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Department of Mathematics

NUMERICAL APPROXIMATION OF SIMILARITY IN NONLINEAR DIFFUSION EQUATIONS

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Abstract

This dissertation investigates the numerical solution of similarity solution of the Porous Medium and the Thin Film equations. Scaling transformations are introduced to reduce the original equations to ordinary differential equations. Self-similar solutions are found for all $n > 0$ in the case of the Porous Medium equation, but only for $n = 1$ in the Thin Film equation. The ordinary differential equations are solved numerically and the numerical results are compared with the self-similar solutions to verify the accuracy of the numerical schemes used. The main idea is to find a numerical self-similar solution for $n > 1$ in the Thin Film equation.

Declaration

I con rm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

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equations than the original PDE. The importance of similarity solutions lies in their ease of calculation, the fact that they often act as attractors for the more general solutions of the PDE.

The examples below show different types of PDEs for which they are invariant under certain groups of transformations [1].

Example 1: Fisher's equation, given by

$$u_t = u_{xx} + u(1 - u); \qquad u \in C^2(-\infty; \infty)$$

plays an important role in the study of mathematical biology and in probability. It is invariant under:

i. translations in time, $t \to t + \epsilon$

ii. translations in space, $x \to x + \epsilon$

iii. reflexions in x, $x \to -x$

Example 2: The universal heat equation

$$u_t = u_{xx}; \qquad u \in C^2(-\infty; \infty)$$

is invariant under the same transformations as in Example 1, and also the stretching groups given by

$$\begin{aligned} t &\to \lambda t; \\ x &\to \lambda^{\frac{1}{2}} x; \end{aligned}$$

where $\lambda > 0$ is arbitrary.

Example 3: The blow-up equation, of the form

$$u_t = u_{xx} + u^2; \qquad u \in C^2(-\infty; \infty)$$

is used in modelling combustion processes in which materials become hot very quickly. It is invariant under the action of the same groups as in Example 2, with an additional stretching group given by

$$u \to \frac{u}{\lambda}.$$

1.2 Scale Invariance

Scale invariance is a basic idea which originates from the analysis of the consequences of changes of units of measurement on the mathematical form of the laws of physics. It is viewed as a particular aspect of study of differential equations under groups of transformations [7].

Basically, this type of transformation maps all the variables in the original differential equation (DE) to newly transformed variables by different scaling parameters for each of the original variables. The DE is then said to be scale-invariant if the system remains unchanged by the transformations.

1.2.1 Scaling Invariance on an ODE

Suppose a ODE is given by

$$\frac{dy}{dx} = F(x; y): \tag{1.1}$$

Introduce a mapping from the original system $(x; y)$ to a new system $(x; y)$ by the transformations

$$\begin{aligned} x &= x \\ \text{and} \qquad y &= y \end{aligned} \tag{1.2}$$

where and are the scaling parameters.

Substituting (1.2) into equation (1.1), the left-hand side becomes

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \, \frac{y}{}}{d \, \frac{x}{}} \\ &= -\frac{dy}{dx} \end{aligned}$$

and the right-hand side gives

$$F(x; y) = F\left(\frac{x}{}; \frac{y}{}\right):$$

The new system is therefore

$$\frac{dy}{dx} = -F\left(\frac{x}{}; \frac{y}{}\right):$$

The definition of invariance is then

$$-F \quad \frac{x}{\;};\frac{y}{\;} \quad = F\,(x;y):$$

and

$$= \frac{y}{y}:$$

Assuming there exists a functional relationship between and such that = ( ), self-similar solutions can be found to satisfy the ODE in (1.1), which is invariant under the transformations in (1.2).

1.3.2 Self-Similarity for a PDE

Making the subject from (1.4), we have

$$= \frac{u^{\frac{1}{}}}{u^{\frac{1}{}}} = \frac{t}{t} = \frac{x^{\frac{1}{}}}{x^{\frac{1}{}}}$$

De ne similarity variables to be

$$= \frac{u}{t} = \frac{u}{t}; \tag{1.5}$$

$$= \frac{x}{t} = \frac{x}{t}; \tag{1.6}$$

which are independent of and hence scale-invariant under (1.4).

A functional relationship between the similarity variables is assumed to take the form = ( ). We can then nd self-similar solutions satisfying the ODE, that is derived by the transformation of the original PDE into the variables and with scaling exponents and .

1.4 Outline of Dissertation

In Chapter Two, we look at non-linear di usion equation and its applications. An illustration of the method of similarity under scaling transformation to this equation is presented. A detailed account of the construction of similarity variables to obtain self-similar solutions are shown for both the Porous Medium equation, for all n, and the Thin Film equation, in the case of $n = 1$, with given initial conditions. We then go on to consider the applications of both di usion equations. Furthermore, we develop another approach to solving both equations by including the time variable in the relationship between the new transformed variables. This results in scale-invariant but not self-similarity solutions. It was hoped that when method is run to convergence, within some tolerance, the solutions would converge to give the numerical self-similar solutions for both equations. In practice, we found another way of achieving our aims so this was left to further work.

Chapter 2

Non-Linear Diffusion

We focus mainly on non-linear diffusion equations of general form

$$\frac{\partial u}{\partial t} = (-1)^m \frac{\partial}{\partial x}\left(u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right); \tag{2.1}$$

where u^n represents the diffusion coefficient and $n > 0$ is a diffusion growth we require

$$1 = (n + 1) \quad (2m + 2) \; : \tag{2.5}$$

To determine and we need another equation. Integrating equation (2.1) over the domain gives

$$\int_{a(t)}^{b(t)} \frac{\partial u}{\partial t}\, dx = \int_{a(t)}^{b(t)} (\;\; 1)^m \frac{\partial}{\partial x} \;\; u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \;\; dx;$$

which simplifies to

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u\, dx = (\;\; 1)^m \;\; u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \Bigg|_{a(t)}^{b(t)} :$$

Taking the boundary conditions at $u(a(t)) = u(b(t)) = 0$, we have

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u\, dx = 0$$

and therefore

$$\int_{a(t)}^{b(t)} u\, dx = k; \tag{2.6}$$

where k is a constant. This shows that mass is conserved over the whole domain.

Transforming the integral in (2.6) to the variables $(u; x; t)$, by (1.4), we obtain

$$\int_{a(t)}^{b(t)} \quad u \quad d \quad\quad x \quad = \quad k$$

$$+ \int_{a(t)}^{b(t)} u\, dx \quad = \quad {}^{0}k:$$

For (2.6) to be invariant under the transformation (1.4), we require

$$+ \quad = 0: \tag{2.7}$$

Solving (2.5) and (2.7) simultaneously, we find that

$$= \frac{1}{n + (2m + 2)}$$

$$\text{and} \qquad = \frac{1}{n + (2m + 2)} \tag{2.8}$$

By (1.5) and (1.6), the self-similar solutions for non-linear di usion equations are of the form

$$= \quad (\ )$$
$$\text{i.e.} \qquad u(x;t) \quad = \quad t \qquad \frac{x}{t} \tag{2.9}$$

under the scalings and de ned in (2.8).

