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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Elena Panti.

Abstract

In this dissertation we explore the Boundary Element Method for heat transfer in a buried pipe. We are interested in modelling the steady-state heat transfer from buried pipes. We are studying the temperature through Laplace's

Contents

Chapter 1

Introduction

This dissertation explores the Boundary Element Method for the Heat Transfer in a buried pipe. Heat transfer occurs due to temperature difference between the pipeline fluid and the ambient fluid which is air or water, overcoming thermal resistances offered by the pipe, coatings and ground.
The state of the fluid (oil, liquid or gas) i.e. the density and the viscosity of the fluid, is defined by the temperature.

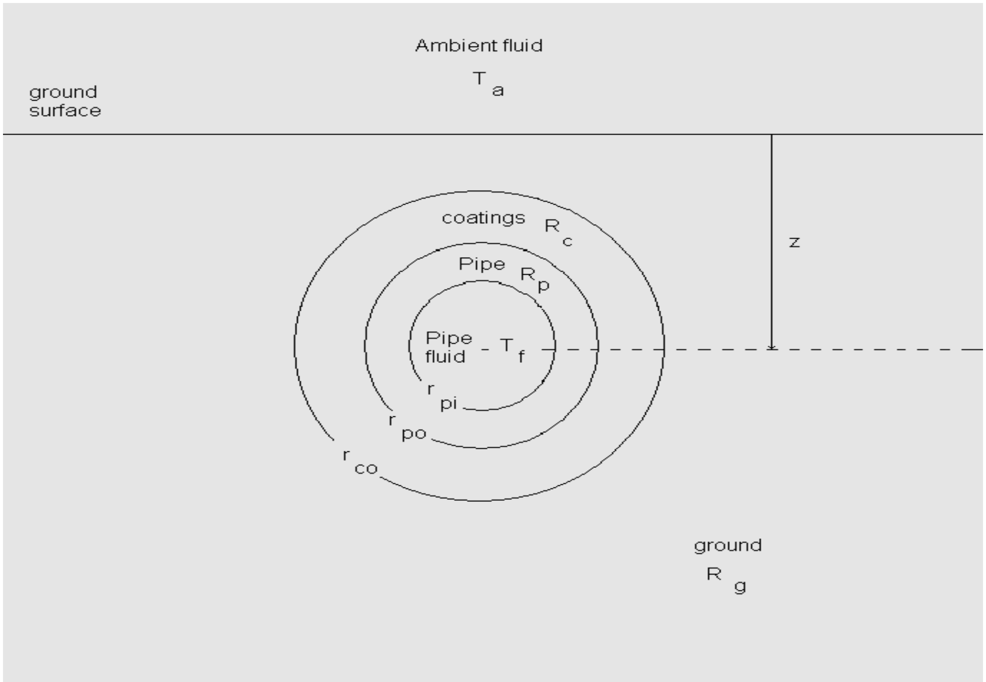


Figure1: Buried pipe (source: Partially Buried Pipe Heat Transfer (Powerpoint), Chuk Ovuworie from Schlumberger Company)

=radius of the pipe inside

=radius of the pipe outside

We can express, at steady state, the rate of heat flow between the pipeline and the ambient fluid as:

$$= -2\pi \qquad (\quad - \quad)$$

where

The temperature difference between the pipe inside and outside walls is given by the following equation:

$$- \qquad = \frac{\ln(\quad / \quad)}{2\pi}$$

where

$=$ radius of the pipe inside

$=$ radius of the pipe outside

=the pipe thermal conductivity

= the temperature of the pipe inside

$=$ the temperature of the pipe outside

The easiest way to approach this problem is to assume radial symmetry.

In this case, simplifying the problem within the pipe to one-dimension, dependent only on   we can easily solve the boundary value problem using traditional techniques. Below are the assumptions made in this analysis:

Assumptions:

- Heat flow, denoted by Q (radial heat flow per length of pipe), is radially symmetric within the pipe and the coatings such that T=T(r).
- Heat flow is in a steady state (dQ/dt = 0)
- Conservation of heat energy reduces down to Laplace's equation

$$\nabla^2 \quad = \frac{\partial^2}{\partial \quad ^2} + \frac{1}{\ }\frac{\partial}{\partial} = 0$$ $\longrightarrow$ It is Laplace's equation, but now the $\frac{\partial}{\partial\theta}$ term has disappeared.

(where r, θ are polar coordinates on the centre of the pipe).

Solving Laplace's equation without the $\frac{\partial}{\partial\theta}$ term, we obtain the general solution for temperature:

$$(\quad) = \quad \ln \quad +$$

we have:

In this dissertation we begin by introducing the Boundary Element Method and we separate our problem in three stages:

(1) Interior problem (bounded problem)

(2) Exterior problem (unbounded problem)

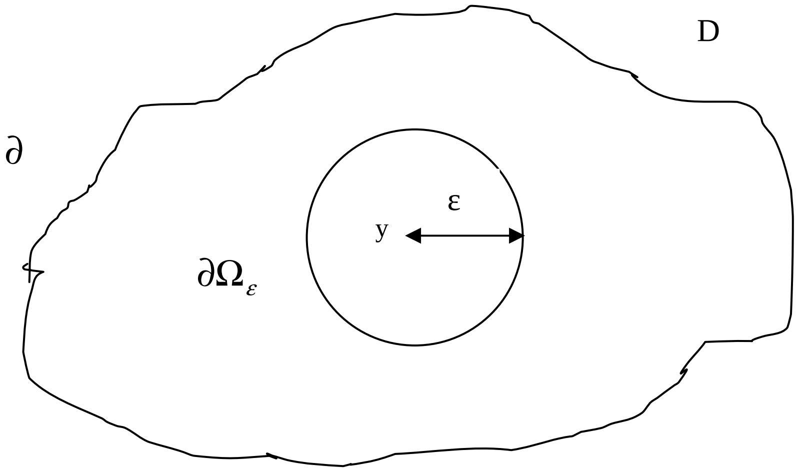
(3) Full problem.

