Singular Values and t e Distance to Instability

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September 1994

Submitted to the

The University of Reading

Department of Mathematics

in partial fulfilment of the requirements for the

Degree of Master of Science

Abstract

A numerical algorithm for constructing a state feedback for a controllable stable linear system is presented and tested. Main attention is given to maximizing the distance to instability such that the system remains stable. Two methods are considered, namely robust eigenstructure assignment and singular value assignment. xamples are looked at to illustrate the theoretical results discussed. A comparison between these two methods is considered and conclusions drawn from the numerical results.

Ac no ledgements

I would like to acknowledge the help and guidance of Dr. N.K. Nichols and the help and support of Professor M.J.Baines throughout the year.

I would also like to acknowledge the help and support of my parents throughout the year.

I also acknowledge the financial support of the S RC.

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Not tion

Symbols	Meaning
A $\in R^{n\times n}$	state matrix
B $\in R^{n\times m}$	input matrix
C $\in R^{p\times n}$	output matrix
K $\in R^{m\times n}$	state feedback matrix
F $\in R^{p\times n}$	output feedback matrix
X $\in R^{n\times n}$	matrix of eigenvectors
$\mathcal{N}$	null space
$\mathcal{U}$	the set of unstable matrices
A+BK $\in R^{n\times n}$	state closed loop matrix
A+BFC $\in R^{n\times n}$	output closed loop matrix
$\Lambda \in R^{n\times n}$	diagonal matrix of eigenvalues (λ_j)
$\mathbf{x} \in R^{n\times 1}$	state vector
$\mathbf{u} \in R^{m\times 1}$	control vector
$\mathbf{y} \in R^{p\times 1}$	output vector
$\mathbf{v} \in R^{m\times 1}$	input vector
$\lambda_j(*)$	jth eigenvalues of *
$\kappa_2(X)$	condition number of X
$\sigma_i(*)$	ith largest singular value of *
$\sigma_{min}(*)$	minimum singular value of *
$\sigma_{max}(*)$	maximum singular value of *
$\|*\|$	the 2-norm of *

The aim of this project is to construct a numerical algorithm for finding a state feedback, for a linear time-invariant control system, which will increase the distance to instability and keep the eigenvalues stable.

Before we consider this work in more detail, it is necessary to give some basic definitions.

A control system may be defined as an arrangement of physical components connected or related in such a manner as to command or regulate itself or another the room. For example, changes in the outside temperature or the opening and closing of doors. A closed loop system is one in which the control action depends on the output in some way. The use of thermostats in order to control the heating system of a room or a house is a well known example of a closed loop system. Both of these systems are given in the following figures.

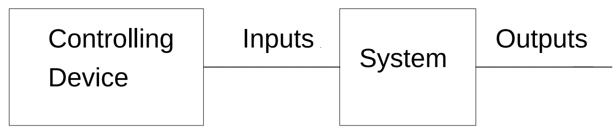

Figure 1.1: An open loop system

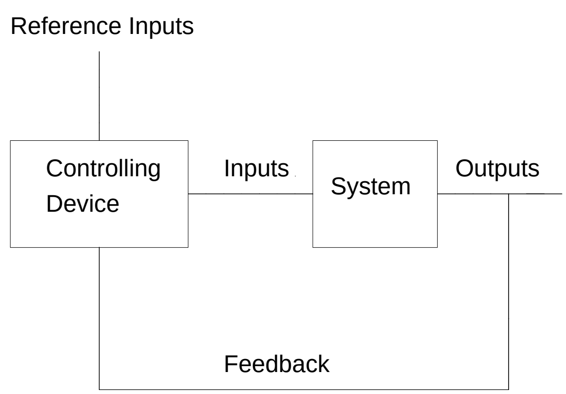

Figure 1.2: A closed loop system

In practical control problems, analysis starts with the formulation of a mathematical model of the physical system under investigation. This is done in chapter 2, and there the problem is formulated and the conditions given for the system to be controllable and observable.

In chapter 3 we look at the method of eigenstructure assignment. This method is used most commonly and is known to produce good results. In this method we are given a set of eigenvalues that we wish to assign and a feedback is sought that will assign these eigenvalues to our system. A numerical algorithm is given for finding a state feedback. Some examples are also given of finding a state feedback. It is known that if we assign the eigenvalues robustly we get a good distance to instability.

In chapter 4 we consider the method of singular value assignment. Here we have some fixed singular values and we wish to assign the remaining singular values. A numerical algorithm is given with some examples. In this chapter we look at finding a state feedback, since algorithms are available that assign singular values in this case. The area of finding an output feedback is not discussed as it is a recent area of research. It is not known what happens to the eigenvalues if we try to increase our distance to instability.

In chapter 5 we look at the distance to instability. A definition is given as well as some theory. We see that the distance to instability depends on the minimum singular value and so the method of singular value assignment would be a good way of increasing this singular value.

In chapter 6 the numerical algorithm for increasing the distance to instability is discussed. We look at some examples where we use this algorithm. We then use robust eigenstructure assignment to find a state feedback and compare whether this method give a better distance to instability than the algorithm discussed.