2.1 The Porous Medium Equation

In the case when $m = 0$, equation (2.1) becomes the Porous Medium equation (PME), a second-order non-linear di usion equation. It has the form

$$\frac{@u}{@t} = \frac{@}{@x} \quad u^n \frac{@u}{@x} \quad ; \tag{2.10}$$

where n is as described in equation (2.1), with $u = u^n \frac{@u}{@x} = 0$ at the boundaries. Such problems with zero boundary conditions are degenerate in the sense that $u = 0$ is a su cient boundary condition.

Equation (2.10) has been widely used to model many di erent applications. In physical problems, it is used to model the ow in thin saturated region in a porous medium, the percolation of gas through porous media, the spreading of thin viscous spreading under gravity over a horizontal plane, and many other processes [2],[10].

Further applications of the second-order case arise in the modelling of bacterial growth on agar plates and in medicine. An example for the latter is the development of a tumour inside a human body. The tumour gains nutrients and oxygen for growth by di usion from already existing vasculature surrounding them. Thus the size of the tumour is limited by di usion through a porous medium [3].

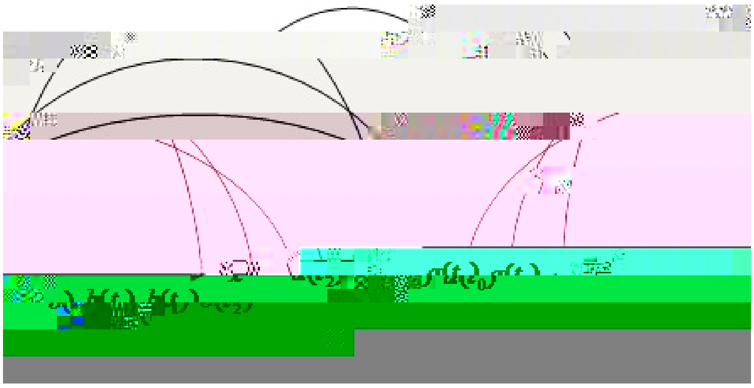


Figure 2.1: Evolution of PME at Di erent Times for a xed n

The above gure represents a self-similar solution at three points in time. The solution at time t_0 is transformed onto the solution at a di erent time, say at time t_1 by the scaling transformation in equation (1.4).

2.1.1 Self-Similar Solutions

From (2.8), and for the second-order equation (2.10) are

$$= \frac{1}{(n+2)} \quad \text{and}$$
$$= \frac{1}{(n+2)}. \qquad (2.11)$$

To construct self-similar solutions in the form given in (2.9), we rst derive the ODE in terms of the similarity variables = ( ) and , which are de ned in (1.5) and (1.6).

and hence

$$
\begin{aligned}
\frac{d}{d} &= \frac{d}{d} \quad {}^{n}\frac{d}{d} \\
+\ \frac{d}{d} &= \frac{d}{d} \quad {}^{n}\frac{d}{d} \\
\frac{d}{d}(\quad) &= \frac{d}{d} \quad {}^{n}\frac{d}{d} \quad ;
\end{aligned}
\tag{2.16}
$$

which is a second-order ODE in the function ( ) with boundary conditions = 0 at = 1:

In order to solve the ODE, we integrate (2.16) to get

$$
{}^{n}\frac{d}{d} = \quad (\quad) + C;
$$

where C is an integration constant. Since = 0 at the boundary, $C = 0$ and thus

$$
{}^{n}\frac{d}{d} = \quad (\quad):
$$

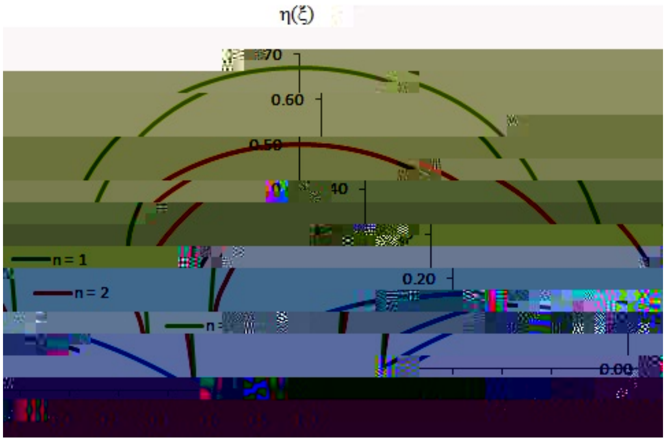
zero boundary conditions,

$$\eta(\xi) = \begin{cases} \left(A_n - \frac{n\xi^2}{2}\right)^{\frac{1}{n}} & \frac{n\xi^2}{2} \leq A_n; \\ 0 & \frac{n\xi^2}{2} > A_n. \end{cases}$$

Mapping this back to the original variables u, x and t using the definitions in (1.5) and (1.6) with α and β given by (2.11), we obtain

$$u(x;t) = \frac{1}{t^{\frac{1}{(n+2)}}}\left(A_n - \frac{nx^2}{2(n+2)t^{\frac{2}{(n+2)}}}\right)_+^{\frac{1}{n}}, \tag{2.17}$$

where the notation $(.)_+^{\frac{1}{n}}$ indicates that we take the positive solution of the argument. Equation (2.17) is a self-similar solution of the original PME with $u = 0$ at the boundaries, for all values of $n > 0$. The original derivation is due to Barenblatt[6].



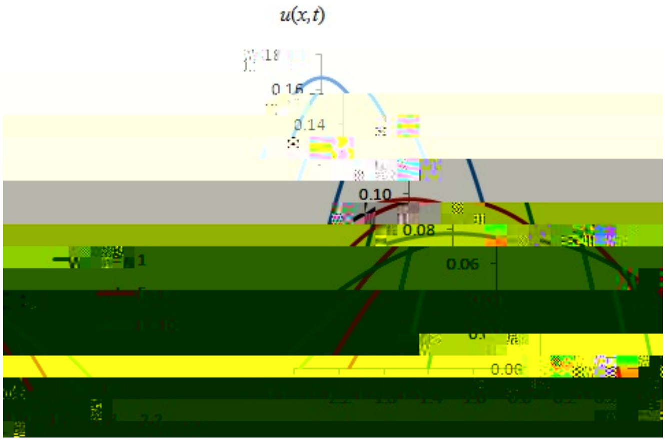


Figure 2.3: Self-Similar Solutions of the original PME when $n = 1$ at di erent times

2.1.2 Non-Self-Similar Solutions

More generally, the equation (2.10) can be solved with as a function of and the time t, giving scale-invariant solutions but not self-similarity. This results in an additional term to the tranformed left-hand side expressed in (2.12),

