The No Response Test for the Reconstruction of Polyhedral Objects in Electromagnetics

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Abstract

We develope a *No Response Test* for the reconstruction of some polyhedral obstacle from one or few time-harmonic electromagnetic incident waves in electromagnetics. The basic idea of the test is to probe some region in space with waves which are small on some test domain and, thus, do not generate a response when the scatterer is inside of this test domain.

This is the rst formulation of the No Response Test for electromagnetics. We will prove *convergence* of the method for testing a non-vibrating domain B whether the far eld pattern of some scattered time-harmonic eld is analytically extendable into the interior of B. We will describe algorithmical realizations of the No Response Test. Finally, we will show the feasibility of the method by reconstruction of polygonal objects in three dimensions.

Key words: Electromagnetic Waves, Maxwell Equations, Inverse Scattering, Object Reconstruction, Sampling Method, No Response Test

1 Introduction

Using electromagnetic waves for probing and investigation of unknown regions in space is widely employed in the natural sciences, ranging from optics and microscopy via X-Ray science to radar and electromagnetic tomography. An introduction into the mathematical theory of inverse problems for acoustic and electromagnetic waves can be found in (Colton and Kress, 1998). A survey about several more recent methods is given by (Potthast, 2006) and a comparative study of some of these methods can be found in (Honda et al., 2007).

Our goal here is to formulate and analyse the *No Response Test* rst suggested in acoustics by (Luke et al., 2003) for object identi cation in r

In particular, we will provide a convergence analysis for the reconstruction of a polygonal perfectly conducting object in three dimensions from the far field pattern of two incident time-harmonic electromagnetic waves.

Let D be a polyhedral domain in $\mathbb{R}^3$. We consider the following electromagnetic scattering problem. The propagation of time-harmonic electromagnetic fields in a homogeneous media is governed by the *Maxwell equations*

$$\operatorname{curl} E \quad i \quad H = 0; \tag{1}$$
$$\operatorname{curl} H + i \quad E = 0; \tag{2}$$

in $\mathbb{R}^3 \setminus \overline{D}$ where   is the real positive *wave number*. At the boundary of the scatterers the total field E satisfies the *Dirichlet boundary condition*

$$E = 0 \text{ on } \partial D. \tag{3}$$

We look for solutions of the form $E := E^i + E^s$, and $H = \frac{1}{i}\operatorname{curl} E$, of (2) and (3) where the *scattered field* $(E^s; H^s)$ is assumed to satisfy the Silver-Müller radiation condition

$$\lim_{r \to \infty} (H^s \quad x \quad rE) = 0; \tag{4}$$

$r = |x|$

where $(E^\infty(\cdot;d;p); H^\infty(\cdot;d;p))$ defined on the unit sphere $\mathbb{S}$ is called the far field pattern associated to the incident field $(E^i(\cdot;d;p); H^i(\cdot;d;p))$.

We will study and solve the shape reconstruction problem for polygonal domains.

Definition 1.1 (Shape reconstruction problem) *Given $E^\infty(\cdot;d;p)$ on $\mathbb{S}$ with N directions, $N \geq 1$ of incidence $d_i; i = 1;\dots;N$ and polarization $p_j; j = 1;\dots;M$ for the scattering problem (2) - (4) reconstruct the obstacle D.*

2 The No Response Test in Electromagnetics

2.1 The Idea of the No Response Test

We consider scattering of incident plane waves with direction of incidence d and with polarization p_i for $i = 1;2$. We assume that we have

$$p_i \perp d; \quad i = 1;2 \text{ and } p_1 \text{ and } p_2 \text{ are not co-linear}. \tag{7}$$

For every $g \in L^2(\mathbb{S})$, we set $v_g(x) := \int_{\mathbb{S}} e^{i\kappa\theta\cdot x} g(\theta)\,ds(\theta)$ to be the scalar Herglotz wave corresponding the density g.

Then we define

$$I(B) = \lim_{\varepsilon\to 0} \left\{ \sum_{i=1}^{2} \left| \int_{\mathbb{S}} E^\infty(\theta;d;p_i) g(\theta)\, ds(\theta) \right| : \|v_g\|_{C^1(B)} \le \varepsilon \right\} \tag{8}$$

for any nonvibrating domain B, i.e. B is in the set

$$\mathcal{B} := \left\{ B : \begin{array}{l} \text{the homogeneous interior Maxwell problem for } B \text{ does} \\ \text{have at most the trivial solution} \end{array} \right\} \tag{9}$$

The idea of the No Response Test is to test if the unknown obstacle D is included in some $B \in \mathcal{B}$ by computing $I(B)$. In the next subsection, we show how this idea can be used to reconstruct the convex hull of D.