Controlling
Device

System

Figure 2.1 shows a simple closed loop feedback control system which is employed in order to achieve or maintain a prescribed behaviour (ie stability). The controller examines the difference between the output of the process and the input and so employs a function to control the system. The equations describing the system in Figure 2.1 are

$$\frac{d}{d} = A \; (\;) + B \; (\;) \tag{2.1}$$

$$(\;) = \quad (\;) \qquad (2\ 2)$$

where,

$^{n\times 1}$ is the system state vector,

$^{m\times 1}$ is the system input vector,

$^{p\times 1}$ is the system output vector,

A $^{n\times n}$ is the state matrix,

B $^{n\times m}$ is the input matrix,

C $^{p\times n}$ is the output matrix.

Both the matrices B and C are assumed to be of full rank. If A, B and C are constant then our system is known to be ; otherwise it is

.

Additionally we have the feedback

$$= \quad + \qquad (2\ 3)$$

where,

F $^{m\times p}$ is the constant gain matrix and

^{m} is a reference input.

Then equation (2.1) becomes :

$$— = (\quad + \qquad) \quad + \qquad (2\ 4)$$

$m \times n$

$—$

0

0 1 0

0 1 1

|

$$^{t}[\mathrm{B},\mathrm{AB},.....\ ^{n-1}\mathrm{B}]=[\ ^{t}\mathrm{B},\ \ ^{t}\mathrm{B},...,\ ^{n-1}\ ^{t}\mathrm{B}]=0.$$

t t

t t t t

0

0 o 1 0

0 1 o

We have now looked at conditions that are needed for a system to have either a controller or an observer, that is, the system to be completely controllable or completely observable. In the next chapter we define the problem for pole assignment and singular value assignment.

2.4 Motivation for Pole and Singular Value Assignment

Now we look at the motivation behind pole assignment and we examine the use of a feedback in a particular way to achieve some property. A general time-continuous system can be described by the differential problem

$$\frac{d\mathbf{x}}{d\mathbf{t}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x_0}, \tag{2.9}$$

where the matrix A is of dimension $n \times n$ and is constant.

quation (2.9) has the following solution:

$$\mathbf{x}(t) = exp(At)\mathbf{x_0} \tag{2.10}$$

If we expand the exponential term and take norms, we have

$$\|exp(At)\| \leq \sum_{k=1}^{q}\sum_{j=1}^{\alpha_k} t^{j-1} exp(Re(\lambda_k)t)\|Z_{kj}\| \tag{2.11}$$

where,

Re(λ_k) denotes the real part of the eigenvalues of A,

q is the number of distinct eigenvalues,

α_k is the order of the largest Jordan block associated with the eigenvalues of A

plan to use will assign eigenvalues or poles to precise locations and is commonly known as pole placement. So the motivation behind pole assignment is to find a feedback matrix which makes the system stable by moving the eigenvalues to new locations.

We now look at the motivation behind singular value assignment. It is assumed that we have a stable matrix A in the sense that all the eigenvalues of A have negative real parts. We consider the set of matrices $\mathcal{U}$ which have at least one eigenvalue on the imaginary axis and so are unstable. Then the distance from A to set $\mathcal{U}$ is defined to be:

$$\beta(A) = \min_{E \in \mathbb{C}^{n\times n}} [\|E\| | A + E \in \mathcal{U}]$$

This is a measure of how 'nearly unstable' is the stable matrix A (ie the distance to instability). So we need to find a feedback such that we maximise the distance between our closed loop matrix and the set of the set of unstable matrices. In a later chapter we see that the distance to instability is related to singular values and so the motivation behind singular value assignment is that we wish to find a feedback such the distance to instability is as large as possible. More details on how this is done will be discussed in a later chapter. In either case we require that our new system matrix is stable. Before we look at these methods, we need to give some basic matrix theory.

2.5 Basic Matrix Theory

In this section we describe two basic decompositions of a matrix. Throughout this dissertation we make extensive use of the singular value decomposition(SVD) and the QR decomposition of a matrix M $^{n\times m}$. In the usual notation the SVD is given by:

$$\mathrm{M}=\mathrm{U} \begin{matrix} \Sigma & 0 \\ 0 & 0 \end{matrix} \; ^{t}$$

where U and V are n n and m m orthogonal matrices, respectively, and Σ is a rank(M) rank(M) diagonal matrix with positive diagonal entries. Also we refer to the orthogonal reduction of M to diagonal form:

$$^{t}\,\mathrm{M}\;\mathrm{V} = \begin{matrix} \Sigma & 0 \\ 0 & 0 \end{matrix}$$

as an SVD of M because we always need it in this form.

The $_i$ are the singular values of and the vectors $_i$ $_i$ are the th

and the th , respectively. It is easy to see

^{t} ^{t}

$$_i \quad _i \; _i$$

$$^t{}_i \quad _i \; _i$$

$^{n\times n}$ $^{n\times m}$

Ch pter 3

Robust Eigenstructure Assignment

3.1 Introduction

In this chapter we look at a way of assigning eigenvalues and eigenvectors by state feedback in the linear time invariant system described by the equations (2.1)-(2.2). There are two approaches for doing this:

- by Linear State Feedback.
- by Output Feedback.

In the following section we discuss how we assign eigenvalues and eigenvectors by linear state feedback. We could find a feedback by output feedback but this is not discussed here.