Green’s second identity:

The Green’s first identity is also valid when interchanging then ^{2} ( )

<u>Suppose ΔG=0 inside a domain D</u>

Suppose $\in$



We choose G to be the solution of $\Delta \quad = 0$ in D/Ω_ε, hence we have:

$$\int_{\partial \ \varepsilon} \left(\frac{\partial}{\partial} - \frac{\partial}{\partial} \right) \quad = 0 \qquad \partial \quad \text{——} \quad \frac{\partial}{\partial} \quad + {}_{\partial\Omega} \quad \frac{\partial}{\partial} - \frac{\partial}{\partial}$$

Θ·Θ⎪ (·γ∂≤⎪ (≥⇔γ≈(Θ⇔9 ⎪∑⎪ ⎪ ΦΚ ·≤≥≥Θ≤Θ9·≠ ⎪∑(Φ+Κ (]Θγ≈≤λ∑≠⎪]·∑≈∑≠≥⎪ (Φ∂Κ ((((Θ∑

Let $\varepsilon \to 0$

Then the 2nd part of (2.6) which is $\int_{\partial\Omega_\varepsilon} (\;\frac{\partial}{\partial} - \;\frac{\partial}{\partial}) \;\_$ can be separated in two parts:

(i) $\int_{\partial\Omega_\varepsilon} \;(\;)\frac{\partial}{\partial}(\;) \;\_ = -\frac{1}{2\pi\varepsilon}\int_{\partial\Omega_\varepsilon} \;(\;) \;\_$

$$= -\frac{1}{2\pi\varepsilon}\int_{\partial\Omega_\varepsilon} (\;(\;) + \varepsilon \;'(\;) + \;(\varepsilon^2)) \;\_$$

$$= \left[-\frac{1}{2\pi\varepsilon}\;(\;) - \frac{\;'(\;)}{2\pi} + \;(\varepsilon)\right]\int_{\partial\Omega_\varepsilon} \;\_$$

$$= \left[-\frac{1}{2\pi\varepsilon}\;(\;) - \frac{\;'(\;)}{2\pi} + \;(\varepsilon)\right]2\pi\varepsilon$$

$$= -\;(\;) - \varepsilon \;'(\;) + \;(\varepsilon^2) \qquad (\;)$$

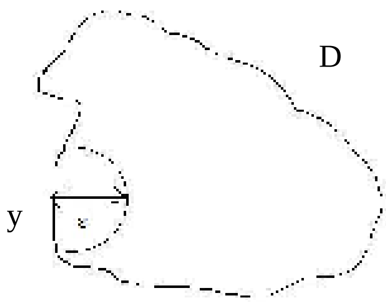
Hence,

$$(2.6)\quad \int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)_{-} = -\int_{\partial\Omega_{\varepsilon}}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)_{-} = -[-\ (\ )] = \ (\ )$$

$\in$ as $\varepsilon \to 0$.

which is the same result as (2.5) but is defined in a slightly more rigorous answer.

Suppose now, $\in \partial$ (y is on the boundary).In this case, the same procedure as before can be applied with the difference that now we have a semicircle instead of a circle. Therefore the length of the boundary is π instead of 2π in the derivations above.



ε=radius of the circle
y=centre of the circle

Hence,

$$\int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)_{-} = -\int_{\partial\Omega_{\varepsilon}}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)_{-} = \frac{1}{2}\ (\ ) \qquad \in \partial \qquad (2.7)$$

2.1.2 Unbounded Problem

<u>Suppose ΔT=0 and ΔG=0 are outside the domain D</u> (exterior problem)

Suppose $\notin$

The following equation is equal to zero because is outside the domain as we have mentioned in equation (2.5) when $({}_{1.}, {}_{2}) \notin$.

$$\int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) + \int_{\partial\Omega_{\varepsilon}}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) + \int_{\partial\Omega}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) = 0$$

(2.8)

$$\Rightarrow \int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) = -\int_{\partial\Omega_{\varepsilon}}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) - \int_{\partial\Omega}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right)(\ ) = 0$$

Same as before

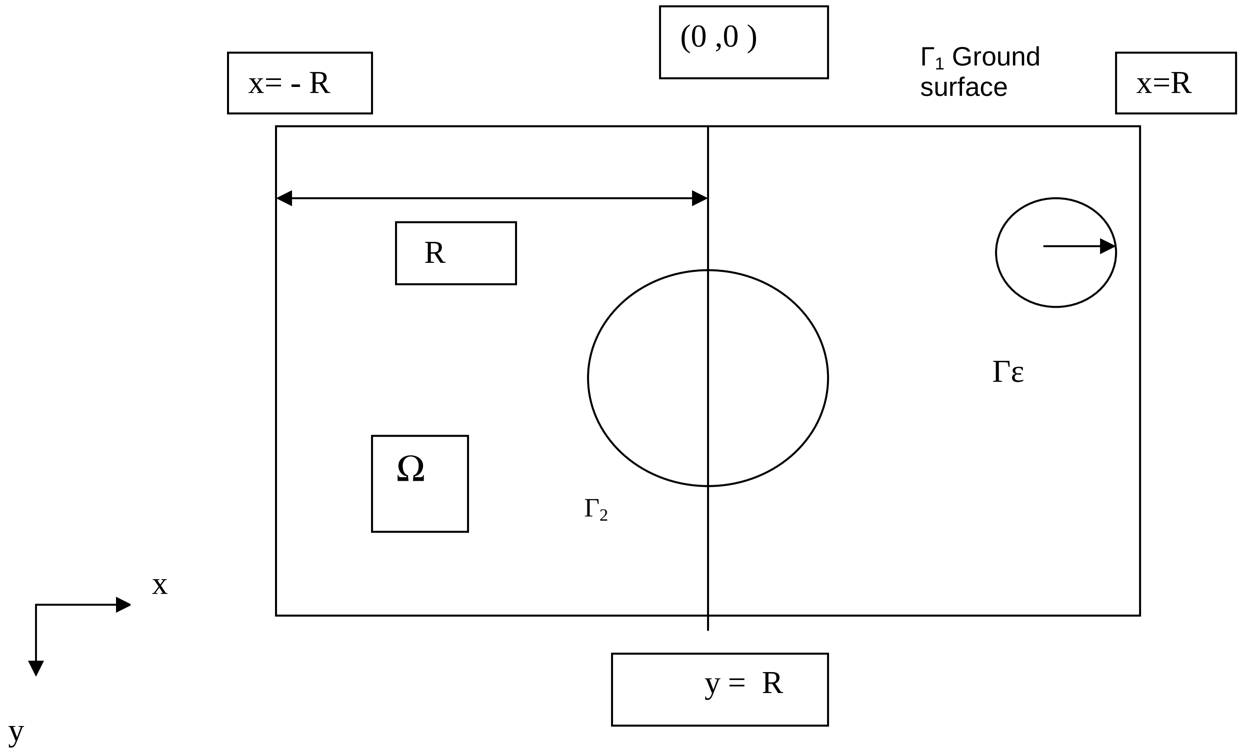
= - T(y)