2.2 Convergence of the NRT.

Our key goal is to prove the following reconstruction of the convex hull of D.

Theorem 2.1 (No-response characterization) *The convex hull of D is characterized by*

$$\overline{CH(D)} = \bigcap_{B\in\mathcal{B};\, I(B)=0} B. \tag{10}$$

Further, as a consequence of this results we immediately obtain the following uniqueness statement.

Corollary 1 *The convex hull of a polygonal domain in $\mathbb{R}^3$ is uniquely determined by the scattered field for one ($N = 1$) directions of incidence and $M = 2$ polarizations.*

Definition 2.2 (Admissible vertices) *We call a convex vertex of ∂D admissible if we can continue at least one of the faces of ∂D to the infinity without crossing ∂D, again.*

We call a vertex an exterior convex vertex *if it is in the boundary $\partial CH(D)$ of the convex hull $CH(D)$ of D.*

Remark 2.3 *The exterior convex vertices characterize the convex hull of D.*

We will need the following identity

$$E^{\infty}(\hat{x}; d; p) = \frac{i\kappa}{4\pi}\int_{\partial D}\Big\{\nu(y)\times E^s(y; d; p) + [\nu(y)\times H^s]\times\hat{x}\Big\}\, e^{-i\kappa\hat{x}\cdot y}\, ds(y) \tag{11}$$

given by using the Straton-Shu formula in $\mathbb{R}^3\setminus\overline{D}$ for $E^s(\cdot\,; d; p)$, $H^s(\cdot\,; d; p)$ and $\Phi(\cdot\,; y)$ and their asymptotic behavior at infinity (see (Colton and Kress, 1998), Theorem 6.8) where ν is the outward normal of ∂D. Let $g\in L^2(\mathbb{S})$, then

$$\begin{aligned}\int_{\mathbb{S}} E^{\infty}(-\hat{x}; d; p)g(\hat{x})\,ds(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\Big\{&\nu(y)\times E^s(y; d; p)\cdot\operatorname{curl} v_g \\ &+ \frac{1}{i\kappa}[\nu(y)\times H^s]\cdot\operatorname{curl}\operatorname{curl} v_g\Big\}\, ds(y)\end{aligned} \tag{12}$$

Let $B\subset\mathbb{R}^3$ be a convex non-vibrating domain for the Maxwell equation, i.e. let the interior homogeneous boundary value problem with boundary condition $\nu\times E = 0$ be uniquely solvable. We consider two cases:

(A) $\overline{D}\subset\overline{B}$. Suppose that $|v_g|\le\epsilon$, then from (12), we have , for $d = d_i$ and

$p = p_i^j$,

$$\left| \int_S E^\infty(\hat{x}; d; p) g(\hat{x}) ds(\hat{x}) \right| \le C$$

with a uniform constant C. This implies that $I(B) = 0$.

(B) $\overline{D} \not\subset \overline{B}$. In this case, we can find at least one exterior convex point of ∂D which is not in $\overline{B}$. We denote by z_0 one of these points. We consider a sequence of points z_q included in $\mathbb{R}^3 \setminus \overline{D}$ tending to z_0.

We consider the *multipole fields*

$$\psi_q := \frac{}{2\gamma(z_q;\alpha_q)} (h_q \cdot \nabla_z)^{\alpha_q} \Phi(x; z_q) \qquad (13)$$

where h_q is a unit vector, α_q is a multi-integer and

$$\gamma(z_q;\alpha_q) := \sup_{y \in \overline{B}} \{ |(h_q \cdot \nabla_z)^{\alpha_q} \Phi(x; z_q)| \}.$$

For every q we take $g_n^q \in L^2(S)$ such that $v[g_n^q]$ tends to ψ_q in $C^1(B \cup D)$.

From (12), we get:

$$\lim_{n \to \infty} \int_S E^\infty(\hat{x}; d; p) g_n^q(\hat{x}) ds(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \Big\{ (\nu(y) \times E^s(y; d; p)) \cdot \operatorname{curl} \psi_q + \frac{1}{i\omega} [\nu(y) \times H^s] \cdot \operatorname{curl}\operatorname{curl} \psi_q \Big\} ds(y) \qquad (14)$$

Using the Stratton-Chu formula and due to the form of ψ_q, we have:

$$\lim_{n \to \infty} \int_S E^\infty(\hat{x}; d; p) g_n^q(\hat{x}) ds(\hat{x}) = \frac{}{2\gamma(z_q;\alpha_q)} (h_q \cdot \nabla_z)^{p} E^s(z_q; d; p) + \int_{\partial\Omega_R} (\nu(y) \times E^s(y; d; p)) \cdot \operatorname{curl} \psi_q + \frac{1}{i\omega} [\nu(y) \times H^s] \cdot \operatorname{curl}\operatorname{curl} \psi_q \, ds(y)$$

where Ω_R is a ball of radius R

is uniformly bounded in a compact set V, where here the boundedness is understood componentwise. Then $E^s(z;d;p)$ is analytically extensible into an open neighbourhood $V = \{x : d(x;V) < \}$ of V.