The state feedback pole assignment problem in control system design is essentially an inverse eigenvalue problem; that is, we assign eigenvalues and find the system which has these assigned eigenvalues. A desirable property of any system design is that the poles should be insensitive to perturbations in the coefficients matrices of the system equations. There are many way of assigning eigenvalues discussed in earlier papers [4, 11] but in this section we look for ways of obtaining a robust solution that is '

We now consider the completely controllable, time invariant, linear, multivariate system (2.1)-(2.2). In this section = the identity matrix. The behaviour of the system (2.1) is governed by the eigJB2228c/l

—

The state feedback pole assignment problem for system (2.1) can be formulated as:

Δ $_1$ $_2$ $_n$

$_j$ $= 1$ 2

Given the Problem 3.1 can we find a solution to this?. Conditions for a solution to exist are well known and the following theorem is well established.

Δ

^{t} ^{t} ^{t} $^t =$

^{t} ^{t}

^{t} ^{t} ^{t} ^{t}

1 2

1

( )

2 2

2 2

where $_2($ $)$ is the condition number of the matrix $=[$ $\mathbf{1}$ $\mathbf{2}$ $\mathbf{n}]$

Bearing in mind what we have just discussed, we can now reformulate Problem 3.3 such that we have a pole assignment problem to solve. It can be stated as follows:

Δ

$$(\quad + \quad) \quad = \quad \Lambda \tag{3 4}$$

Λ $_1$ $_2$ $_n$

We could take this measure to be $_1 =$ $_\infty$ where $^t = [$ $_1$ $_2$ $_n]$ is the vector of the condition numbers (ie. the condition number $_j$ defined above) corresponding to the selected matrix of eigenvectors. Alternatively, we could take the measure of robustness to be $_2 =$ $_2($ $)$, the condition number of the

2 1

1 2 n

$$U_1{}^t(AX \quad X\Lambda) = 0, \tag{3.5}$$

$$B = [U_0, U_1] \begin{matrix} Z \\ \\ 0 \end{matrix} \quad , \tag{3.6}$$

$U = [U_0, U_1]$ K

$$K = Z^{-1} U_0{}^T (X \Lambda X^{-1} \quad A). \tag{3.7}$$

See ref [6]

The assumption that B is of full rank implies the existence of the decomposition (3.6). From (3.4), K must satisfy

$$BK = X \Lambda X^{-1} \quad A \tag{3.8}$$

and pre-multiplication by U^t then gives the two equations

$$ZK = U_0{}^t(X\Lambda WW^t P\omega P)PA..i\zeta SSPU \quad Ca.\tau i S(3.9)PU \quad YCiHSHW \tag{3.9}$$

$$0 = \ _1{}^t(\ \Lambda \ ^{-1} \quad) \tag{3 10}$$

from which (3.5) and (3.7) follow directly, since is invertible from our condition that is nonsingular.

We observe that the decomposition of B in (3.6) is in fact a QR decomposition in which is an upper triangular matrix. Alternatively we could take the decomposition to be the Singular Value Decomposition in which we have $= \Sigma \ ^t$, where $\Sigma = \quad (\ _1 \ \ _2 \quad \ _n)$ is a positive matrix and is orthogonal.

Now we have looked at some of the theory behind the assigning of eigenvalues and eigenvectors, we present an algorithm that will do this.

3.3 Numerical Algorithm

We now consider the practical implementation of the theory discussed in the previous section for the linear state feedback design. The following algorithm can be found in [6]. The procedure consists of three basic steps :

•**Step 1:**

Compute the decomposition of matrix B by either using SVD or QR, to find the matrices U_0, U_1 and Z , and construct the orthonormal bases, comprised of the columns of the matrices S_j, $\hat{S}_j$, for the null space $S_j = \mathcal{N}[{U_1}^t(A-\lambda_j I)]$ and its complement $\hat{S}_j$ for $\lambda_j \in \Delta, j = 1, 2, ..., n$.

Standard library software is available to compute the decomposition of B using either SVD or QR. We see that QR is less expensive to compute than SVD but doesn't give as much information as SVD does about the system.

We consider two methods to find the orthonormal bases S_j and $\hat{S}_j$:

Case 1(SVD):

We determine the singular value decomposition of ${U_1}^t(A-\lambda_j I)$ in the form:

$$ {U_1}^t(A-\lambda_j I) = Z_j[\Gamma, 0][\hat{S}_j, S_j]^t. \tag{3.11} $$

Then the columns of S_j and $\hat{S}_j$ give the required orthonormal bases.

Case 2(QR):

We determine the QR decomposition of $({U_1}^t(A-\lambda_j I))^t$ partitioned as the

1^t j t j j j

j j

j j 2

j j $+$

j

1 0^t 1

1

•**Step 4:**

All of the above steps can be carried out using the system MATLAB [7]. This system uses standard library routines from software packages such as LINPACK and ISPACK.

3.4 Examples

In this section we look at some examples which have been collected from the literature [6] for which the numerical procedures in the earlier sections have been used. In two of the examples a linear state feedback control has been used.