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial (\ t \ )}{\partial t} \\
&= \quad t \quad ^{1} + t \quad \frac{\partial}{\partial t} \\
&= \quad t \quad ^{1} + t \quad \frac{\partial}{\partial} \quad \frac{d}{dt} + \frac{\partial}{\partial t} \\
&= \quad t \quad ^{1} + t \quad \frac{\partial}{\partial} \quad \frac{x}{t^{\ +1}} + \frac{\partial}{\partial t} \\
&= \quad t \quad ^{1} + t \quad \frac{\partial}{\partial} \quad \frac{}{t} + \frac{\partial}{\partial t} \\
&= \quad t \quad ^{1} \quad t \quad ^{1} \frac{\partial}{\partial} + t \quad \frac{\partial}{\partial t} .
\end{aligned} \tag{2.18}$$

The right-hand side has the same transformed expression as in (2.13),

$$\frac{\partial}{\partial x} \quad u^{n} \frac{\partial u}{\partial x} \quad = t^{\ n+ \quad 2} \frac{\partial}{\partial} \quad ^{n} \frac{\partial}{\partial} \quad . \tag{2.19}$$

Equating (2.18) and (2.19), we have

$$
\begin{aligned}
t^{\;\;1}\quad t^{\;\;1}\frac{\partial}{\partial} + t\,\frac{\partial}{\partial t} &= t^{\,n+\;\;2}\,\frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \\
t^{\;\;1}\qquad\quad \frac{\partial}{\partial} + t\frac{\partial}{\partial t} &= t^{\,n+\;\;2}\,\frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \\
\frac{\partial}{\partial} + t\frac{\partial}{\partial t} &= t^{\,n+1\;\;2}\,\frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \quad . \qquad (2.20)
\end{aligned}
$$

Using (2.11) and that $\;=\;$, and because of (2.15), gives

$$
\begin{aligned}
\frac{\partial}{\partial} + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \\
+\;\frac{\partial}{\partial}\quad + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \\
\frac{\partial}{\partial}(\quad) + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \\
t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}(\quad) + \frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \quad ; \qquad (2.21)
\end{aligned}
$$

which is a PDE for the function $(\;;t)$, with chosen domain $|\;|\;\;1$ and boundary conditions $\;= 0$. This PDE can be solved numerically.

2.2 The Thin Film Equation

The Thin Film equation (TFE) is a fourth-order non-linear diffusion equation, given by

$$\frac{\partial u}{\partial t} = \;\frac{\partial}{\partial x}\;\; u^{n}\frac{\partial^{3}u}{\partial x^{3}} \quad ; \qquad (2.22)$$

where n is as stated in equation (2.1), and with zero boundary conditions. This is when $m = 1$ in equation (2.1). When $n = 1$, the equation is used to describe flow in a Hele-Shaw cell. The fluid placed between two parallel plates moves in response to pressure gradients due to surface tension and other externally imposed forces.

With $n = 3$, it models the lubrication of a surface tension driven thin viscous liquid on a horizontal surface with a no-slip condition at the interface. However, the no-slip condition implies that an infinite force occurs at the interface; but this can be avoided by having more realistic models allowing slip, which are of Navier-type slip condition type [8]. Other applications involve the spreading of uid on textiles and the over ow of rainwater over soils.

2.2.1 Self-Similar Solutions

Similarly, we follow the same procedure in nding self-similar solutions for TFE. and de ned in (2.8) become

$$
\begin{aligned}
&= \frac{1}{(n+4)} \\
\text{and} \qquad &= \frac{1}{(n+4)}: \qquad (2.23)
\end{aligned}
$$

A similarity solution of the form = ( ) is sought, by obtaining an ODE for in terms of . The transformed left-hand side of (2.22) is the same as in (2.12),

$$\frac{@u}{@t} = \quad t \quad ^{1} \quad t \quad ^{1} \frac{d}{d} \qquad (2.24)$$

To transform the right-hand side, rst consider

$$
\begin{aligned}
\frac{@u}{@x} &= \frac{@u}{@} \frac{d}{d} \frac{@}{@x} \\
&= t \frac{d}{d} \frac{1}{t} \\
&= t \quad \frac{d}{d}
\end{aligned}
$$

from which,

$$
\begin{aligned}
\frac{@^2u}{@x^2} &= t \quad \frac{d^2}{d \ ^2} \frac{@}{@x} \\
&= t \quad \frac{d^2}{d \ ^2} \frac{1}{t} \\
&= t \quad ^{2} \frac{d^2}{d \ ^2}
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial^3 u}{\partial x^3} &= t^{\;\;2} \frac{d^3}{d^{\;3}} \frac{\partial}{\partial x} \\
&= t^{\;\;2} \frac{d^3}{d^{\;3}} \frac{1}{t} \\
&= t^{\;\;3} \frac{d^3}{d^{\;3}} :
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial x} \quad u^n \frac{\partial^3 u}{\partial x^3} &= \frac{\partial}{\partial x} \frac{d}{d} \quad {}^n t^{\;n} t^{\;\;3} \frac{d^3}{d^{\;3}} \\
&= \frac{1}{t} \frac{d}{d} \quad {}^n t^{\;n} t^{\;\;3} \frac{d^3}{d^{\;3}} \\
&= t^{\;n+\;\;\;4} \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \quad : \qquad (2.25)
\end{aligned}$$

Equating (2.24) with (2.25) gives

$$\begin{aligned}
t^{\;\;1} \quad t^{\;\;1} \frac{d}{d} &= t^{\;n+\;\;\;4} \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \\
t^{\;\;1} \quad \frac{d}{d} &= t^{\;n+\;\;\;4} \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \\
\frac{d}{d} &= t^{\;n+1\;\;4} \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \quad : \qquad (2.26)
\end{aligned}$$

From (2.23), since $\;=\;$, the power of t in (2.27) becomes

$$\begin{aligned}
n + 1 \quad 4 &= n + 1 + 4 \\
&= \frac{n}{(n+4)} + 1 \qquad \frac{4}{(n+4)} \\
&= 0 \qquad (2.27)
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{d} &= \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \\
+ \;\frac{d}{d} &= \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \\
\frac{d}{d}(\quad) &= \frac{d}{d} \quad {}^n \frac{d^3}{d^{\;3}} \quad ; \qquad (2.28)
\end{aligned}$$

a fourth-order ODE for in terms of . We impose boundary conditions = 0

To evaluate K_3, we note that

$$
\begin{aligned}
&= 0 \quad \text{at} \quad = 1 \\
) \quad \frac{u}{t^{\frac{1}{5}}} &= 0 \quad \text{at} \quad \frac{x}{t^{\frac{1}{5}}} = 1.
\end{aligned} \tag{2.32}
$$

Substituting (2.32) into (2.31) gives $K_3 = \frac{1}{120}$. Hence the self-similar solution of the original TFE for $n = 1$ is

$$u(x;t) = \frac{1}{120t^{\frac{1}{5}}} \left(1 \quad \frac{x^2}{t^{\frac{2}{5}}}\right)^2. \tag{2.33}$$

For values of $n > 1$, we nd their approximate solutions from the numerical schemes presented in the next chapter.