Proof of Lemma 2.4. The basic result can be found in (Honda et al., 2007) or (Potthast, 2007). The authors use (16) as a bound for the Taylor coe cients of the function and construct an analytic extension into the open neighbourhood of V by multi-dimensional Taylor series. □

Lemma 2.5 *Consider the scattered elds $E^s(;d;p_i)$ for $i = 1;2$ in a neigbourhood of an exterior c. Then there exists at least one pair $(d;p_i)$ such that $E^s(z;d;p_i)$ is not analytically extensible into an open neighbourhood of the point z_0.*

Proof of Lemma 2.5. By de nition of the exterior vertex, there exists at least one face around z_0 which can be extended to in nity without crossing again $@D$. On this face we have $E = 0$. Since E is extendable near z_0 then it satis es, with H, the Maxwell equations around z_0. Hence it is real analytic near z_0. This means that $E = 0$ on an in nite part of the plan having as a normal

>From (15) and Corollary 2, we have

$$\lim_{q\to\infty}\lim_{n\to\infty} \left| \int_S E^{\infty}(\ ; d; p) g_n^q(\ ) ds(\ ) \right| = \infty.$$

For $\ > 0$ xed, we can take $q; n$ large enough such that

$$\|v_{g_n^q}\|_{C^1(B)} \quad \|v_{g_n^q} \quad {}_q\|_{C^1(B)} + \| \ {}_q\|_{C^1(B)} \qquad .$$

This implies that $I(B) = \infty$. □

3 The Realization of the No Response Test

The basic goal of this chapter is to develop the numerical realization of the No Response Test. We will rst describe general preparation steps which are uniform for all subsequent realizations of the No Response Test. Then, we will describe an e cient approach to realize the No Response Test numerically.

We consider an electromangetic *Herglotz wave function*

$$V[a](x) := \frac{i}{\ }\mathrm{curl}\,\mathrm{curl} \int_S e^{i \ x}\ a(\ )\, ds(\ ); \quad x \in \mathbb{R}^3 \tag{18}$$

with density $a \in T(S)$, where $T(S)$ denotes the set of all vector elds $a \in L^2(S)$ with $(\hat{x})\ a(\hat{x}) = 0$ for all $\hat{x} \in S$. Clearly, rly, rly, rly, rl TfTd [(2)]TJ/F43 11.9552 Tf 11.985 0 Tc

With $\operatorname{curl}_x(\varphi(x)a) = \operatorname{grad}_x \varphi \quad a$ when a does not depend on x we obtain

$$(Ha)(x) = i \int_{S} e^{i \quad x} \; (\quad a(\;)) \qquad ds(\;); \quad x \in \partial B; \tag{22}$$

and for tangential field $a(\;) \in T(S)$ this reduces to

$$(Ha)(x) = i \int_{S} e^{i \quad x} \; a(\;) \, ds(\;); \quad x \in \partial B; \tag{23}$$

First, we note important properties of equation (21).

Lemma 3.1 *The equation (21) does not have a solution $a \in L^2(S)$.*

Proof. Assume that there is a solution $a \in L^2(S)$ of equation (21). Then both fields $V[a]$ and $(\; ; z)$ solve the Maxwell equations in B with identical boundary values. By the well-posedness of the interior Dirichlet problem in B the two fields will coincide in B

and Kress, 1998) the function

$$W[\;](x) := \operatorname{curl}\operatorname{curl}\int_{\partial B} \;(x;y)\;\;(y)\,ds(y); \quad x \in \mathbb{R}^3 \tag{25}$$

has far eld $1/4\;\;H\;\; = 0$. According to Rellichs lemma Theorem 6.9 of (Colton and Kress, 1998) the eld $W[a]$ vanishes in $\mathbb{R}^3 \setminus$

scalar equation

$$Hg = \ (\ ;z) \text{ on } \partial B \tag{30}$$

with some parameter $\ > 0$, $B \ := \{x \in \mathbb{R}^3 : d(x;B) \ \ \}$ and

$$(Hg)(x) := \int_S e^{i \ x} \ g(\ )\, ds(\ ); \ \ x \in \mathbb{R}^m: \tag{31}$$