Example 1: Chemical Reactor [6]

$$n = 4, m = 2$$

$$A = \begin{bmatrix} 1.380 & -0.0277 & 6.715 & -5.676 \\ -0.5814 & -4.290 & 0 & 0.6750 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.0480 & 4.273 & 1.343 & -2.104 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}$$

IG(A)=(1.991, 6.351 $\times 10^{-2}$, -5.057, -8.6666)

This system can be seen to be unstable (ie. $Re(\lambda_j) > 0$) and a feedback gain matrix is required to stabilize the system. We therefore assign the following eigenvalues Δ=(-0.2, -0.5, -5.0566, -8.6659). If the procedure of Section 3.3 is carried out to find a state linear feedback control of the form **u**= K**x**, we get the following feedback gain matrix (using Method 2/3 in Step 2 [6]):

$$K=\begin{bmatrix} 0.10277 & -0.63333 & -0.11872 & 0.14632 \\ 0.83615 & 0.52704 & -0.25775 & 0.54269 \end{bmatrix}$$

The conditioning of the results are given in the following table [6]:

(a) sol. after two sweeps					(b) sol. at convergence				
Method	$\|\mathbf{c}\|_\infty$	κ_2(X)	$\|\mathbf{c}\|_2$	$\|K\|_2$	$\|\mathbf{c}\|_\infty$	κ_2(X)	$\|\mathbf{c}\|_2$	$\|K\|_2$	Sweeps
0	1.82	3.43	3.28	1.47	-	-	-	-	-
1	1.79	3.38	3.27	1.44	1.76	3.32	3.23	1.40	106
2/3	2.36	4.56	3.71	1.16	2.37	4.54	3.68	1.17	6

Table 3.1 Conditioning

The last column in the table is the number of sweeps needed for convergence. From the table the magnitude of the gain matrix using Method 2/3 is $\|K\|_2$=1.17 and the condition number of the matrix of eigenvectors is κ_2(X)=4.54. The matrix which has these assigned eigenvalues is:

$$A+BK=\begin{bmatrix} 1.38 & -0.20770 & 6.715 & -5.6760 \\ 0.0022062 & -7.8867 & -0.67420 & 1.5059 \\ -1.4468 & 1.8955 & -5.9780 & 4.3519 \\ 0.16474 & 3.5535 & 1.2081 & -1.9378 \end{bmatrix}.$$

The condition number of X is not too large so we conclude that we have fond a well-conditioned solution. If Method 0, is used the best result is obtained after one sweep; if Method one is used, then we have convergence within 106 sweeps compared with 6 when Method 2/3 is used. Although Method 1 gives a better condition number for X which is 3.32, we use a lot of sweeps to achieve this. The maximum condition number $\|\mathbf{c}\|_\infty$ using method 2/3 is increased slightly as is the magnitude of $\|K\|_2$ of the gains.

We now go on and look at a different example which comes from the area of aircraft control. We wish to move the eigenvalues such that they are all real.

<u>**Example 2**</u>: Aircraft control [6]

$$n = 4, m = 3$$

$$\text{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.00014 & -2.04 & -1.95 & 0.013 \\ -0.00025 & 1 & -1.32 & -0.024 \\ -0.56 & 0 & 0.36 & -0.28 \end{bmatrix} \quad \text{B} = \begin{bmatrix} 0 & 0 & 0 \\ -5.33 & 0.0065 & -0.27 \\ -0.16 & -0.012 & -0.25 \\ 0 & 0.11 & 0.086 \end{bmatrix}$$

IG(A)=(-3.12×10^{-2}, -2.46×10^{-1}, $-1.68 \pm 1.35i$)

This time we assign the eigenvalues Δ=($-1, -2, -3, -4$), and so we want all the eigenvalues to be real. Again, if Method 2/3 is used, then we get a state feedback matrix which has $\|K\|$=28.255 after two sweeps and has converged at this point. With the other methods we get the same sort of results with the condition number of K ranging from 25-30. The errors introduced are due to rounding error . More details about this example can be found in [6].

In the examples we have illustrated the method of eigenstructure assignment. In the next chapter we look at the method of singular value assignment. The theory is discussed and then the numerical algorithm is stated to achieve this. In all of the examples either the system given is unstable or we just wish to move the eigenvalues to obtain different system behaviour.

Ch pter

Singul r V lue Assignment

In this chapter we again consider the time invariant continuous dynamical system of the form (2.1)-(2.2) with state feedback (3.1). The closed loop system takes the form (3.2).

The closed loop matrix $A + BK$ gives us the response of the system and therefore we have to choose K to obtain the required behaviour. In this chapter we are interested in assigning singular values which give the system certain properties (i.e. to make the matrix $A+BK$ as well-conditioned as possible or, equivalently, to make the distance to instability as large as possible). The method presented is a numerically stable method. To obtain the feedback matrix we apply a method which employs a number of orthogonal matrix decompositions.

4.1 Preliminary Theory

Again our system has to be completely controllable. The following theorem gives us the basic tool and provides a 'canonical form ' for our system, which can be obtained in a numerically stable way. The theorem is a modification of the theory presented in [2].

Theorem 4.1 *Let $A \in R^{n\times n}$, $B \in R^{n\times m}$ and let rank(B)=$m \leq n$. Then there exists orthogonal matrices Q, U, V such that*

$$QAU=\begin{bmatrix} \Sigma_1 & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad QBV=\begin{bmatrix} 0 \\ \Sigma_B \\ 0 \end{bmatrix}$$

where Σ_1, Σ_B are $l \times l$ and $m \times m$ diagonal matrices respectively with positive diagonal entries and A_{22} is a matrix with full column rank. The partitioning in QAU and QBV is conformable.

