High Frequency Boundary Element Methods for Scattering by Convex Polygons

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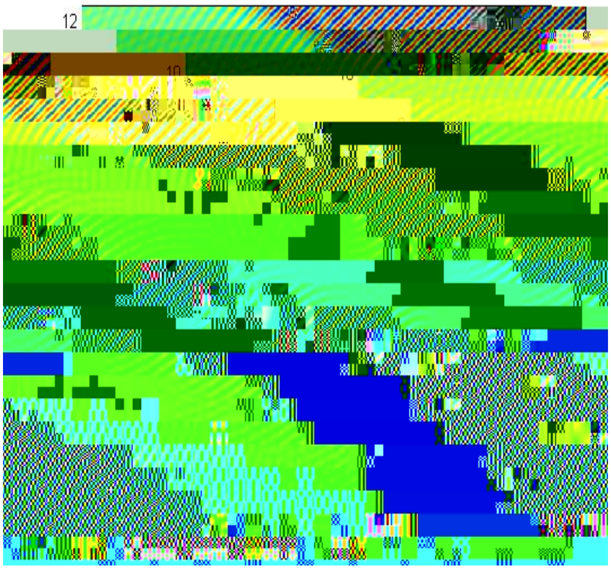
Declaration

I confirm that this work is my own and the use of all other material from other sources has been properly and fully acknowledged.

Abstract



We consider a numerical approximation to the scattering of a high-frequency plane wave by a sound soft convex polygon. By reformulating the domain problem to a boundary problem, we approximate the solution on the boundary by piecewise polynomials multiplied by waves. Using theory of the best approximation of polynomials we aim to show an error bound and how it varies with the number of mesh points and the polynomial degree. We discover that we can achieve exponential convergence, as well as seeing the optimum values and ratios of the parameters involved.

Contents

1 Introduction

1.1 Background

In engineering and physics there are many problems where the acoustic scattering of objects in a medium by a given wave require simulating, for example sonar and noise reduction. In a homogeneous medium the pressure $P(\mathrm{x}, t)$ satisfies the wave equation

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0$$

where c is the wave speed in the medium. By considering only the time-harmonic case with angular frequency ω, the pressure can be written as

$$P(\mathrm{x}, t) = \mathrm{Re}\left[u(\mathrm{x})\mathrm{e}^{-\mathrm{i}\omega t}\right]$$

The function $u(\mathrm{x})$ is known as the *complex acoustic pressure*. By substituting this back into the wave equation we get the Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{1.1}$$

Here

$$k := \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

where f is the frequency and λ is the wavelength of the incoming wave. A result of Green's second theorem is that if (1.1) is satisfied in a domain D with boundary ∂D, the solution $u(\mathrm{x})$ must satisfy the following integral equation for all $\mathrm{x} \in D$ with $\mathrm{x} \neq \mathrm{x}_0$,

$$u(\mathrm{x}) = G(\mathrm{x}_0, \mathrm{x}) + \int_{\partial D}\left[G(\mathrm{y}, \mathrm{x})\frac{\partial u}{\partial n}(\mathrm{y}) - u(\mathrm{y})\frac{\partial G(\mathrm{y}, \mathrm{x})}{\partial n(\mathrm{y})}\right] ds(\mathrm{y})$$

where

$$G(\mathrm{x}, \mathrm{x}_0) := -\frac{\mathrm{i}}{4}H_0^{(1)}(k|\mathrm{x} - \mathrm{x}_0|)$$

is a fundamental solution of the Helmholtz equation. The function $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and its real and imaginary parts are Bessel functions. Note the expression $\mathrm{n}(\mathrm{y})$ denotes the normal direction at y, and the expression

$$\frac{\partial G(\mathrm{y}, \mathrm{x})}{\partial n(\mathrm{y})}$$

is the rate of increase of $G(\mathrm{y}, \mathrm{x})$ as y moves o the boundary in the direction $\mathrm{n}(\mathrm{y})$. The integral is sometimes called a *Green's representation formula*, and has the crucial property that if we know the values of u and $\frac{\partial u}{\partial n}$ on ∂D, we have an explicit formula for computing the solution throughout the domain D, though not on ∂D.