Now, we are going to find (2.9) in the limit as $\mathrm{R} \longrightarrow \infty$

$$\int_{\partial\Omega} \left(\frac{\partial}{\partial} - \frac{\partial}{\partial} \right) \quad (\quad) = -\int_{\partial\Omega} \quad (\quad) \frac{1}{2\pi} \quad (\quad) - \int_{\partial\Omega} \frac{1}{2\pi} \ln \quad \frac{\partial}{\partial} \quad (\quad)$$

=0 (

2.1.4 Neumann Green's function for a half plane Problem



where Γ_2=pipe

Γ_ε =small circle

ε=radius of small circle

Ω =domain

We want to find the integral equation of the domain Ω .

<u>In the Domain :</u>

<u>Known:</u>

$$\nabla^2 \quad = 0$$

$$\nabla^2 \quad (\_ , \_) = 0$$

.

We choose $\hat{\ }$ such that $\frac{\partial \hat{\ }}{\partial} = 0$ on $\Gamma_{1.}$ (*)

$$\hat{\ }(\underline{\ },\ \underline{\ }) = \frac{1}{2\pi}\ln\frac{1}{|\underline{\ } - \underline{\ }|} + \frac{1}{2\pi}\ln\frac{1}{|\underline{\ } - \underline{\ }'|}$$

where $'$ is the reflection of in the line = - z and where $\underline{\ } = (\ _1,\ _2)$

$\underline{\ } = (\ _1,\ _2)$

$\hat{\ }(\underline{\ },\ \underline{\ })$ satisfies $\nabla^2\hat{\ } = \delta(\underline{\ } - \underline{\ }) + \delta(\underline{\ } - \underline{\ }')$.

To find the integral equation of the domain we add the integral equations of the pipe, the circle, theyecnnn BU3B3RUB3vuKj'EU'4'7vrK'EW94U9v j-EfW'BUvaKj'EURB7Rv

We consider the asymptotic condition as $\to \infty, \quad \pm \infty$

$$\to \quad + \quad . \tag{2.11}$$

Therefore,

$$\frac{\partial}{\partial} = 0 \quad \text{and} \quad \frac{\partial}{\partial} =$$

We solve for $= \quad - \quad -$ as $\to \infty, \quad \pm \infty$. (2.12)

Hence if we substitute (2.11) into (2.12) we have :

$= \quad - \quad - \quad = \quad + \quad - \quad - \quad = 0$ .Thus $\to 0$ as $\to \infty, \quad \pm \infty$

$$\frac{\partial}{\partial}, \frac{\partial}{\partial} \to 0$$

Thus the 3rd and 4th integrals are:

$$\int_{= \ , \ \in[0, \ ]} \Big(\underbrace{\cancel{\frac{\partial \hat{}}{\partial}}}_{=0} - \hat{} \underbrace{\cancel{\frac{\partial}{\partial}}}_{=0} \Big) \ (\ ) + \int_{=- \ , \ \in[0, \ ]} \Big(\underbrace{\cancel{\frac{\partial \hat{}}{\partial}}}_{=0} - \hat{} \underbrace{\cancel{\frac{\partial}{\partial}}}_{=0} \Big) \ (\ ) = 0$$

Therefore, the result of the addition of the 3rd and 4th integral is zero as R $\to \infty$.

In the 5th integral we considerr:

$$\hat{}(\underline{\ }, \underline{\ }) = \frac{1}{2\pi} \ln \frac{1}{|\underline{\ } - \underline{\ }|} + \frac{1}{2\pi} \ln \frac{1}{|\underline{\ } - \underline{\ }'|}$$

where $\Phi(\underline{\ }, \underline{\ }) = \frac{1}{2\pi} \ln \frac{1}{|\underline{\ } - \underline{\ }|}$, $\Phi(\underline{\ }, \underline{\ }') = \frac{1}{2\pi} \ln \frac{1}{|\underline{\ } - \underline{\ }'|}$

$$- \ (\ _{1},\ \ _{2}) - \int_{\Gamma_2} \frac{\partial\ \hat{}}{\partial}\ (\ )\ \ (\ ) = \int_{\Gamma_2} (-\ \hat{}\ \ _{2})\ \ (\ ) + \int_{\Gamma_1} (-\ \hat{}\ \ _{1})\ \ (\ )$$