Then, the $a := p \ \ g(x)$ is a solution to (21). >From a algorithmical point of view to solve a scalar equation is clearly much more e cient. With the same arguments as above we can employ Tikhonov regularization for its solution, i.e. we calculate

$$g_{z;} \ := (\ I + H \ H)^{\ 1} H \ \ (\ ;z) \text{ on } \partial B \tag{32}$$

for $\ > 0$. Also, it has been shown in (Ben Hassen et al, 2006) that by inserting the approximation of $\ (\ ;z)$ into the Stratton-Chu formula we obtain an approximation

$$\int_S E^{1}(\hat{x}) g_{z;}\ (\hat{x})\, ds(\hat{x}) \ \to \ E^s(z); \qquad \to 0 \tag{33}$$

in the sense that given $\ > 0$ there is $g_z \in L^2(\mathbb{S})$ such that

$$E^s(z) \ \ \int_S E^{1}(\hat{x}) g_z(\hat{x})\, ds(\hat{x}) \tag{34}$$

which holds under the condition that the eld E^s can be analytically extended into $\mathbb{R}^3 \setminus B$.

We now describe a direct realization of the No Response Test via the functional

$I(B;d;p;\ ) := \sup \Big\{ \int_S E^{1}($ Tf 7. 5 5 - 4 . 338 F23 0 (N0) - 23 0 ((N0) - 23

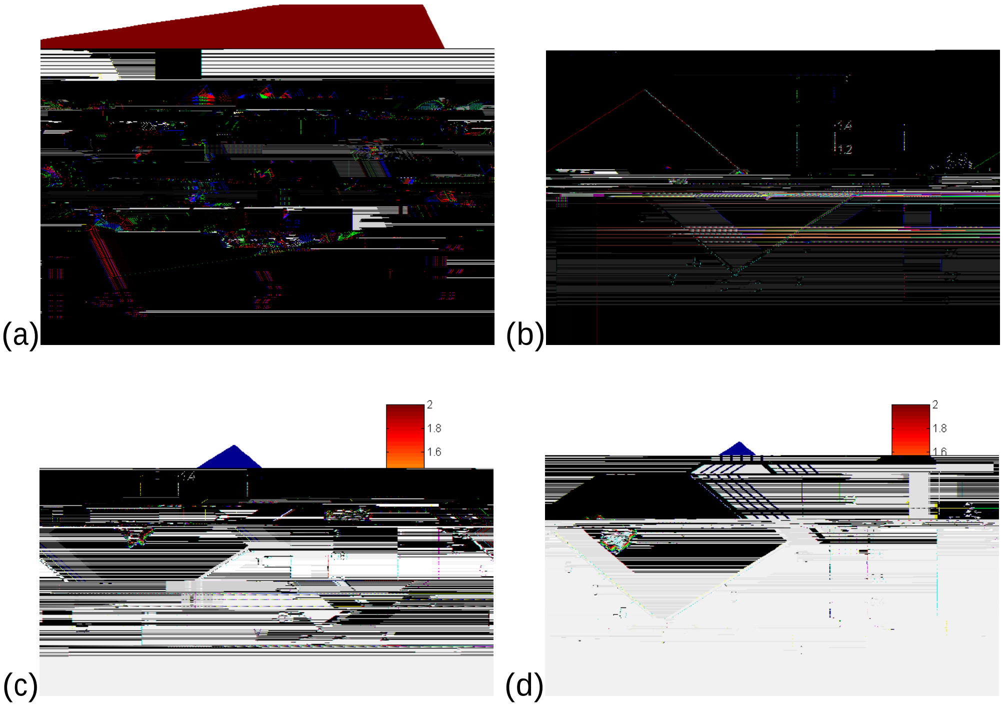

Fig. 2. In (a) we demonstrate the behaviour of the indicator function of the No Response Test for *one* electromagnetic wave only. Here, every image point z corresponds to a test domain $G(z)$ with $z \in \partial G(z)$ and $G(z) \subset \{ y \in \mathbb{R}^3 : y_1 < z_1 \}$. The blue area clearly indicates all such domains for which $D \subset G(z)$, i.e. it indicates a successfull No Response Test for the location of the domain. A second step is then to build the intersections (38). Figure (b) - (d) show reconstructions of some polygonal domain from the far field pattern of *one* wave via the No Response Test functional with balls as test domains. Here, we show a slice of the mask on a plane intersecting the scatterer. The results here have not been optimized to yield good shape reconstructions, but we worked on a grid with cells of size $h = 0.5$. Clearly, we can easily identify the location and size of the scatterer and prove the feasibility of the ideas described above.

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