The boundary condition considered for this dissertation is the 'sound-soft' condition $u = 0$ on the boundary of a convex polygon Ω, and $D = \mathbb{R}^2 \setminus \Omega$. This removes one of the terms of the integral equation, and additionally wave energy is scattered. When this scattered energy is distributed uniformly across a circle of radius r, the modulus of $|u^s|$ should tend as $r^{\frac{1}{2}}$.

In [1] it is shown that the integral equation can be reformulated as

$$\frac{1}{2}\frac{\partial u}{\partial n} + \int_{\partial D} \left(\frac{\partial \Phi(\mathrm{x},\mathrm{y})}{\partial n(\mathrm{x})} + \mathrm{i}\eta\Phi(\mathrm{x},\mathrm{y}) \right) \frac{\partial u}{\partial n} ds(\mathrm{y}) = f(\mathrm{x})$$

where I is the identity operator, $f = \frac{\partial u^i}{\partial n}(\mathrm{x}) + \mathrm{i}\eta u^i(\mathrm{x})$, and $\Phi(\mathrm{x},\mathrm{y}) = -G(\mathrm{x}, \mathrm{y}$I iu0(

1.2 Motivation

spaces determined by lesser values of n and p, then it could be seen that the approximation space tends to the solution space. Furthermore, standard results about normed spaces hold in such a situation (e.g. there exists a best approximation which minimises the error, the error term is orthogonal to the approximation space and the related equations to solve are relatively simple), and the convergence rate with respect to n and p can be calculated separately.

1.3 Overview

In section 2.1 we will consider the error in approximating over $[-1, 1]$ a function which is analytic in a domain around $[-1, 1]$. The best polynomial approximation will be taken using Chebyshev polynomials and the error calculation will be minimised.

In section 2.2 we will use the results from section 2.1 to find the minimum error for a problem on a geometric mesh, and seek to minimise the error bound with respect to the parameters.

In section 2.3 we consider a function which is singular at the origin and seek an error bound from a geometric mesh, using results from the previous two sections.

In section 3.1 we will review current theory about the normal derivative of the total field of a polygon scattering problem, and show how it relates to the work done in the previous sections.

In section 3.2 we take a program designed to solve the *hp* problem, and taking large values of the parameters as an 'exact' solution, we compare how the approximation error varies with the number of degrees of freedom. We then present the results.

In section 3.3 we analyse the results and compare the deductions, theory, and global progress in the field for an overall conclusion.

In section 3.4 we shall describe ideal avenues of further research to improve upon or verify ideas that have resulted from this project.

2 Part I: Theory of *hp*-approximants in 1-D

2.1 Best Polynomial Approximation of Analytic Functions in the L_2-Norm

Definition 2.1 *We denote by* E_ρ, $\rho > 1$ *the ellipse*

$$E_\rho := \Big\{ z \in \mathbb{C} \Leftrightarrow$$

$$= \frac{1}{2\pi i}\oint_{|z|=r_1} (u-u_p)\left(\frac{z+z^{-1}}{2}\right) z^{n-1}dz + \frac{1}{2\pi i}\oint_{|z|=r_2} (u-u_p)\left(\frac{z+z^{-1}}{2}\right) z^{-n-1}dz$$

for some $\rho^{-1} \le r_1, r_2 \le \rho$. Therefore

$$|a_n| \le \frac{1}{2\pi}\oint_{|z|=r_1} \left|(u-u_p)\left(\frac{z+z^{-1}}{2}\right)\right| \left|z^{n-1}\right| dz + \frac{1}{2\pi}\oint_{|z|=r_2} \left|(u-u_p)\left(\frac{z+z^{-1}}{2}\right)\right| \left|z^{-n-1}\right| dz$$

$$\le M\left(r_1^n + r_2^{-n}\right).$$

This bound holds for all r_1, r_2 s.t. $\rho^{-1} \le r_1, r_2 \le \rho$, and is clearly minimised when $r_1 = \rho^{-1}$, $r_2 = \rho$, meaning $|a_n| \le 2M\rho^{-n}$. This leads to

$$\|u-u_p\|^2_{L_2[-1,1]} = \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} a_n T_n(t)\right)^2 dt$$

$$\le \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} |a_n||T_n(t)|\right)^2 dt \le \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} |a_n|\right)^2 dt$$