We set

$$\int_{\Gamma_2} (-\ \hat{}\ \ _{2})\ \ (\ ) + \int_{\Gamma_1} (-\ \hat{}\ \ _{1})\ \ (\ ) = \ \ (\ )$$

Hence,

$$- \ (\ _{1},\ \ _{2}) - \int_{\Gamma_2} \frac{\partial\ \hat{}}{\partial}\ (\ )\ \ (\ ) = \ \ (\ )$$

We are going to solve this integral equation numerically. The general integral equation is of the following form:

$$(\ ) + \int_{\Gamma} \mathrm{K}(\ ,\ )\ (\ )\ \ = \ (\ )$$

Chapter 3

$$\int_0^{2\pi} ( ) \approx \frac{}{2}[\ (0)+2 \ (\ )+2 \ (2 \ )+......+2 \ ((\ -1) \ )+ \ (2\pi)]$$

$$= [\ (0)+ \ (\ )+......+ \ ((\ -1) \ )] \quad \text{[2π periodic function}$$

$$\Rightarrow \ (0)= \ (2\pi)].$$

Where $= \frac{2\pi}{}$ (=quadrature parameter)

In our case we replace by .

Therefore, we have:

$$(0) \quad (\ , \ ) \quad (\ ) \quad .... \quad (\ ,((\quad 1) \ ) \quad ((\quad 1) \ )] \quad (\ )$$

We have equations with unknowns:

$$(0), \quad (\ ), \quad (2 \ ), \quad (3 \ ),......, \quad ((\ -1) \ )$$

We write it as a matrix $=$

$$= \begin{bmatrix} + & & & & - & \\ & + & & & - & \\ & & + & & - & \\ & & & 1r & & \\ - & - & & ....1+ & [\ (\ -1) \ ,(\ -1) \ ] & \end{bmatrix}$$

- The conventional Nyström method is a simple and efficient mechanism for discretizing integral equations with non-singular kernels (K(x,y)).

- With a high-order quadrature rule, the solution one obtains by this method is a high-order approximation to the exact solution.

In the Nyström method we could use midpoint rule, Gaussian

Thus,

$$\cos(\ \ ) + \frac{1}{2\pi}\int_{0}^{2\pi} \sin(2\ \ \ +\ \ \ )\cos(\ \ ) \quad = \quad (\ \ )$$

$$\Rightarrow \cos(\ \ ) + \frac{1}{2\pi}\int_{0}^{2\pi} [(\sin(2\ \ )\cos(\ \ ) + \sin(\ \ )\cos(2\ \ )]\cos(\ \ ) \quad = \quad (\ \ )$$

$$\cos(\ \ ) \quad \frac{1}{2} \ {}_{0}^{2} (\sin(2\ \ )\cos^{2}(\ \ ) \quad \sin(\ \ )\cos(2\ \ )\cos(\ \ )) \qquad (\ \ )$$

TABLE 1

	$\| \; - \; \|_2$
2	3.9356e-016
4	2.8353e-016
8	3.6854e-016
16	5.2234e-016
32	7.5510e-016

We just use mesh points to evaluate the $\| \; - \; \|_2$. We always get zero to machine precision.

We have used $\| \; - \; \|_2$ $\frac{1}{2}$

-u

3.1.2 Example of a single non-periodic integral equation

Analytical solution:

$$(\ \ ) + \int_{\Gamma} K(\ \ ,\ \ )\ (\ \ ) \quad = \quad (\ \ )$$

Where

the Kernel=K(x, y) = $\ ^{2}\quad ^{2}$

Γ is a closed boundary from 0 to 1

u(x)=

Hence,

$$(\quad) = 1 + \frac{5}{6} \quad {}^{2}$$

We program this example of a single periodic integral equation in Matlab and

3.2 Collocation Method

The idea is to choose a finite-dimensional space of candidate solutions and a number of points in the domain (called ***collocation points***), and to select that solution which satisfies the given equation at the collocation points.

To solve

$$(\ ) + \int_{0}^{2\pi} (\ ,\ )\ (\ ) \quad = \ (\ ) \qquad (3.3)$$

We seek an approximation of the form:

$$= \sum_{=0}^{-1} \phi\ (\ )\ (\ ) \qquad \text{where } \phi \ = \text{basis functions}$$

Substitute into (3.3)

$$\sum_{=0}^{-1} \phi\ (\ )\ (\ ) + \int_{0}^{2\pi} (\ ,\ ) \sum_{=0}^{-1} \phi\ (\ )\ (\ ) \quad = \ (\ )$$

$$\Rightarrow \sum_{=0}^{-1} [\phi\ (\ ) + \int_{0}^{2\pi} (\ ,\ ) \phi\ (\ )\ ]\ (\ ) = \ (\ ) \qquad (3.4)$$

So we have one equation with n unknowns $\longrightarrow$ the values of $(\ )$.

To get n equations we fix (3.4) to hold at n points i.e. take $\hat{}_1, \ldots\ldots, \hat{}$

We replace the integral by a quadrature rule as in Nyström Method. The easiest way is to use the trapezium rule.

$$\int_0^{2\pi} (\ ) \quad \approx \frac{}{2}[\ (0)+2\ (\ )+2\ (2\ )+......+2\ ((\ -1)\ )+\ (2\pi)]$$

$$= [\ (0)+\ (\ )+......+\ ((\ -1)\ )] \qquad \text{(periodic function)}$$

Where $\quad = \frac{2\pi}{} \quad$ (=quadrature parameter)

In our case we replace ϕ by ϕ .

Therefore, we have:

$$\phi(\ )+\ [\ (\ ,0)\phi(0)+\ (\ ,\ )\phi(\ )+....+\ (\ ,((\ -1)\ )\phi((\ -1)\ )]\cdot(\ )=\ (\ )$$

$\forall$

We take $\ =0,\ ,2\ ,3\ ,.....,(\ -1)$

We define $(\ )=$

We have equations with unknowns:

$$(0),\quad (\ ),\quad (2\ ),\quad (3\ ),......,\quad ((\ -1)\ )$$

We use the trapezoidal rule in this method and we have exactly the same matrix as for the Nyström Method.

