ nor , ɔ r ɔt , BC^0 G BC G ɩor $<$ α t $BC^{0,a}$ G BC G not , t , Bɔnɔ pɔ , of f n t on t ɔt ɔr , o n , ɔn n for o ɾ ont n o t n , α ɔn t $BC^{1,\alpha}$ G not , t , Bɔnɔ pɔ , of f n t on ψ BC^1 G for ψ $BC^{0,\alpha}$ G

ɩor S $>$ ɔn n $\mathbb{N}_0$ t BC^n_S $\mathbb{R}$ BC^n $\mathbb{R}$ not , t , t t o , f n t on φ BC^n $\mathbb{R}$ t ɔt ɔr , p ɾ o t p ɾ o S , ɔ r ɔt , BC^0_S $\mathbb{R}$ BC_S $\mathbb{R}$ ɔn t BC^∞_S $\mathbb{R}$ $_{n\in\mathbb{N}}BC^n_S$ $\mathbb{R}$ ɩor p $>$ t w_p s s ^{p} s $\mathbb{R}$ ɔn t BC^n_p $\mathbb{R}$ BC^n $\mathbb{R}$ not , t , Bɔnɔ pɔ ,

BC^n_p $\mathbb{R}$ u BC^n

f n t on for t , ʒ f p ʒn , H l

$$_H\ x, y \qquad x, y - \quad x, y^r, \qquad x, y \quad \mathbb{R}^2, x \,/\, y,$$

r ,

$$x, y \qquad -\frac{}{\pi}\ n\ x - y$$

t , f n ʒ ntʒ o t on to ʒp ʒ , q ʒt on n t o n on ʒn y^r $y_1,$ $H - y_2$ t , r , , t on of y n $_H$

n r , ton r , ton t propo , to oo for ʒ o t on to t , l o n ʒr ʒ , prol , n t , for of ʒ o l , ʒ r pot nt ʒ

$$\varphi\ x \qquad _\Gamma \frac{\partial\ _H\ x, y}{\partial n\ y} \mu_\Gamma\ y \quad s\ y, \qquad x \qquad ,$$

for o , n t μ_Γ BC Not , t ʒt t , ʒ f p ʒn , r , n f n t on ,

Theorem 2.2. pp n $I - K_\Gamma$ BC BC s n r
o n n rs r s n C_f > or so ons n C > p n
on on $f_\pm$ H n C_f o s

$$\| I - K_\Gamma^{-1} \| \quad C,$$

n r $\|f\|_{BC^2( )}$ C_f

t on n nt to ntro , n o tr o orp J_Γ BC
BC $\mathbb{R}$, n , $J_\Gamma a_\Gamma$ σ a_Γ σ, f σ σ $\mathbb{R}$ for r a_Γ BC
t μ BC $\mathbb{R}$, , n , μ $J_\Gamma \mu_\Gamma$ r , μ_Γ t , o t on of
φ_0 BC $\mathbb{R}$, , n , φ_0 $J_\Gamma \varphi_\Gamma$ n t k_Ω , , n , for x $\mathbb{R}^2$ n
σ $\mathbb{R}$

$$k_\Omega \ x, \sigma \quad \frac{\partial \ _H \ x, y}{\partial n \ y}\bigg|_{y=(\sigma, f(\sigma))} w \ \sigma$$

$$- \frac{}{\pi} \quad \frac{x - \ \sigma, f \ \sigma}{x - \ \sigma, f \ \sigma \ ^2} - \frac{x - \ \sigma, \ H - f \ \sigma}{x - \ \sigma, \ H - f \ \sigma \ ^2} \quad .n \ \sigma \ w \ \sigma$$

r , w σ $\overline{\quad f' \ \sigma^2}$ n σ n σ, f σ $-f'$ σ , /w σ n ,
not , t t s σ s σ, f σ , f' σ /w σ , n t n r , r t ,

$$\varphi \ x \quad k_\Omega \ x, \sigma \ \mu \ \sigma \ \sigma, \quad x \quad ,$$