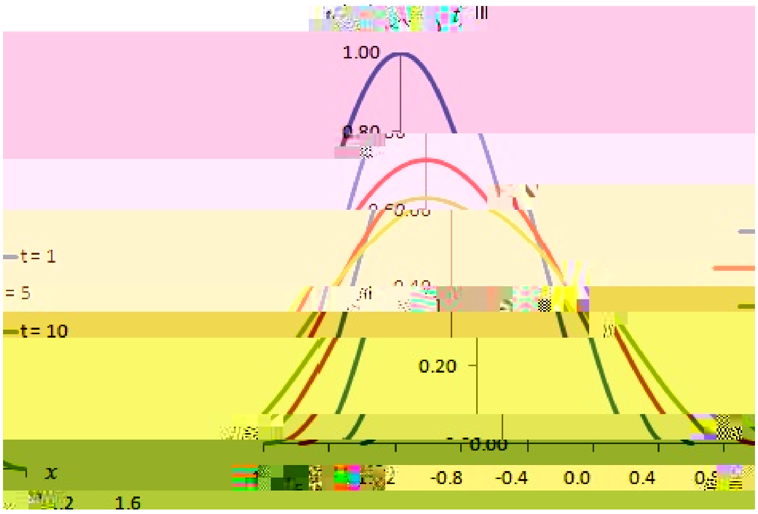


Figure 2.4: Self-Similar Solutions of original TFE when $n = 1$ at di erent times

2.2.2 Non-Self-Similar Solutions

We can also nd the solutions of equation (2.22) with as a function of and t, again giving scale-invariant solutions but not self-similarity. The transformed expression of the left-hand side in (2.24) now has an extra term,

$$\frac{\partial u}{\partial t} = \quad t^{\ 1} \quad t^{\ 1}\frac{\partial}{\partial} + t\frac{\partial}{\partial t}. \tag{2.34}$$

The right-hand side is transformed into the same expression as in (2.25),

$$\frac{\partial}{\partial x}\left(u^n\frac{\partial^3 u}{\partial x^3}\right) = t^{n+\quad 4}\frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right): \qquad (2.35)$$

Equating (2.34) and (2.35), gives

$$\begin{aligned} t^{\ \ 1}\left(t^{\ \ 1}\frac{\partial}{\partial} + t\frac{\partial}{\partial t}\right) &= t^{n+\quad 4}\frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) \\ t^{\ \ 1}\left(\frac{\partial}{\partial} + t\frac{\partial}{\partial t}\right) &= t^{n+\quad 4}\frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) \\ \frac{\partial}{\partial} + t\frac{\partial}{\partial t} &= t^{n+1\ \ 4}\frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right): \qquad (2.36)\end{aligned}$$

Deducing from (2.23) that $\ = \ $, and because of (2.27),

$$\begin{aligned} \frac{\partial}{\partial} + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) \\ + \frac{\partial}{\partial}\quad + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) \\ \frac{\partial}{\partial}(\quad) + t\frac{\partial}{\partial t} &= \frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) \\ \frac{\partial}{\partial}(\quad) + \frac{\partial}{\partial}\left(\ ^n\frac{\partial^3}{\partial\ ^3}\right) &= t\frac{\partial}{\partial t}; \qquad (2.37)\end{aligned}$$

which is a PDE in the function $(\ ; t)$. We choose the domain to be $|\ |\ \ 1$ with zero boundary conditions. The numerical solutions are presented in chapter Four.

Chapter 3

Numerical Results for the Porous Medium Equation

In this chapter, we compare numerical results obtained for PME with the similarity solutions found earlier, in order to check the accuracy of the numerical methods, in approximating nonlinear di usion.

3.1 Self-Similar Solutions

Discretising equation (2.16) using a nite di erence method, we have

$$\frac{(i+1)\ _{i+1} \quad (i \quad 1)\ _{i\ \ 1}}{2h}$$

$$= \quad \frac{_{i+1} + \ _{i}}{2}^{\,n} \ \frac{_{i+1} \quad _{i}}{h^2} \qquad \frac{_{i} + \ _{i\ \ 1}}{2}^{\,n} \ \frac{_{i} \quad _{i\ \ 1}}{h^2}$$

$$) \quad 0 = \frac{(i+1)\ _{i+1} \quad (i \quad 1)\ _{i\ \ 1}}{2h} \ +$$

$$\frac{_{i+1} + \ _{i}}{2}^{\,n} \ \frac{_{i+1} \quad _{i}}{h^2}$$

$$\frac{_{i} + \ _{i\ \ 1}}{2}^{\,n} \ \frac{_{i} \quad _{i\ \ 1}}{h^2}$$

where h is the distance between adjacent points in space.

The system can be represented in a matrix form

$$A(\ )\_ = \underline{0}; \tag{3.1}$$

where $\_ = [\ _1; \ _2; :::; \ _{N} \ _1]^T$ with known boundary conditions $_0 = 0$ and $_N = 0$, and N represents the number of uniformly spaced intervals between $1 < \ < 1$.

From the matrix system in (3.1), is non-unique and one of the solutions for is zero. To remedy this, we use symmetry and solve on $0 < \ < 1$ instead with boundary conditions $= 0$ at $= 1$ and $= 0$ at $= 0$. Using the fact that $_1 = \ _1$

choosing two initial values at = 0 such that the solutions of at the boundary = 1 will result in a positive and a negative at the last point.

For n = 1, the value of at = 0 is the same regardless of the number of intervals between = 1 and = 0 for a fixed value of tolerance. This is because the solution is a quadratic. The numerical method used, Runga-Kutta of order 2, is exact for approximating a quadratic curve.

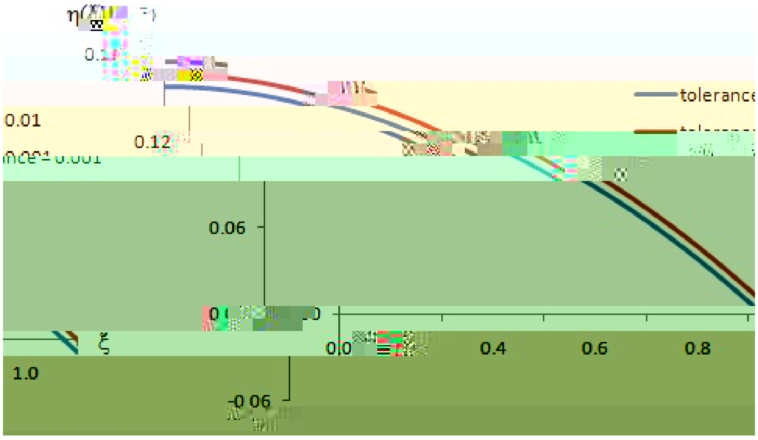


Figure 3.1: Numerical Solution of PME for n = 1 at different tolerances

Comparing two different tolerance values of 0.01 and 0.001, we see that = 0.1580 and = 0.1664 respectively.