Hence the Nyström Method is exact at mesh points.

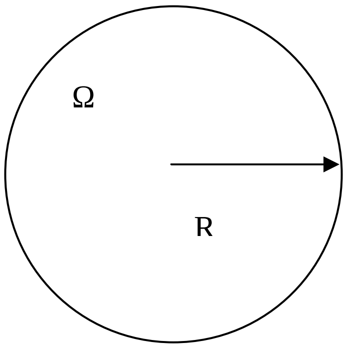
Chapter 4

Interior Problem for Laplace's equation

Consider $\quad \Delta \quad = 0$,

Ω is a circle with radius R

$$\frac{\partial}{\partial} =$$



$$\frac{\partial}{\partial} =$$

We already know that the Fundamental solution $G(\_ , \_) = \frac{1}{2\pi} \ln \frac{1}{| \_ - \_ |}$ and

that the form of the general integral equation is:

$$(\ ) + \int_{\Gamma} K(\ , \ ) \ (\ ) \quad = \quad (\ ).$$

Thus, we set up the interior problem as an integral equation of the above form and we will solve it using a code in Matlab.

From (2.7) of the bounded problem we have:

$$\int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right) {}_{-} = \frac{1}{2} (\quad) \qquad \in \partial$$

$$\Rightarrow -\frac{1}{2} (\quad) + \int_{\partial} \frac{\partial}{\partial} {}_{-} = \int_{\partial} \frac{\partial}{\partial} {}_{-} \qquad \in \partial$$

$$\frac{\quad}{{}_1} \quad \frac{1}{2}\sqrt{(\ {}_1 \quad {}_1)^2 \quad (\ {}_2 \quad {}_2)^2} \left(\quad \frac{1}{2}\right)\left[(\ {}_1 - \ {}_1)^2 + (\ {}_2 - \ {}_2)^2\right]^{\frac{3}{2}} 2(\ {}_1 - \ {}_1)$$

$$\Rightarrow \frac{\partial}{\partial} = -\frac{1}{8\pi}\frac{(1-(\cos\varsigma\cos \; + \sin\varsigma\sin \;))}{\sin^2(\frac{\varsigma-}{2})}$$

Now we are going to simplify the numerator of this fraction.

$$\begin{aligned} 1-(\cos\varsigma\cos \; + \sin\varsigma\sin \;) &= 1-\cos(\varsigma- \;) \\ &= 1-(1-2\sin^2(\frac{\varsigma-}{2})) \\ &= 2\sin^2(\frac{\varsigma-}{2}) \end{aligned}$$

(Note: Trigonometric identities: $\cos(\varsigma- \;) = \cos\varsigma\cos \; + \sin\varsigma\sin$

$2\varsigma \quad \cos^2\varsigma \quad \sin^2\varsigma$

$$(\ )=\int_0^{2\pi} 2 \qquad = \frac{1}{2\pi}\int_0^{2\pi} 2\ln\frac{1}{\left|{}_{-}\ -\ {}_{-}\right|}\,(\ ) \qquad = \frac{1}{\pi}\int_0^{2\pi}\ln\frac{1}{\left|{}_{-}\ -\ {}_{-}\right|}\,(\ )$$

Lets substitute $\hat{\ } = \ -\ \Rightarrow\ = \hat{\ } +$

$$= \ \hat{\ }$$

$$(\ ) = \ (\hat{\ } + \ )$$

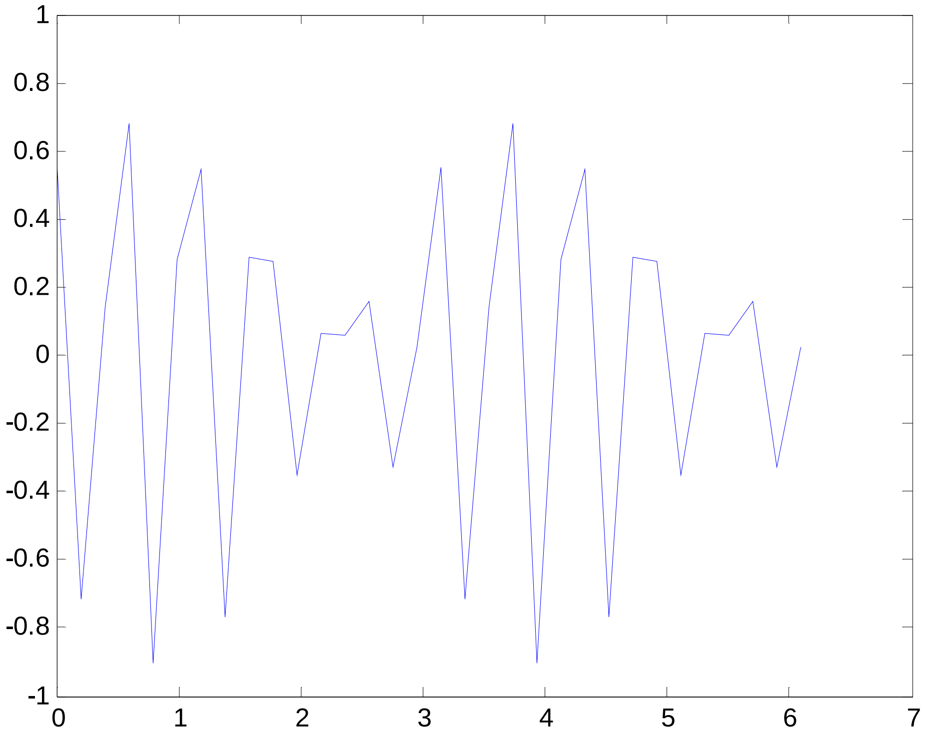
$$(\ )=\frac{1}{\pi}\int_{-}^{2\pi-}\ln\frac{1}{|\hat{\ }|}\,(\hat{\ }+\ )\ \hat{\ } = \int_0^{2\pi}\ln\frac{1}{|\ |}\,(\ +\ )$$

because is periodic

To find $(\ )$ which is equal to $\int_{\partial}\ \ {}_{-}$

Below are the graphs of the numerical solution of the integral equation:

$$- \quad (\quad) + 2\int_{\partial} \frac{\partial}{\partial} \quad _{-} = 2\int_{\partial} \qquad _{-}$$