n

$$\mu \ \tau - \quad k \ \tau, \sigma \ \mu \ \sigma \ \sigma \quad - \varphi_0 \ \tau \ , \quad \tau \quad \mathbb{R},$$

r , k τ, σ k_Ω τ, f τ , σ for τ / σ [( )-0.080868264 Tf 8.2441 p no pro , pp n prop rt for t nt r op r tor $\mathbf{K}$ n o t t t oot n of t rn $\mathbf{k}$ n to t oot n of t o n r t

$$r_1(\tau,\sigma) = \int_0^1 f'(\sigma + (\tau-\sigma)\xi)\, d\xi$$

n

$$r_2(\tau,\sigma) = \int_0^1 f''(\sigma + (\tau-\sigma)\xi)(1-\xi)\, d\xi,$$

for $\tau,\sigma \in \mathbb{R}$ n not t t or t or r pp for $f \in C^2(\mathbb{R})$ t o t t

$$f(\tau) = f(\sigma) + (\tau-\sigma) r_1(\tau,\sigma) = f(\sigma) + (\tau-\sigma) f'(\sigma) + (\tau-\sigma)^2 r_2(\tau,\sigma).$$

Theorem 2.3. $f \in BC^{n+2}(\mathbb{R})$ $\|f\|_{BC^{n+2}(\mathbb{R})} \leq C_f$ *or so* $n \in \mathbb{N}_0$ $C_f >$ *n* $k \in BC^n(\mathbb{R}^2)$ *or* $i,j \in \mathbb{N}_0$ $i + j \leq n$

$$\left|\frac{\partial^{i+j}}{\partial\sigma^i\partial\tau^j} k(\tau,\sigma)\right| \leq \frac{C_k}{1+(\sigma-\tau)^2}, \quad \text{for } \sigma,\tau \in \mathbb{R},$$

r C_k *p s on on* n $f_\pm$ H C_f *F r r or* $K \in BC(\mathbb{R}) \to BC^n(\mathbb{R})$ *r s s* $C_K >$ *p n on on* n $f_\pm$ H C_f *s* $\|K\| \leq C_K$

roo or $\sigma,\tau \in \mathbb{R}^2$ $\sigma \neq \tau$ or t or r

$$\left.\frac{\partial G(x,y)}{\partial n(y)}\right|_{x=(\tau,f(\tau)),y=(\sigma,f(\sigma))} = -\frac{1}{\pi w(\sigma)}\frac{-(\tau-\sigma)f'(\sigma) + f(\tau) - f(\sigma)}{(\tau-\sigma)^2 + (f(\tau)-f(\sigma))^2}$$

$$= -\frac{1}{\pi w(\sigma)}\frac{r_2(\tau,\sigma)}{1 + r_1(\tau,\sigma)^2}.$$

n $f \in BC^{n+2}(\mathbb{R})$ t r t t $w \in BC^{n+1}(\mathbb{R})$ $r_1 \in BC^{n+1}(\mathbb{R}^2)$ n $r_2 \in BC^n(\mathbb{R}^2)$ n $k \in BC^n(\mathbb{R})$ or r t r t on t nt $C_k >$ p n nt on on n $f_\pm$ H n C_f t t $\|k\|_{BC(\mathbb{R})} \leq C_k$

No $G_H(x,y)$ t p q t on f n t on of ot x n y n $\bar{U}_H$ n r ton for $x,y \in \bar{U}_H$ t $x \neq y$ n $y_2 \geqslant f_- -$

$$\nabla_y G_H(x,y) \leq \frac{H - f_-}{\pi |x-y|^2}.$$

n fro t , r , ar t , t at , n ar r n r or ,

for o t on to , pt part a r nt a q at on r , $\mathcal{D}_n$ y H x, y not , an part a r at , of y H x, y of or r , t an or q a to n t r , p , t to t , o pon nt of x an y

$$\mathcal{D}_n \mathbf{1}_{y} \; H \; x, y \qquad \frac{C_n}{x_1 - y_1 \;^2},$$

for x, y $^{-}$ H $x_1 - y_1 \geqslant$ an no r , n t rn t , or , o p t , , pr , on for $M_\Gamma \mu_\Gamma$