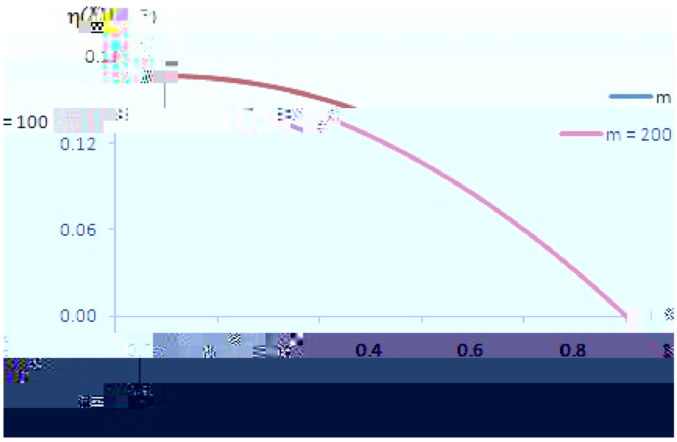


Figure 3.2: Numerical Solution of PME for n = 1 for different number of intervals

Comparing two different numbers of intervals, 100 and 200, the values of is the same.

For n = 2, the values of at = 0 are similar for di erent tolerances but xed number of intervals. As the number of intervals increases, converges to a value of 0.50. Comparing two di erent tolerance values of 0.01 and 0.001,

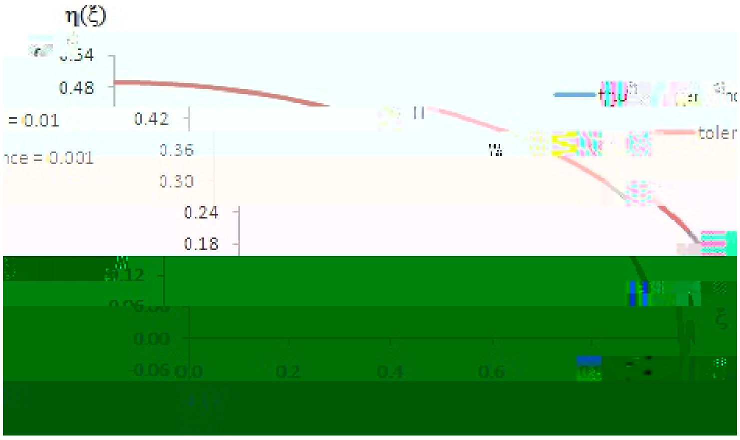


Figure 3.3: Numerical Solution of PME for n = 2 at di erent tolerances

= 0:4881916 and = 0:4881922 respectively.

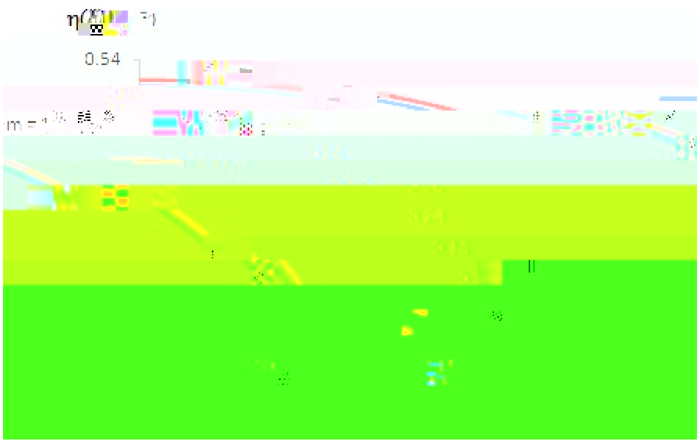


Figure 3.4: Numerical Solution of PME for n = 2 for di erent number of intervals

Comparing two di erent numbers of intervals of 100 and 200, the values of are converging.

Similarly, for $n = 3$, the values of at = 0 are almost the same for di erent tolerances but xed number of intervals. converges to a value of 0.67 as we increase the number of intervals. Comparing two di erent tolerance values of

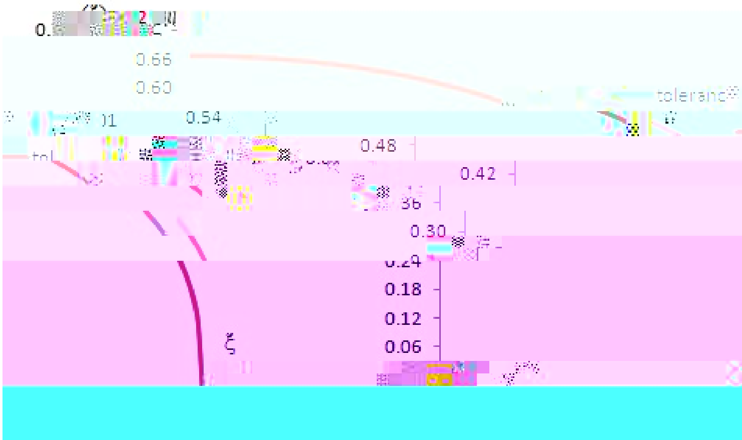


Figure 3.5: Numerical Solution of PME for $n = 3$ at di erent tolerances

0.01 and 0.001, = 0:66814 and = 0:66811 respectively.

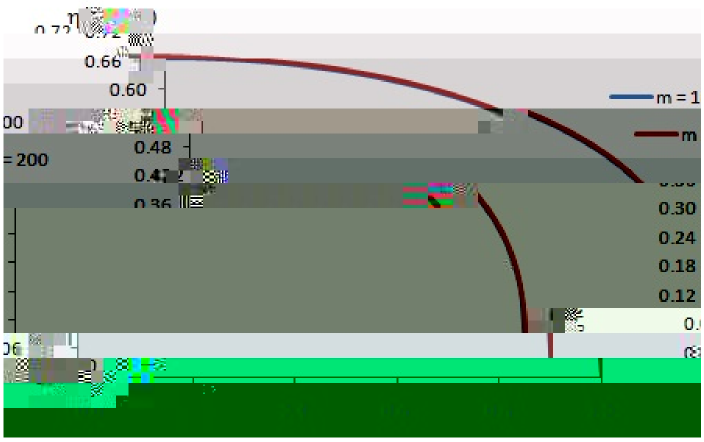


Figure 3.6: Numerical Solution of PME for $n = 3$ for di erent number of intervals

Comparing two di erent numbers of intervals of 100 and 200, the values of are converging.

Another approach of obtaining the solution is by solving (2.21) and run it to convergence, which we look at next.