$$= 2\left(\sum_{n=p+1}^{\infty} |a_n|\right)^2 \le 2\left(\sum_{n=p+1}^{\infty} 2M\rho^{-n}\right)^2 = 8M^2\left(\frac{\rho^{-p}}{\rho-1}\right)$$

$$= \int_{-1}^{1} \left(\sum_{n=p+1}^{\infty} a_n T_n(t) \right) \left(\sum_{n=p+1}^{\infty} \overline{a_n} T_n(t) \right) dt$$

$$= \sum_{n=p+1}^{\infty} |a_n|^2 \int_{-1}^{1} T_n^2(t)dt \; + \; 2 \sum_{n=p+1}^{\infty} \sum_{m=n+1}^{\infty} \mathrm{Re}(a_n \overline{a_m}) \int_{-1}^{1} T_n(t) T_m(t)dt$$

Chebyshev polynomials are orthogonal with respect to the norm

$$(f, g) = \int_{-1}^{1} \frac{f(t)\overline{g}(t)}{\sqrt{1 - t^2}} dt$$

but not the $L_2[-1, 1]$ norm. In this case, with an appropriate coordinate change,

$$= \frac{8M^2}{\rho^2-1}\sum_{n=p+1}^{\infty}\left(\frac{1}{3}+\frac{1}{4(n+1)^2-1}\right)\rho^{-2n} =$$

$$\le \frac{8M^2}{\rho^2-1}\left(\frac{1}{3}+\frac{1}{4(p+2)^2-1}\right)\sum_{n=p+1}^{\infty}\rho^{-2n}$$

$$\le \frac{8M^2\rho^{-2p}}{(\rho^2-1)^2}\left(\frac{1}{3}+\frac{1}{4(p+2)^2-1}\right)$$

and so

$$\|u-u_p\|_{L_2[-1,1]} \le \sqrt{\frac{4M^2\rho^{-2p}}{\rho^2-1}+\frac{8M^2\rho^{-2p}}{(\rho^2-1)^2}\left(\frac{1}{3}+\frac{1}{4(p+2)^2-1}\right)}$$

$$= \frac{2M\rho^{-p}}{\rho-1}\left(\sqrt{\frac{\rho^2-\frac{1}{3}+\frac{2}{4(p+2)^2-1}}{\rho+1}}\right)$$

This is better than the previous bound by a factor of

$$K_{\rho,p} := \frac{\sqrt{\rho^2-\frac{1}{3}+\frac{2}{4(p+2)^2-1}}}{(\rho+1)\sqrt{2}}$$

which, as can be shown by taking limits of ρ and p, satisfies

$$\frac{1}{\sqrt{}}$$

and so the L_2 error on the transformed approximation is bounded by

$$M2\sqrt{2}\frac{\rho_i^{-p}}{\rho_i - 1} = M\sqrt{2}\left(\frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_{i+1}} + \sqrt{x_i}}\right)^p \frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_i}}$$

A simple argument to do with areas shows the square of the L_2 error on $[-1, 1]$ is a linear multiple of the square of the L_2 error on the original mesh. The appropriate scale factor to apply is $\frac{x_{i+1}-x_i}{2}$, meaning the total error bound becomes

$$M^2 x_1 + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |u(t) - \Pi_{p,n}u(t)|^2 \, dt$$

$$\leq M^2 \left(x_1 + \sum_{i=1}^{n-1} \left(\frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_{i+1}} + \sqrt{x_i}}\right)^{2p} \frac{\left(\sqrt{x_{i+1}} - \sqrt{x_i}\right)^2}{x_i} (x_{i+1} - x_i) \right) \tag{2.1}$$

This applies for any mesh on $[0, 1]$, and so in theory it would be possible to differentiate w.r.t. the x_i and thus calculate the optimum mesh of size n on which we approximate with piecewise polynomials of degree $\leq p$. But as is clear from the expression above, the resulting system is nonlinear and would be very difficult to solve (though an iterative process might be possible, noting that $x_n = 1$ and the expressions for the derivatives are identical for $i = 2, 3, \ldots n - 1$). Instead we shall apply a geometric mesh of parameter σ, which makes the sum far easier to evaluate largely due to the simplification:

$$\rho_i = \frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}} \quad \forall i$$