Theorem 2.4. μ_Γ $BC^{1,\alpha}$ *n or* **x**

$$M_\Gamma \mu_\Gamma \; x \quad \int_\Gamma m_\Gamma \; \mu_\Gamma, x, y \quad s \; y$$

r

$$m_\Gamma \; \mu_\Gamma, x, y \qquad \frac{\partial \quad_H \; x,y}{\partial s \; y} \quad n \; x \; .n \; x \; \frac{\partial \mu_\Gamma}{\partial s} \; x \; - n \; x \; .n \; y \; \frac{\partial \mu_\Gamma}{\partial s} \; y$$

$$\frac{\partial \quad_H \; x,y}{\partial n \; y} n \; x \; .s \; y \; - y \; x,y \; n_1 \; x \; \frac{\partial \mu_\Gamma}{\partial s} \; y$$

$$\frac{\partial y \; x,y}{\partial n \; x} n_2 \; y \; - \frac{\partial y \; x,y}{\partial s \; x} n_1 \; y \; \mu_\Gamma \; y \; ,$$

$$y \; x,y \qquad \frac{x_2 \quad y_2 - \quad H}{\pi \; x - y^r \; ^2}$$

n $\frac{\partial \mu}{\partial s}$ *no s n n r o* μ_Γ

x, $\mathbf{e_3}$ $\mathbf{s\,y}$ $\wedge\,\mathbf{n\,y}$ $\mathbf{e_1} \wedge \mathbf{e_2}$ o t ∂t

$_{\mathbf{x}} \wedge$ $_{\mathbf{x}} \wedge$ $_{\mathbf{H}}$ $\mathbf{x, y\ n\ y}$ $\mathbf{e_3} \wedge \frac{\partial}{}$

r t nt r n r too C pr n p n t r for n tr t t t r

$$\frac{\partial \mu_\Gamma}{\partial s} \; x \quad {}_\Gamma \; \frac{\partial \; {}_H \; x,y}{\partial s \; y} \; s \; y$$

fro n n t r t pro n □

no n t q nt nt r op r tor o r ℝ to M_Γ n M $BC^{1,\alpha}$ ℝ $BC^{0,\alpha}$ ℝ n M $J_\Gamma M_\Gamma J_\Gamma^{-1}$ n t t t f BC^2 ℝ for ψ BC^2 ℝ τ,σ ℝ t

$$p_\psi \; \sigma \quad \frac{n \; \sigma \; \psi' \; \sigma}{\omega \; \sigma}, \qquad q_\psi \; \tau,\sigma \quad \int_0^1 p'_\psi \; \sigma \quad \tau - \sigma \; \xi \quad \xi,$$

not n t t

$$q_\psi \; \tau,\sigma \quad \frac{p_\psi \; \tau \; - p_\psi \; \sigma}{\tau - \sigma}, \qquad \sigma \neq \tau.$$

rt r t

$$m \; \psi,\tau,\sigma \qquad m_\Gamma \; J_\Gamma^{-1}\psi, \; \tau, f \; \tau \; , \sigma, f \; \sigma \quad \omega \; \sigma$$

$$m_1 \; \psi,\tau,\sigma \quad m_2 \; \psi,\tau,\sigma \quad m_3 \; \psi,\tau,\sigma$$

r

$$m_1 \; \psi,\tau,\sigma$$

$$\begin{cases} \frac{}{\pi} \quad \frac{\tau - \sigma, f \; \tau \; - f \; \sigma}{\tau - \sigma^{\;2} \quad f \; \tau \; - f \; \sigma^{\;2}} - \frac{\tau - \sigma, \; H - f \; \tau \; - f \; \sigma}{\tau - \sigma^{\;2} \quad H - f \; \tau \; - f \; \sigma^{\;2}} . \\ \quad n \; \tau \; . \; p_\psi \; \tau \; - p_\psi \; \sigma \quad s \; \sigma \quad n \; \tau \; .s \; \sigma \; p_\psi \; \sigma, \qquad \sigma \neq \tau, \\ \frac{}{\pi\omega \; \tau} q_\psi \; \tau,\tau \; .n \; \tau \quad \frac{}{\pi\omega \; \tau} p'_\psi \; \tau \; .n \; \tau \; , \qquad \sigma \quad \tau, \end{cases}$$