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7

4.1 Separation of variables in polar coordinates

We will use polar coordinates and separation of var

The above equation has the form:

f(r)=f(θ)

where f(r) is a function of r and f(θ) is a function of θ. The only way in which the above equation can be satisfied, for general r and θ, is if both sides are equal to the same constant. Thus,

$$\Rightarrow -\frac{\partial}{\partial}\left(\frac{\partial}{\partial} \right) = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = \quad \text{(constant)}$$

The ordinary differential equations we get are then:

(a) $\frac{\partial}{\partial}\left(\frac{\partial}{\partial} \right) - \quad = 0$

(b) $\frac{\partial^2\Theta}{\partial\theta^2} + \quad \Theta = 0$

We take (b) $\frac{\partial^2\Theta}{\partial\theta^2} + \quad \Theta = 0$

Try Θ= $^{\lambda\theta}$ so $\lambda^2 \, ^{\lambda\theta} + \quad ^{\lambda\theta} = 0 \Rightarrow \lambda^2 + \quad = 0 \Rightarrow \lambda = \pm\sqrt{-\quad}$

$$\Rightarrow \Theta = \quad ^{\theta\sqrt{-}} + \quad ^{-\theta\sqrt{-}}$$

We know that Θ(θ) must be 2π periodic because is around a periodic boundary.

If c<0 then is not periodic

If c=0 then is not periodic

Thus when c>0 $\Rightarrow \Theta =$ $\theta\sqrt{\ }$ $+$ $-\theta\sqrt{\ }$

$= \ \cos($

As r $\longrightarrow 0$ the term involving $^{-\nu}$ is unbounded. The only way to fix this is to take $_2 = 0$.

Therefore,

$$) =$$

The interior Neumann problem is solvable if and only if:

$$\int_{0}^{2\pi} (\theta) \quad \theta = 0$$

but there is no existence of a unique solution.

(Theorem 6.26, Kress 'Linear Integral Equations') .

The coefficients A and B

Fix an integer value $\neq 0$

Chapter 5

Exterior Problem for Laplace's equation

For the exterior problem (unbounded problem) as we have seen in chapter 2 (2.8) the general integral is:

$$\int_{\partial}\left(\frac{\partial}{\partial} - \frac{\partial}{\partial}\right) (\ ) = (\ ) - \frac{1}{2\pi}\int_{\partial\Omega} (\ ) \quad (\ ) \qquad (5.1)$$

The above equation (5.2) is the same equation as (2.6) .
Thus, we are going to solve it numerically with the same way as the interior problem.

For the exterior problem the analytical solution is the same as (4.6) :

$$) =$$

$$-\ (\ ) + 2\int_{\partial} \frac{\partial}{\partial} \ \ _{-} = 2\int_{\partial} \ \ _{-}$$ for different values of n.

Where

$$\frac{\partial}{\partial} = \frac{1}{4\pi}$$ which is the same with the $\frac{\partial}{\partial}$ of the interior problem but with different sign.

$$(\ ) = 2\int_{\partial} 2 \ \ _{-} = \int_{0}^{2\pi} \ln \frac{1}{|\ |} (\ \ +\ \ )$$

$$(\theta) = \cos\theta$$

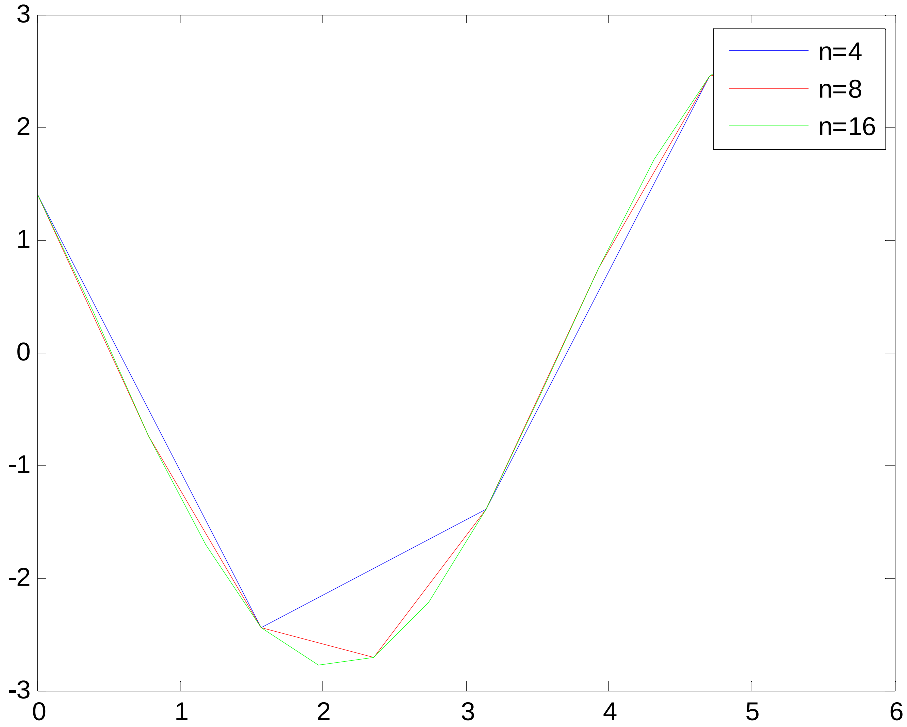


Figure 5.1: The numerical solution of the exterior problem for Laplace's equation for n=4,8,16.