Proof. Let

$$\hat{P}\text{BV}=\begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix}$$

be an SVD of the matrix B . Now let

$$\text{P}=\begin{bmatrix} 0 & I_{n-m} \\ I_m & 0 \end{bmatrix}\hat{P}$$

Then we obtain

$$\text{PBV}=\begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}, \qquad \text{PA}=\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

with a compatible partitioning. Let

$$\text{W}A_1Z_1=\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

be an SVD of A_1, where Σ_1 is an $l \times l$ diagonal matrix with positive entries. Then

$$\begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix} \mathrm{PA} Z_1 = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \\ A_{21} & \hat{A}_{22} \end{pmatrix}$$

where $[A_{21}, \hat{A}_{22}]$ is a compatible partitioning of $A_2 Z_1$. Let Z_2 be an orthogonal matrix which does a 'column compression'

$$\hat{A}_{22} Z_2 = [A_{22}, 0]$$

on $\hat{A}_{22}$, such that A_{22} has full column rank. This matrix could, for example, be derived from an QR decomposition of $\hat{A}_{22}^t$.

Then from the above matrices we get the desired transformation as :

$$\begin{pmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \\ A_{21} & \hat{A}_{22} \end{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & Z_2 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix} \mathrm{PBV} = \begin{pmatrix} 0 \\ \Sigma_B \\ 0 \end{pmatrix} .$$

A, B

$$\begin{matrix} \Sigma_1 & 0 \\ A_{21} & [A_{22}, 0] \end{matrix} \qquad \begin{matrix} 0 \\ \Sigma_B \end{matrix}$$

Z_2 such that $\hat{A}_{22}Z_2$=[A_{22},0] where A_{22} is of full rank. This is achieved by the Q-R decomposition of A_{22}^t.

•Step 5:

Then let

$$Q=\begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-l-m} & 0 \end{bmatrix}\begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix}P\hat{P},\ U=Z_1\begin{bmatrix} I_l & 0 \\ 0 & Z_2 \end{bmatrix}$$

•Step 6:

Now we have to choose our assigned singular values. We choose them to be such that Σ_2=diag($\sigma_{l+1},...,\sigma_n$) where $\sigma_l(\Sigma_1) \leq \sigma_j(\Sigma_2) \leq \sigma_1(\Sigma_1), j = l+1,..,n$

•Step 7:

We now find the feedback matrix K such that A+BK has these assigned singular values. Let $\hat{K} = [\hat{K}_1, \hat{K}_2]$ where

$$\hat{K}_1 = -{\Sigma_B}^{-1}A_{21},$$

$$\hat{K}_2 = {\Sigma_B}^{-1}(\Sigma_2 - [A_{22}, 0])$$

and set $K=V\hat{K}U^t$.

4.3 Examples

Example 1

In this section we consider the following numerical example:

$$A=\begin{bmatrix} 4 & 2 & 1.2 \\ 2 & 1.2 & 0.8 \\ 1.2 & 0.8 & 0.5663 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$_{max}$(A)=5.462, $_{min}$(A)=0.003108, $_{2}$(A)=1757.

The matrix A can been seen to be fairly ill conditioned. We therefore want to design a K that will modify this system such that the matrix becomes well conditioned. When the numerical algorithm of Section 4.2 is applied, we get the state feedback matrix:

$$K = \begin{matrix} 2\ 5996 & 3\ 2883 & 2\ 7685 \\ 0\ 8547 & 2\ 5265 & 1\ 1601 \end{matrix}$$

and

$$A+BK= \begin{matrix} 1\ 4004 & 1\ 2884 & 1\ 5685 \\ 2 & 1\ 2 & 0\ 8 \end{matrix}$$

$_{1}$ $_{2}$ $_{3}$

$_{max}$ $_{min}$ $_{2}$

and

$$A+BK=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.0239 & 0 & 0.9993 & -0.0283 \\ 0.7121 & 0 & 0.0369 & 0.7010 \\ -0.7015 & 0 & 0.0034 & 0.7125 \end{bmatrix}.$$

The singular values of this closed loop matrix $A + BK$ are then $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$. Again this matrix is well conditioned with condition number equal to one.

We now look at the distance to instability and how singular value assignment can be used to widen the distance to instability.

Ch pter 5

Dist nce to Inst bility

If we consider a matrix A that is stable in the sense that all its eigenvalues lie in the open left half plane, then the distance to instability is a measure of 'how stable' matrix A is. In this chapter we describe a bisection method which enables us to find this distance.

Suppose that A $\in \mathbb{C}^{n\times n}$ has no eigenvalues on the imaginary axis. Let $\mathcal{U} \subset \mathbb{C}^{n\times n}$ be the set of matrices with at least one eigenvalue on the imaginary axis. The distance from A to $\mathcal{U}$ is defined to be:

$$\beta(A) = \min_{E\in\mathbb{C}^{n\times n}} [\|E\| | A + E \in \mathcal{U}]$$

<u>Theorem</u> 5.1 [3]

$$\beta(A) = \min_{\omega\in R}(\sigma_{min}(A - i\omega I)$$

Proof See [3] for reference.