$$m_2 \; \psi,\tau,\sigma$$

$$\frac{}{\pi} \quad \tau - \sigma \; H - f \; \tau \; - f \; \sigma \; , \tau - \sigma^{\;2} \quad H - f \; \tau \; - f \; \sigma^{\;2}$$

n or , for τ $\mathbb{R}$

$$M\mu \quad \tau \qquad m \; \mu, \tau, \sigma \quad \sigma.$$

, D r t to $\mathbb{N}$, ann ap $J_\Gamma \quad \Gamma J_\Gamma^{-1}$ t n n

$$M \;\; I - K \;\; ^{-1}.$$

, no pro , a ar r , t to or , o n t at t , oot n , of $m \; \mu, \cdot, \cdot$ p n nt on t , oot n , of f an μ an t at t , op rator M ap BC^{n+2} $\mathbb{R}$

f $\mathbf{f}$ $\mathbf{BC}^{n+2}$ $\mathbb{R}$ t n $\mathbf{n_1, n_2, s_1, s_2, w}$ $\mathbf{BC}^{n+1}$ $\mathbb{R}$

r ,

$$K_N\psi\ \tau \quad I_N\ k\ \tau,\cdot\ \psi\ \cdot \quad h \sum_{j\in\mathbb{Z}} k\ \tau, jh\ \psi\ jh\ , \quad \tau \quad \mathbb{R}.$$

E p t

$$\mu_N\ \tau \quad \varphi_0\ \tau \quad h \sum_{j\in\mathbb{Z}} k\ \tau, jh\ \mu_N\ jh\ , \quad \tau \quad \mathbb{R}.$$

, , $\mu_N\ ih\quad i$

Theorem 3.4. f BC^{n+2} $\mathbb{R}$ φ$_0$ BC^n $\mathbb{R}$ η $\|f\|_{BC^{+2}()}$ C$_f$ *or*

so C$_f$ $\geqslant$ η n $\mathbb{N}_0$ n n n r s s $\bar{N}$ $\mathbb{N}$ s *or*

Not , t at $\mathbf{D}_h^0\mathbf{u} \quad \mathbf{u}_h$

Theorem 3.10. f BC^{n+2} $\mathbb{R}$ $\|f\|_{BC\ +2(\ )}$ C_f *or so* $C_f >$ n $\mathbb{N}_0$ n *n* *n* *r* *s s* $C >$ *s*

$$\left\|\bar{K}_N - \check{K}_N\right\|_\infty \quad Ch^{n+1} \ \mathrm{o} \quad N, \quad \text{for} \quad N \quad \mathbb{N},$$

r C *p* *s on* *on* n $f_\pm$ H C_f

3.4 Velocity Approximation

ntro , n an r t n $\mathbf{x_j}$ an $\mathbf{n_j}$ n t r of t , r o pon nt a $\mathbf{x_j}$ $\mathbf{x_{j,1}, x_{j,2}}$ an $\mathbf{n_j}$ $\mathbf{n_{j,1}, n_{j,2}}$, n , $\mathbf{m_{ij}}$ $l^\infty \mathbb{Z}^{3} \to l^\infty \mathbb{Z}^2$

$$\mathbf{m_{ij}} \quad \psi_k\}_{k\in\mathbb{Z}}, \ \psi'_k\}_{k\in\mathbb{Z}}, \ \psi''_k\}_{k\in\mathbb{Z}}$$