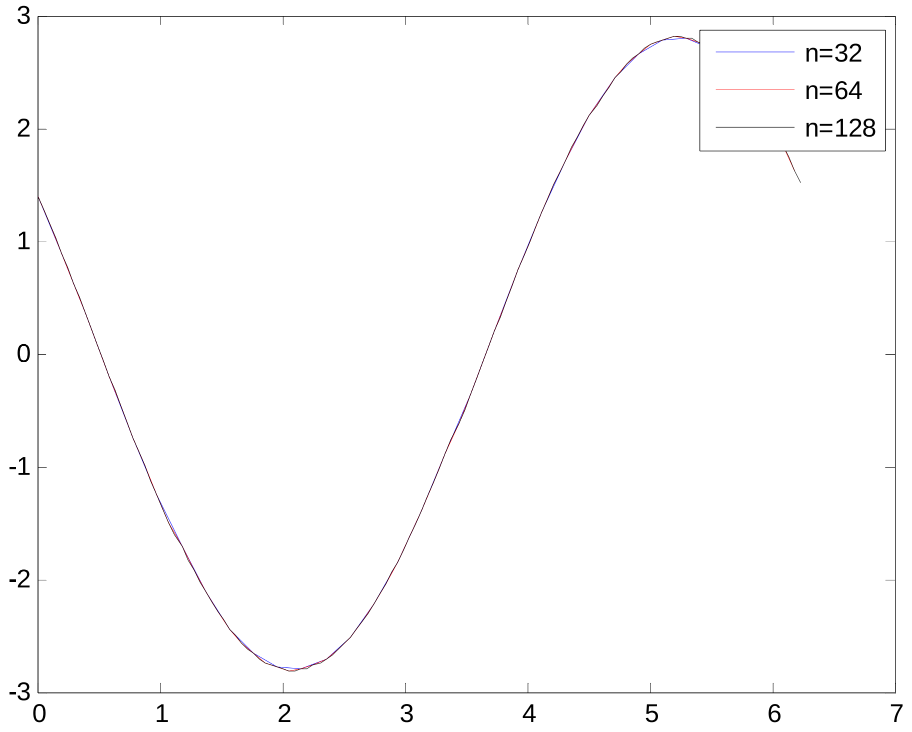


Figure 5.2: The numerical solution of the exterior problem for Laplace's equation for n=32,64,128.

Again we can see a convergence in both diagrams and as the value of n increases the solution becomes more accurate.

The following is an integral equation approach:

$$\int (\frac{\partial}{\partial} - \frac{\partial}{\partial}) \quad (\quad) + \int (- \quad {}_3) \quad (\quad) + \frac{1}{2} \quad (\quad) = 0$$

Thus, now we want to find $\frac{\partial}{\partial}$.

We know $P(\underline{\ },\underline{\ }') = \frac{1}{2\pi}\ln\frac{1}{|\underline{\ } - \underline{\ }'|}$

After that we replace $\frac{\partial}{\partial \;{}_1}$, $\frac{\partial}{\partial \;{}_2}$, ${}_1$ and ${}_2$ into equation (6.2)

And finally we have:

$$\frac{\partial}{\partial} = -\frac{1}{2\pi}\frac{({}_1 - \;{}_1)}{[({}_1 - \;{}_1)^2 + ({}_2 + 2 \;\; + \;\; {}_2)^2]}\cos(\varsigma) - \frac{1}{2\pi}\frac{({}_2 + 2 \;\; + \;\; {}_2)}{[({}_1 - \;{}_1)^2 + ({}_2 + 2 \;\; + \;\; {}_2)^2]}\sin(\varsigma)$$

$$\Rightarrow \frac{\partial}{\partial} = -\frac{1}{2\pi[({}_1 - \;{}_1)^2 + ({}_2 + 2 \;\; + \;\; {}_2)^2]}\left[\cos(\varsigma)({}_1 - \;{}_1) + \sin(\varsigma)({}_2 + 2 \;\; + \;\; {}_2)\right]$$

We also replace ${}_1 = \cos(\varsigma)$

$$= 4\sin^2\left(\frac{\varsigma +}{2}\right) + 4 \quad ^2 +$$

$$4 \quad \sin\varsigma + 4 \quad \sin$$

).

$$\Rightarrow \frac{\partial}{\partial} = \frac{-\cos^2\varsigma + \cos\varsigma\cos \quad - \sin^2\varsigma - \sin\varsigma\sin \quad - 2 \quad \sin\varsigma}{2\pi \cdot [4\sin^2(\frac{\varsigma +}{2}) + 4 \quad ^2 + 4 \quad \sin\varsigma + 4 \quad \sin \quad]}$$

$$\Rightarrow \frac{\partial}{\partial} = \frac{-1 + \cos\varsigma\cos \quad - \sin\varsigma\sin \quad - 2 \quad \sin\varsigma}{2\pi \cdot [4\sin^2(\frac{\varsigma +}{2}) + 4 \quad ^2 + 4 \quad \sin\varsigma + 4 \quad \sin \quad]}$$

$$\Rightarrow \frac{\partial}{\partial} = -\frac{1}{2\pi}\frac{(1 - (\cos\varsigma\cos \quad - \sin\varsigma\sin \quad) + 2 \quad \sin\varsigma}{4\sin^2(\frac{\varsigma +}{2}) + 4 \quad ^2 + 4 \quad \sin\varsigma + 4 \quad \sin}$$