If matrix A is stable, let B be the closest unstable matrix to A (i.e. B is unstable and minimizes $\|A - C\|$ over all unstable C.) Then B has an eigenvalue on the imaginary axis with the same imaginary part as some of the eigenvalues of A, then one may conclude that,

$$\|A - B\| = \min_{\omega \in R}(\sigma_{min}(A - i\omega I)).$$

where $\sigma_{min}(A - i\omega I)$ is the smallest singular value of $A - \omega I$ (i.e. the distance from A to B an unstable matrix). □.

So for any real ω, an upper bound on β(A) is

$$\beta(A) \leq \sigma_{min}(A - i\omega I)$$

In the next section we describe a bisection method which will enable us to find this distance.

5.1 Bisection Method

If we are given $\sigma \geq 0$ and A $\in R^{n \times n}$, then we may define the 2n $\times$ 2n matrix H=H(σ) by:

$$\text{H=H}(\sigma)= \begin{bmatrix} A & -\sigma I_n \\ \sigma I_n & -A^H \end{bmatrix},$$

where I_n denotes the n by n identity matrix and A^H represents the complex transpose.

The following theorem shows how the eigenvalues of H(σ) distinguish the cases $\sigma \geq \beta(A)$ from $\sigma < \beta(A)$.

<u>**Theorem**</u> **5.2** *H(σ) has an eigenvalue whose real part is zero if and only if $\sigma \geq \beta(A)$.*

Proof Can be found in [3] □.

Suppose that α is a lower bound and γ is an upper bound on β(A). The bounds can be improved by choosing a number σ that lies between α and γ and checking to see if H(σ) has any eigenvalue with a zero real part. The following algorithm gives an estimate of the distance to instability, β(A) to within a factor of ten. Also this algorithm uses the naive upper bound $\beta(\mathrm{A}) \leq 1/2\|A + A^H\|$ found in [3].

Bisection Algorithm.

•Step 1:

Input $A \in \mathbb{C}^{n\times n}$ *and a tolerance* $\tau > 0$

•Step 2:

Finding α *and* γ

$$\alpha = 0 \ , \ \gamma = 1/2\|(A + A^H)\|$$

$$\mathbf{WHILE}\ \gamma > 10\mathbf{MAX}(\tau, \alpha)$$

$$\sigma = \sqrt{\gamma \mathbf{MAX}(\tau, \alpha)}$$

IF *H(σ) has an eigenvalue with zero real part* **THEN** $\gamma = \sigma$ **ELSE** $\alpha = \sigma$

•Step 3

Output $\alpha \in R$ *and* $\gamma \in R$ *such that either* $\gamma \setminus 10 \leq \alpha \leq \beta(A) \leq \gamma$ *or* $0{=}\alpha \leq \beta(A) \leq \gamma \leq 10\tau$.

With the choice of $\tau = 1/2(10^{-8}\|A + A^H\|)$, then at most we require three bisection steps.

Now we require to know the value of ω which gives the smallest singular value, as it is this that we are trying to maximise. There are two way of doing this: either by plotting ω against $\sigma_{min}(A - i\omega I)$ for some range of ω or by simply modifying the bisection algorithm.

Modified Bisection Algorithm

•Step 1:

Input $A \in \mathbb{C}^{m \times n}$, ζ *and a tolerance* $\tau > 0$.

•Step 2:

Finding α *and* γ

$$\alpha = 0 \;,\; \gamma = 1/2\|(A + A^H)\|$$

$$\textbf{WHILE } \gamma > \zeta \textbf{MAX}(\tau, \alpha)$$

$$\sigma = \sqrt{\gamma \textbf{MAX}(\tau, \alpha)}$$

IF *$H(\sigma)$ has an eigenvalue with zero real part* **THEN** $\gamma = \sigma$ **ELSE** $\alpha = \sigma$

•Step 3:

Finding ω:

Take the singular value $\hat{\sigma} = \gamma$

Calculate the eigenvalues of $H(\hat{\sigma})$ *and find the eigenvalues* $\lambda = \pm\omega i$ *which have real part which is zero.*

Calculate $\sigma_{min}(A - i\omega I)$ *for each* ω *and take* ω *for which* $\sigma_{min}(A - i\omega I) = \hat{\sigma}$.

•Step 4:

Output $\alpha \in R$ *and* $\gamma \in R$ *such that either* $\gamma \setminus \zeta \leq \alpha \leq \beta(A) \leq \gamma$ *or* $0 = \alpha \leq \beta(A) \leq \gamma \leq \zeta\tau$, *and* ω *and* $\hat{\sigma}$

In the modified algorithm we again take the tolerance τ to be as before, but this time ζ is taken to be less than 10 as we want the error on $\beta(A)$ to be quite small. Then our estimate of the minimum singular value will be as accurate as possible, and our estimate of ω will be close to the real value of ω.

5.2 Examples

In the examples to follow we use the modified bisection algorithm of Section 5.1 to calculate the distance to instability and then find the corresponding value of ω.