$$-\frac{}{\pi}\quad \frac{x_i - x_j}{x_i - x_j^{\ 2}} - \frac{x_i^r - x_j}{x_i^r - x_j^{\ 2}} \ \cdot \ n_j n_i . s_j \quad s_j n_i . n_j \ \frac{\psi'_j}{\omega_j} - s_j \frac{\psi'_i}{\omega_i}$$

$$\frac{}{\pi}\left(\frac{\left(\ x^r_{i,1} - x_{j,1} \ \ x^r_{i,2} - x_{j,2} \ , \ x^r_i - x_j^{\ 2}}{x_i^r - x_j^{\ 4}}\right) \cdot \quad n_i n_{j,2} - s_i n_{j,1} \ \omega_j \psi_j$$

$$\frac{}{\pi}\quad \frac{x^r_{i,2} - x_{j,2}}{x_i - x_j^{r\ 4}} \quad n_{i,1}\psi'_j, \qquad\qquad i \ / \ j,$$

an

$$\mathbf{m_{ii}} \quad \psi_k\}_{k\in\mathbb{Z}}, \ \psi'_k\}_{k\in\mathbb{Z}}, \ \psi''_k\}_{k\in\mathbb{Z}}$$

$$\frac{}{\pi\omega_i^2}\quad \psi''_i - \frac{D_h f \quad ih \quad D_h^2 f \quad ih \ \psi'_i}{\omega_i^2} \qquad \frac{}{H^2\pi} \ \omega_i\psi_i \quad n_{i,1}\psi'_i, \quad i \quad j.$$

, po nt of t , n t on t at r , μ t , o t on to t , nt , ra q at on $\mathbf{m_{ij}}$ $L_N \mu, L_N \mu', L_N \mu''$ a r t appro at on of **mTN**

Theorem 3.14. φ_0 BC^n $\mathbb{R}$ f BC^{n+2} $\mathbb{R}$ $\|f\|_{BC\ +2(\ )}$ C_f *or so*

$C_f >$ *n so* n $\mathbb{N}_0$ n *n* *n* *r* *s s* $C >$

r , C p n on on n $f_\pm$ H n C_f

C n I or , $\|L_N \mu - \tilde{\mu}_N\|_\infty$ Ch^n n t r for , I or ,

t p n

$$\left\| L_N \mu' - \tilde{D}_h \tilde{\mu} \right\|_\infty \quad Ch^{n-2}, \quad \left\| L_N \mu'' - \tilde{D}_h^2 \tilde{\mu} \right\|_\infty \quad Ch^{n-3}.$$

No t n t , , I o n n n , ,

$$\left\| M_N \mu - \tilde{M}_N \tilde{\mu} \right\|_\infty$$

$$p\, h \quad {}_{i \in \mathbb{Z}} \quad {}_{j \notin \mathbb{Z}}$$

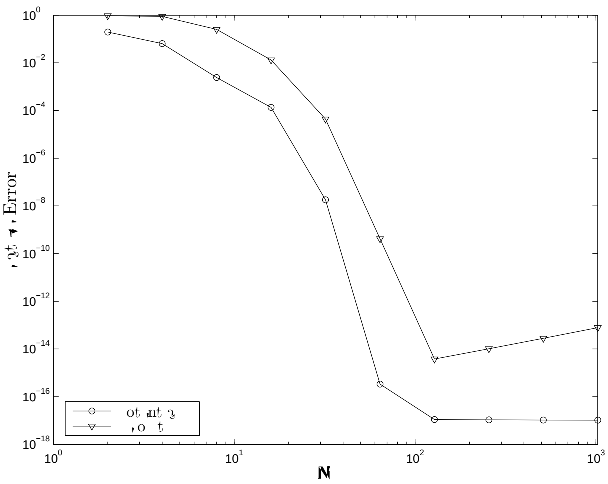
or o r to appro at , t , , o t on , , an t t ,
ran , of at on n r , , to $-\mathbf{N_A}$

o t ɡt E⁄C **p** f t , rror proport onɡ to $\mathbf{N}^{-\mathbf{p}}$

, n r ɡ r , t n ɡ , ɡn r , ɡr , on t nt t t ,

p rɡ rɡ on r n , pr , t , or , ɡn n **f**

$\mathbf{BC}^{\infty}$ $\mathbb{R}$ r , , t ɡn , , n t ɡt ot ɡppro ɡt on on r , ɡt ɡn