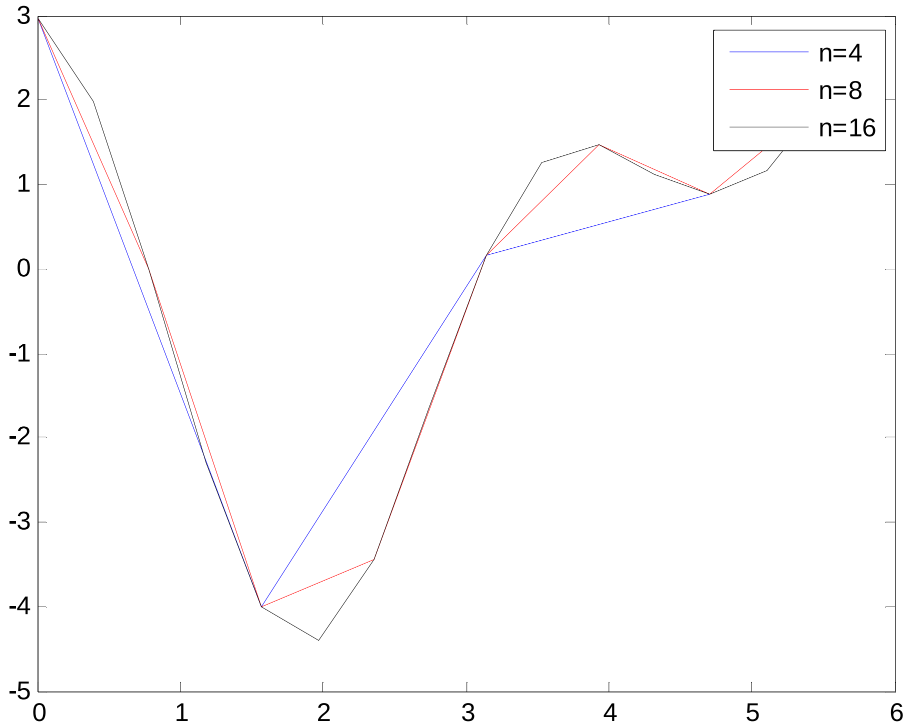
(Note: Trigonometric identity: $\cos^2\varsigma + \sin^2\varsigma = 1$)

Now we are going to simplify the numerator of this fraction.

$1 \quad (\cos \quad \cos \quad \sin \quad \sin \ ) \quad 1 \quad \cos(\quad)$

Finally,

$$\frac{\partial\ \hat{}}{} = \frac{\partial}{\partial} + \frac{\partial}{\partial} = -\frac{1}{4\pi} + \quad \frac{}{)\quad 4 \qquad 4\ \sin \qquad 4\ \sin}$$


n=4
n=8
n=16
3
2
1
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6

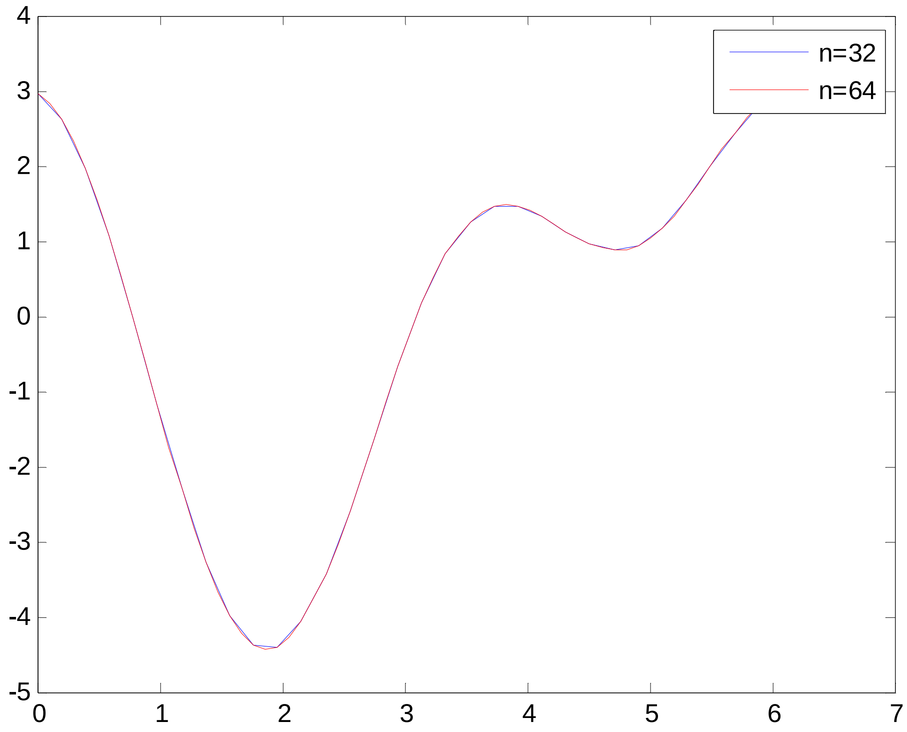


Figure 6.2: The numerical solution of the full pipe flow problem for n=32,64.

As we can see from the diagrams the solution converges and by increasing the value of n , the solution becomes more accurate.

Chapter 7

Summary and Conclusions

Bibliography

[1] Bau, H.H. and Sadhal,S.S. (1982) ***Heat losses from fluid flowing in a Buried pipe,***International Journal of Heat and Mass Transfer, 25 (11): 1621-1629

[2] Bau, H.H. (1984) ***Convective heat losses from a pipe buried in a Semi-infinite porous medium,*** International Journal of Heat and Mass Transfer, Vol 27,No.||, pp.2047-2056

[3] Beatrice Pelloni (2006) ***Fourier Series and Applications***

[4] Carslaw, H.S. and Jaeger, J.C. (1986) ***Conduction of Heat in Solids,***