Example 1[10]

$$A=\begin{bmatrix} * & 4 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -10 & 4 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & * & 4 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & * & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & * & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & -4 & * & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & * \end{bmatrix}$$

Note in this example $^*=-1\times10^{-5}$

The eigenvalues are -10^{-5}, -10, $-10^{-5} \pm 2i$, $-10^{-5} \pm 4i$, and $-10^{-5} \pm 6i$ and can be seen to be distinct.

From the modified algorithm we get a value of $0.29738124\ \times10^{-5}$ for $\beta(A)$ with $\omega = \pm 3.99$ and ζ=1.00001. This is verified by plotting and can been seen in Figure 5.1.

$A, B, \zeta, \gamma, \tau, \alpha, H(\sigma)$.

In this step we calculate the minimum singular values and the corresponding ω.

$\sigma = \overline{\gamma MAX(\tau, \alpha)}$ λ_j $H(\sigma)$

λ_j $Re(\lambda_j) > 0$ $\gamma = \sigma$ $\alpha = \sigma$

$\gamma < \zeta MAX(\tau, \alpha)$.

$\hat{\sigma} = \gamma$ $H(\hat{\sigma})$ $\omega_j = Im(\lambda_j)$

$Re(\lambda_j) = 0$

A $i\omega_j I$ $\sigma_{min}(A$ $i\omega_j I)$ $\sigma_{min}(A$ $i\omega_j I)$ $\hat{\sigma}$ $\omega_j = \omega$

j

In this step we aim to find the state feedback which increases the distance to instability.

$\hat{} =$

$\hat{} = \Sigma_1 \quad 0 \quad {}_{21} \quad [\ {}_{22}\ 0] \quad = 0 \quad \Sigma_B$

Σ_1 ${}^{l\times l}$ $[\ {}_{21}\ 0]$ ${}^{n-l\times n-l}$

$$_l(\Sigma_1) \quad _i(\Sigma_2) \quad _1(\Sigma_1) \quad = \quad +1$$

$$\Sigma_2 = \quad (\; _{l+1} \quad _n)$$

$$= \quad \hat{} \quad ^t \qquad \hat{} = [\,\hat{}_1 \quad \hat{}_2]$$

$$\hat{}_1 = \quad \Sigma_B{}^{-1} \quad _{21}$$

$$\hat{}_2 = \Sigma_B^{-1}(\Sigma_2 \quad [\quad _{22} \; 0])$$

$+$ $\hat{}_1$

$_1$

$\hat{}_1 \quad _1 \quad \hat{} \quad +$

The following examples were computed using the MATLAB package. The programs are given in Appendix 1. In each example we apply the numerical algorithm described in Section 6.1.

___________[6]

$$= \begin{matrix} 0 & 1 & 0 & 0 \\ 0\;00014 & 2\;04 & 1\;95 & 0\;013 \\ 0\;00025 & 1 & 1\;32 & 0\;024 \end{matrix}$$

When the algorithm of Section 6.1 is applied we get the state feedback:

$$K = \begin{bmatrix} 0.15786 & -0.6184 & -0.2843 & 0.2045 \\ 1.0597 & -3.5707 & 0.8932 & 11.837 \\ -3.0017 & 4.5672 & -5.2887 & -3.59913 \end{bmatrix}$$

where the singular values of A+BK are $\sigma_i = 1, i = 1, 2, 3, 4$.

The results are as follows:

*	$\beta(*)$	ω	$\lambda_j(*)$
A	0.010912	0	-0.031, -0.2473, $-1.6809 \pm 1.3504i$
$A+BK$	0.53813	± 0.85	$-0.535 \pm 0.8443i, 0.9106 \pm 0.4312i$

Table 6.1 xample 1

As we can see from the Table 6.1 we have managed to increase our distance to instability, but the eigenvalues have moved from eigenvalues which were stable to eigenvalues that are unstable. We observe that $\|K\|$=13.8512. Let see if we get the same sort of results with another example.

Example 2[6]

$$A = \begin{matrix} 0 & 1 & 0 & 0 \\ -10.940 & -6.4894 & 1.5838 & 0.023645 \\ -1.5163 & 0.16176 & -0.51425 & 0.042692 \\ -0.44748 & -0.087530 & 0.20686 & -2.9964 \end{matrix},$$

$$B = \begin{matrix} 0 & 0 \\ -0.172 & 0.0000745 \\ -0.0238 & -0.0000778 \\ 0 & 0.00369 \end{matrix}.$$

When the algorithm is applied we get the state feedback matrix:

$$K = 10^2 \times \begin{bmatrix} -0.5785 & -0.3689 & 0.085 & 0.001585 \\ 1.20939 & 0.29728 & -0.6476 & 9.47286 \end{bmatrix}$$

where the singular values of A+BK are $\sigma_1 = 1.54989, \sigma_2 = 1, \sigma_3 = 0.5$ and $\sigma_4 = 0.466129$.

The results for this example are:

$*$	$\beta(*)$	ω	$\lambda_j(*)$
A	0.463111	0	$-1,\ -2,\ -3,\ -4$
A+BK	1.265059	± 0.90	2.86922, -1.2999$\pm$6.47732i -3.3044

Table 6.2 xample 2

We see from Table 6.2 that we have increased the distance to instability but again the eigenvalues have moved. In this example we find $\|K\|$=9.5767 $\times 10^2$. If all we wanted was to maximise the distance to instability and were not worried about the eigenvalues this, would be fine. Unfortunately we require the distance to instability to be increased and our eigenvalues to remain stable. We now consider $A + \alpha BK$, instead of $A + BK$, and find the value of α where the eigenvalues change from a stable set to an unstable set. In the examples to follow we wish to find this α. In all of the examples $\alpha \in [0, 1]$.