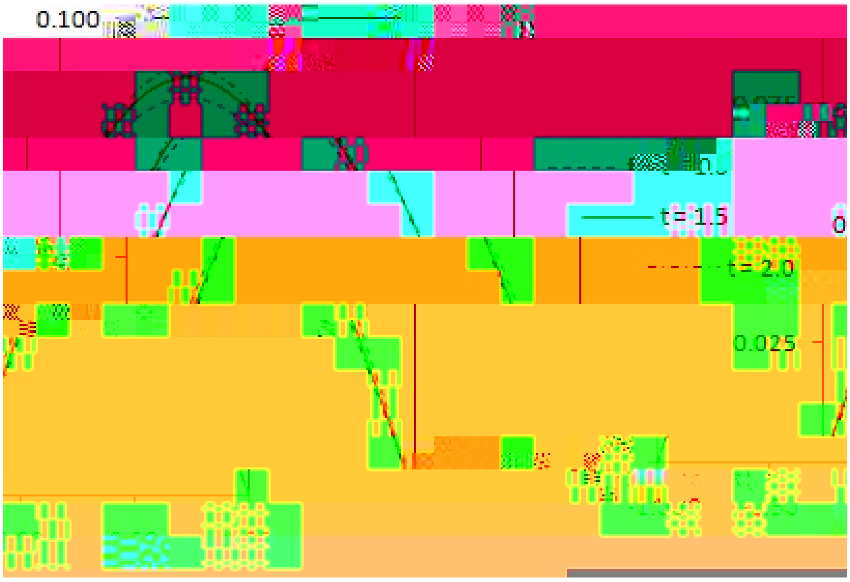
3.2 Another Approach via Non-Self-Similar Equation

We discretise the equation (2.21) to give

$$ {}^{k+1} = {}^{k} + \frac{t}{t} \quad \frac{(i+1)\ {}_{i+1} \quad (i \quad 1)\ {}_{i \ 1}}{2h} $$

$$ + \frac{t}{t} \quad \frac{{}_{i+1} + {}_{i}}{2}^{n} \quad \frac{{}_{i+1} \qquad {}_{i}}{h^2} \qquad \frac{{}_{i} + {}_{i \ 1}}{2}^{n} \quad \frac{{}_{i} \qquad {}_{i \ 1}}{h^2}^{k} ; \qquad (3.3) $$

where k represents the time level and t is the local distance between time steps. We regard (3.3) as an interation and perform two iterative methods, Jacobi and Gauss-Seidel. We run the program for a few time steps for each of $n = 1; 2; 3$. The evolutions are shown below:



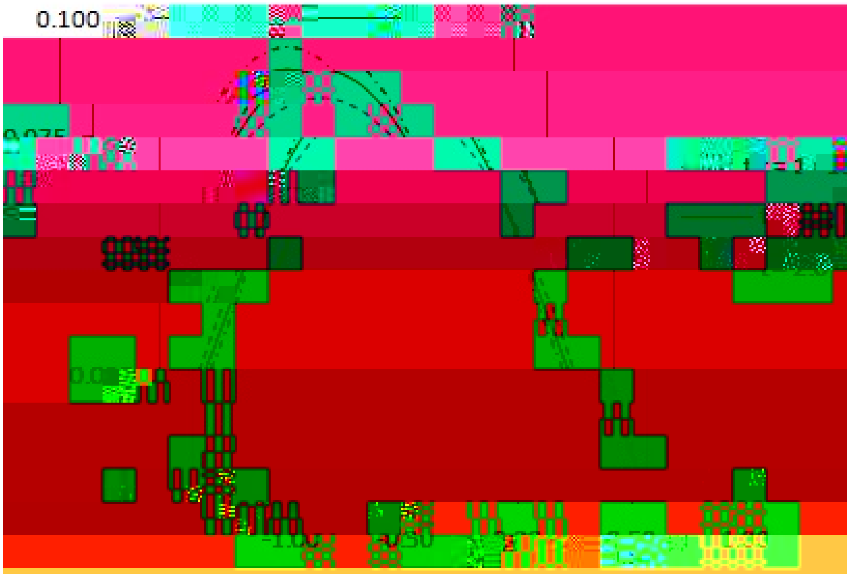


Figure 3.8: Evolution of Self-Similar Solution of PME for $n = 2$

method converges about twice as fast as Jacobi does because we use updated values of $_{i}$ and $_{i\ 1}$.

Figure 3.9: Evolution of Self-Similar Solution of PME for $n = 3$

We check the results against the exact solutions and they both have the give same graphs.

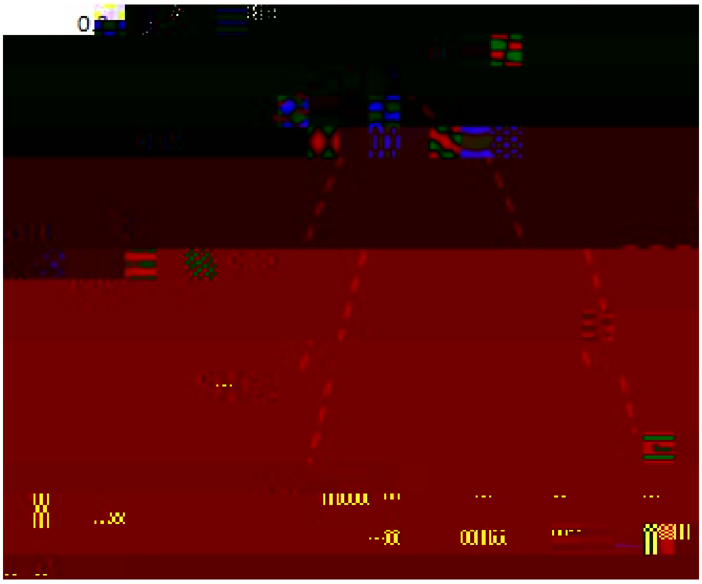


Figure 3.10: Exact and Approximate Solutions for PME for $n = 1$

3.3 Accuracy

We then investigated the accuracy of the method used in nding self-similar solutions by evaluating the sum of the global errors taken at all points for di erent number of intervals with value of n xed. The global error is the di erence between the exact and the approximate (self-similar) solutions at $_i$. Table 1 below shows the total error at intervals of 200, 400, and 800. The error can be

No. of Intervals	Total Exact	Total Approx	Total Global Errors
200	112.5393	112.0696	0.4725
400	225.2001	224.7100	0.4913
800	0.0000	0.0000	0.0000

represented graphically as given below:

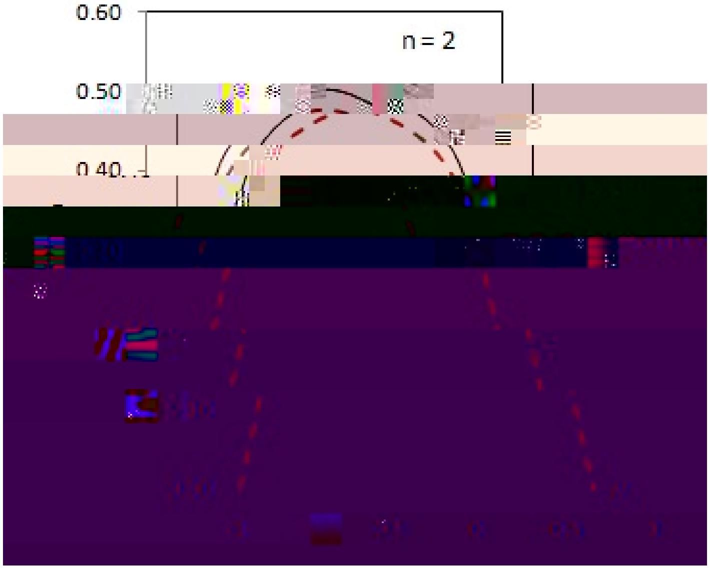


Figure 3.11: Exact and Approximate Solutions for PME for n = 2

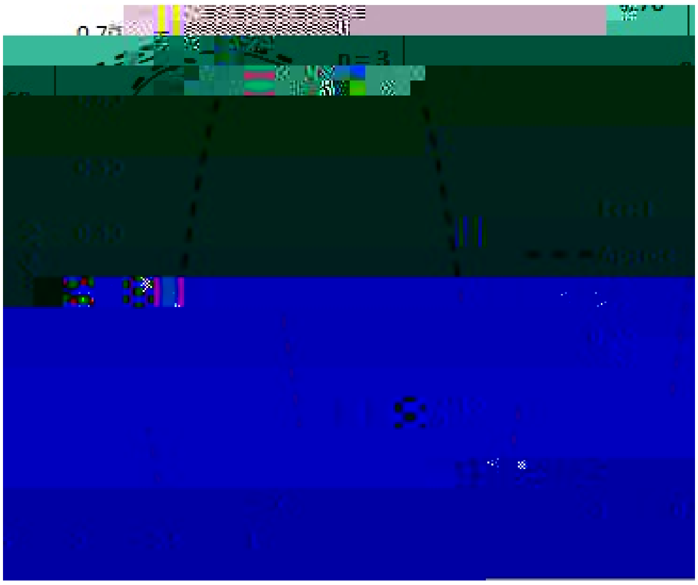


Figure 3.12: Exact and Approximate Solutions for PME for n = 3

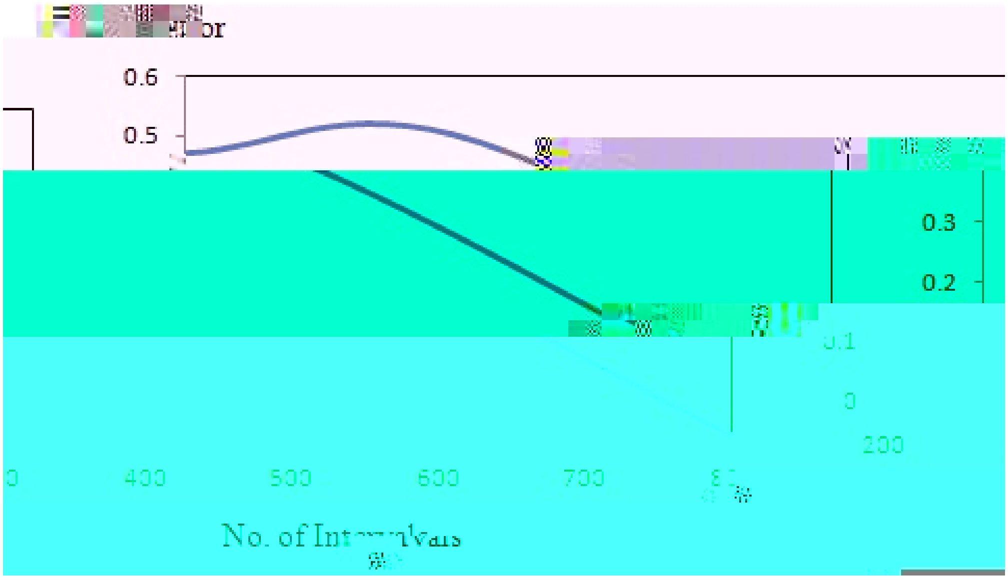


Figure 3.13: Error Analysis for PME for n = 3

Chapter 4

Numerical Results for the Thin Film Equation

running it to convergence.

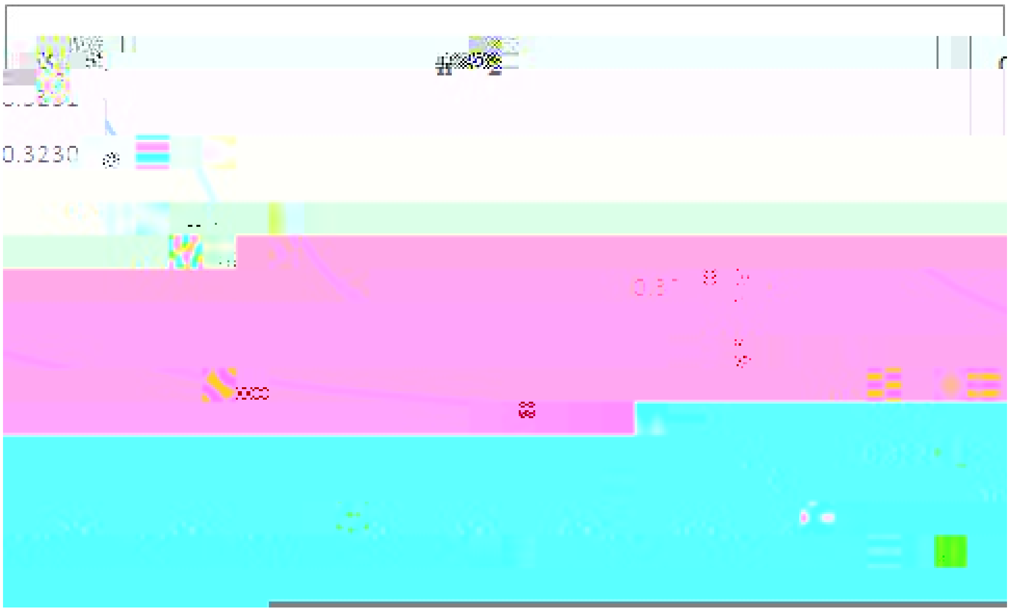


Figure 4.2: Numerical Solution of TFE at for $n = 2$

4.2 Another Approach via Non-Self-Similar Equation

Substituting (4.1) into equation (2.37), we have

$$\frac{\partial}{\partial}(\quad) + \frac{\partial}{\partial}\quad {}^{n}\frac{\partial}{\partial} \quad = \quad t\frac{\partial}{\partial t}. \tag{4.6}$$

Then we discretise (4.6) and this gives a similar expression with that for equation (2.21) but with . Before running the program, the boundary values of are determined by linear extrapolation, that is

$$\begin{aligned} {}_0 &= 2 \ {}_1 \quad {}_2; \\ 2 \ {}_{n \ 1} &= \ {}_{n \ 2}. \end{aligned}$$

We run the program for a few time steps for each $n = 1; 2; 3$ using self-similar solution as initial conditions. The evolutions are shown below:

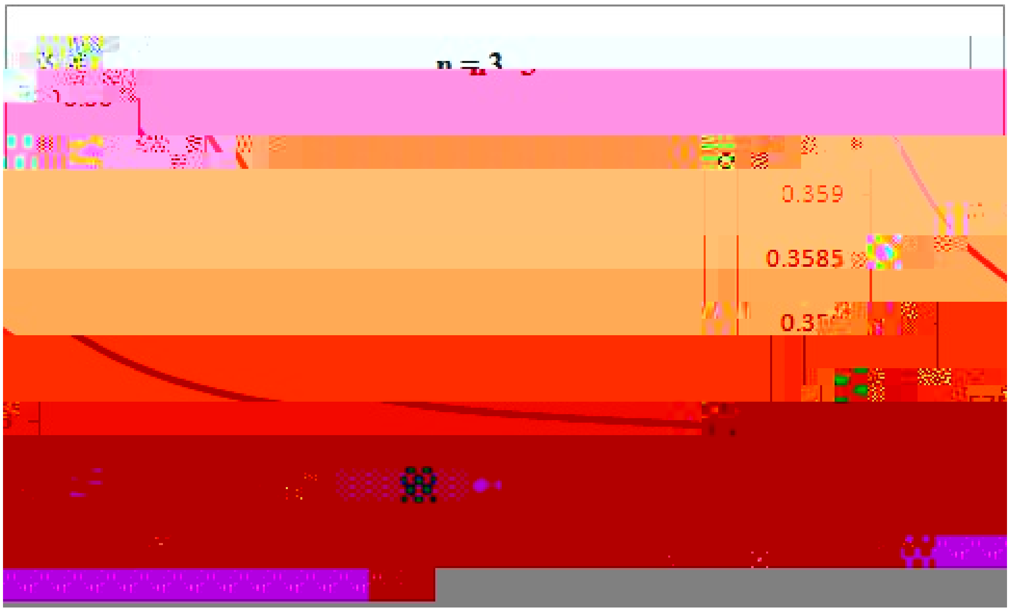


Figure 4.3: Numerical Solution of TFE for $n = 3$

However, when running this program to convergence, that is when $^{k+1} = {}^{k}$, within some tolerance, we find that the numerical solutions were unstable. If we had more time, we could refine the method by using smaller t.

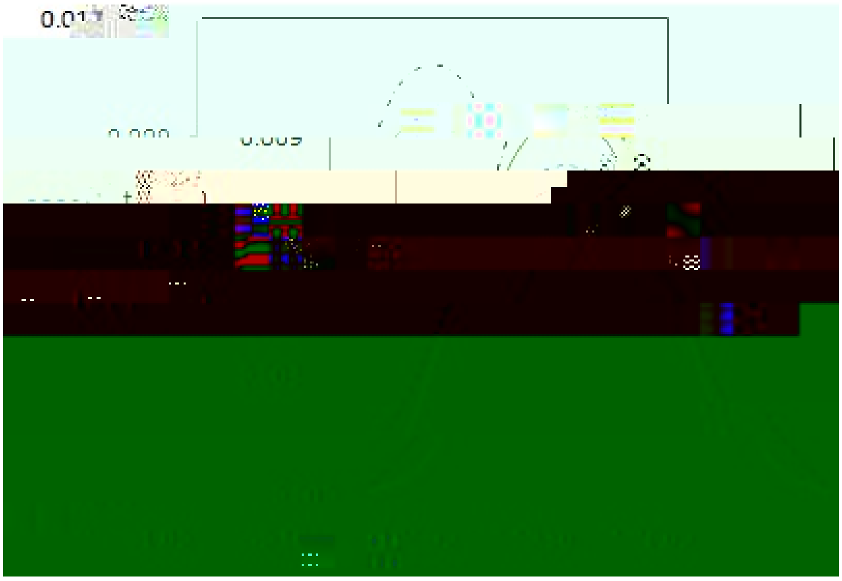


Figure 4.4: Evolution of Self-Similar Solution of TFE at $n = 1$

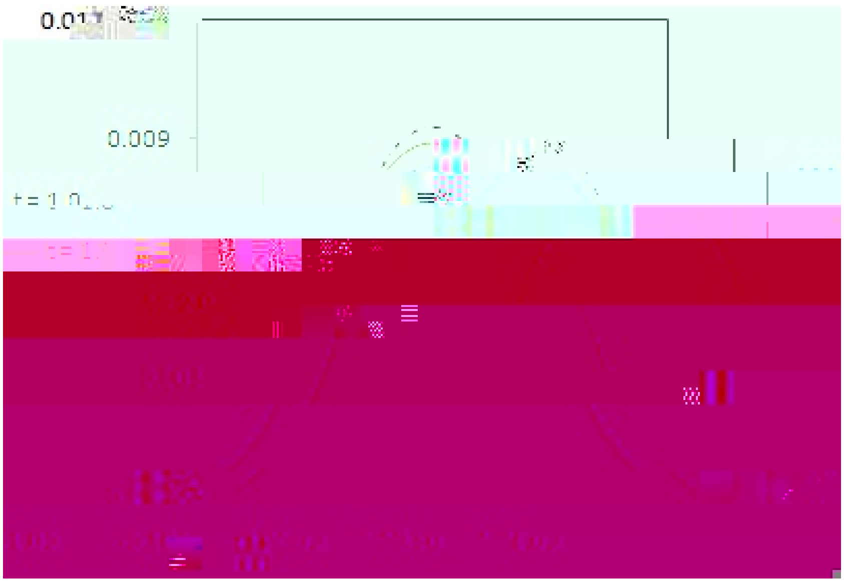


Figure 4.5: Evolution of Self-Similar Solution of TFE at $n = 2$

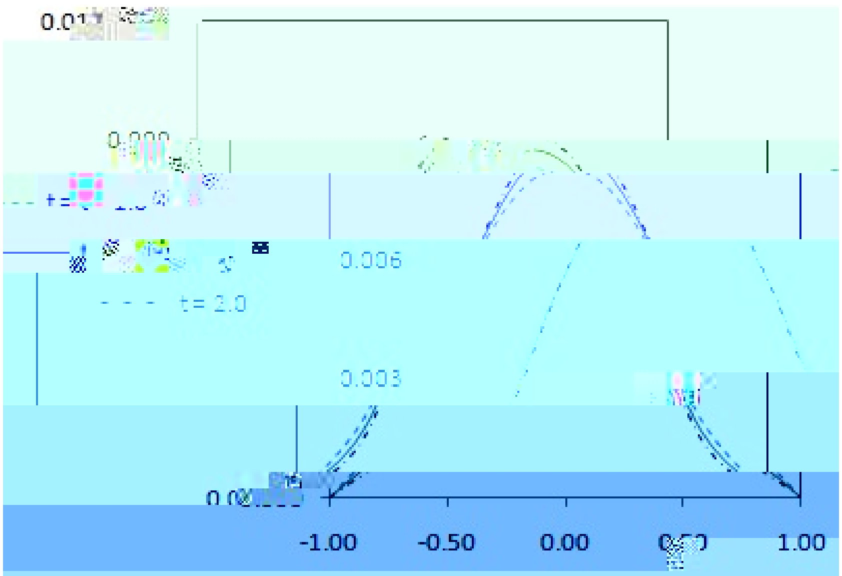


Figure 4.6: Evolution of Self-Similar Solution of TFE at $n = 3$

Chapter 5

Conclusions and Further Work

5.1 Summary

This chapter summarises the work carried out in this dissertation. We then

Unfortunately, the solutions blew up.

5.2 Further Work

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