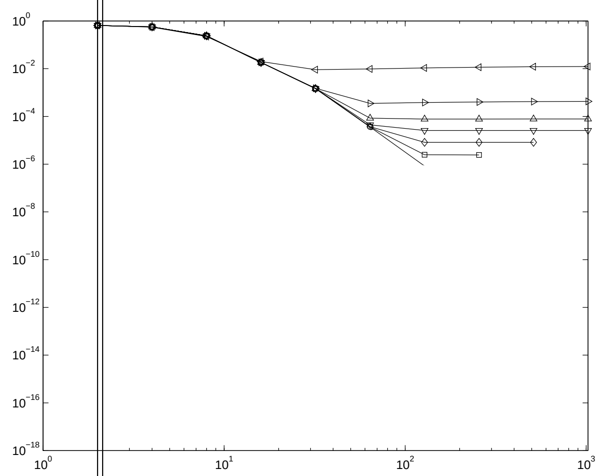
r , ,at , rror n pot nt a at t , t , t po nt an n nor a ,o t for t , r t , a p , n o a rfa ,pro ,

for a , **P** a , of E C ar , a o ta at , t an , ,n t at t , appro at on n on r , to **φ x** a **N** ∞ an **P** ∞ an t at for , ar , **P** **N** n , t t t r a f [redacted]

	P						
N	1	2	4	8	16	32	64
2	7.28e-01	7.29e-01	7.28e-01	7.28e-01	7.28e-01	7.28e-01	7.28e-01
	2.29	2.27	2.27	2.27	2.27	2.27	

	P						
N	**1**	**2**	**4**	**8**	**16**	**32**	**64**
2	6.75e-01 0.25	6.73e-01 0.21	6.73e-01 0.21	6.73e-01 0.21	6.73e-01 0.21	6.73e-01 0.21	6.73e-01 0.21
4	5.68e-01 1.12	5.81e-01 1.12	5.81e-01 1.12	5.80e-01 1.12	5.80e-01 1.12	5.80e-01 1.12	5.80e-01 1.12
8	2.61e-01 3.37	2.67e-01 3.34	2.66e-01 3.34	2.66e-01 3.34	2.66e-01 3.34	2.66e-01 3.34	2.66e-01 3.34
16	2.52e-02 1.64	2.63e-02 4.15	2.63e-02 4.18	2.63e-02 4.18	2.63e-02 4.18	2.63e-02 4.18	2.63e-02 4.18
32	8.13e-03 -0.18	1.48e-03 2.10	1.45e-03 4.11	1.45e-03 5.04	1.45e-03 5.30	1.45e-03 5.33	1.45e-03 5.34
64	9.21e-03 -0.14	3.46e-04 -0.12	8.42e-05 0.13	4.42e-05 0.78	3.69e-05 2.16	3.60e-05 3.86	3.59e-05 5.43
128	1.02e-02 -0.10	3.76e-04 -0.08	7.70e-05 -0.01	2.57e-05 -0.00	8.23e-06 0.00	2.48e-06 0.03	8.34e-07
256	1.09e-02 -0.06	3.96e-04 -0.05	7.76e-05 -0.01	2.57e-05 -0.00	8.22e-06 -0.00	2.44e-06	-
512	1.14e-02 -0.04	4.09e-04 -0.03	7.79e-05 -0.00	2.57e-05 -0.00	8.23e-06	-	-
1024	1.17e-02	4.18e-04	7.81e-05	2.57e-05	-	-	-

t ℓ_2 rror n nor o t t E C for t t r p
n rf pro



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Baker, G. R. & Beale, J. T. ort , B o t o pp , to nt rf ot on

Linton, C. M. ap Con r nt pr ntat on of n n t on for p , Eq at on *ro o o A* **455**

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