				$j0$	1.040	-1.04	1.01-3.62

[6]

$$A = \begin{array}{cccc} 1.38 & 0.20770 & 6.715 & 5.676 \\ 0.0022062 & 7.8867 & 0.67420 & 1.5059 \\ 1\ 4468 & 1\ 8955 & 5\ 9780 & 4\ 3519 \\ 0\ 16474 & 3\ 5535 & 1\ 2081 & 1\ 9378 \end{array} \qquad = \begin{array}{cc} 0 & 0 \\ 5\ 679 & 0 \\ 1\ 136 & 3\ 146 \\ 1\ 136 & 0 \end{array}$$

We get the state feedback matrix,

$$= \begin{array}{cccc} 0\ 6616 & 1\ 4225 & 0\ 7077 & 0\ 7158 \\ 2\ 1271 & 1\ 0166 & 2\ 7255 & 0\ 93764 \end{array}$$

when the algorithm of Section 6.1 is applied. The closed loop matrix $+$ has the singular values $_1 = 9\ 3212$ $_2 = 8\ 5$ $_3 = 7\ 5$ and $_4 = 4\ 926141$

The results are shown in the following table:

unstable at 0.08. In all of the cases we notice that when the eigenvalues go from a real pair to a complex pair the condition number increases and then decreases. When our eigenvalues become unstable then our distance to instability decreases to zero and increases soon afterwards. If a different set of assigned singular values were assigned, as in the next example, we observed the same behaviour as in the

---

1

2 3 4

2 3 i

The matrices A and B of xample 1 in Section 6.2 are used. The following

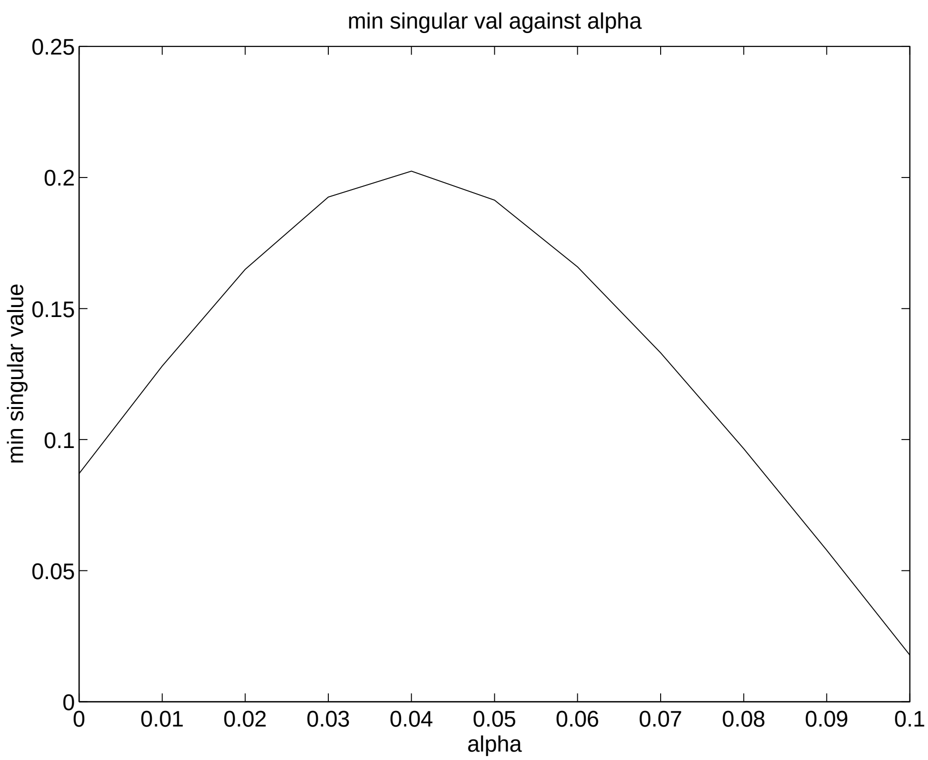

Figure 6.3: xample 5

Conclusions

In this dissertation we have described an algorithm that aims to construct a state feedback to maximize the distance to instability. The method is based on a numerically stable approach. This method however turns out not to be a good one if we want to increase the distance to instability as well as keeping the eigenvalues stable. We see from the numerical results that in fact we have an optimum value where the distance to instability is increased as well as the eigenvalues staying stable, although this value tends to be lower than the distance observed by the algorithm. Unfortunately there is no time in this dissertation to construct an algorithm that will find the optimum state feedback and so opens up a new area of research. If on the other hand we obtain this feedback by robust eigenstructure assignment then we can guarantee that we have a stable set of eigenvalues and have increased the distance to instability.

Appendix 1

rogr ms in M tl b Not tion

Throughout this dissertation the following programs were used to generate the results. The programs were written using the MATLAB package [7].

- **Plot1.m**
- **Eigplot.m**
- **Byers1.m**
- **Sing1.m**

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