The overall error bound becomes

$$\leq M^2 \left(\sigma^{n-1} + \sum_{i=1}^{n-1} \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \sigma^{n-i-1} \left(1 - \sqrt{\sigma}\right)^2 \left(\sigma^{-1} - 1\right) \right)$$

$$= M^2 \left(\sigma^{n-1} + \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \left(1 - \sqrt{\sigma}\right)^2 \left(\sigma^{-1} - 1\right) \sigma^{n-1} \sum_{i=1}^{n-1} \sigma^{-i} \right)$$

$$= M^2 \left(\sigma^{n-1} + \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \left(1 - \sqrt{\sigma}\right)^2 (1 - \sigma)\, \sigma^{n-2} \frac{\sigma^{-1}(\sigma^{1-n} - 1)}{\sigma^{-1} - 1} \right)$$

$$= M^2 \quad \sigma^{n-1} + \quad 1 -$$

$(0,1)$, the function resulting from taking the logarithm would have a stationary value at the same point. Thus

$$\log\sigma\sqrt{\frac{2N}{\log\sigma}\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)} = \sqrt{2N\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)\log\sigma}$$

Squaring this function will not change the stationary value of σ, and will make the result much easier to differentiate. Though it is obvious at this point that the constant $2N$ can be removed. Noting that

$$\frac{d}{d\sigma}\log\left(1\pm\sqrt{\sigma}\right) = \frac{\pm 1}{2\sqrt{\sigma}\,(1\pm\sqrt{\sigma})}$$

we find

$$\begin{aligned}
&\frac{d}{d\sigma}\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)\log\sigma \\
&= \frac{1}{\sigma}\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) + \log\sigma\left(\frac{-1}{2\sqrt{\sigma}\,(1-\sqrt{\sigma})} - \frac{1}{2\sqrt{\sigma}\,(1+\sqrt{\sigma})}\right) \\
&= \frac{1}{\sigma}\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) - \frac{\log\sigma}{\sqrt{\sigma}\,(1-\sigma)}
\end{aligned}$$

The minimum value is therefore reached when

$$(1-\sigma)\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) = \sqrt{\sigma}\log\sigma$$

which is at precisely $\sigma = \sigma_{opt} = \left(\sqrt{2}-1\right)^2 \approx 0.17$ which matches the result in [3]. Finally, with a value of σ

where

$$C_\sigma := \left(\frac{2}{\log \sigma} \log \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right) \right)$$

Note C_σ is a strictly increasing function of σ over $(0,1)$. Note further that I have excluded the factor $K_{\rho,p}$ from the working out simply because it is a constant multiplied onto the second term in the expression for $\| u - \Pi_{p,n} u \|_{L_2[0,1]}$ and does not affect the results of the decay rates or the optimum value of σ.

2.3 *hp*-approximations of Smooth Functions

We now draw our attention to the case when u is analytic for $\mathrm{Re}(z) > 0$ and

- As $A \to 0$, $F(A) = A^{-\alpha} \to \infty$
- As $A \to x_a$, $\rho \to 1$ and so $\frac{1}{\rho-1} \to \infty$

Therefore there exists a minimum value between 0 and x_a, which will occur at a stationary point. In fact there is only one stationary point in $(0, x_a)$, and the function is meaningless outside it. Di erentiating w.r.t. A results in the following equation to be solved

$$\frac{\rho}{\rho}\left(1 + p + \frac{1}{\rho - 1}\right) = \frac{F\ (A)}{F(A)} \tag{2.3}$$

where we have assumed that F is smooth. Particular results are

$$\frac{d\rho}{dA} = - \qquad \rho(x$$

and so (2.2) becomes

$$\frac{\rho^{-p}}{\rho - 1}\left(\frac{v}{2} - \frac{w}{4}(\rho + \rho^{-1})\right)^{-\alpha} \tag{2.4}$$

Note that for any three functions f, g, h the chain rule dictates that

$$(fgh)' = fgh\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)$$

and thus the product fgh is only stationary when the term in the brackets is zero. For (2.4) this results in

$$\frac{\alpha(\rho - \rho^{-1})}{\rho^2 - 2\frac{v}{w}\rho + 1} + p$$

show us that the square of the L

$$\frac{M^2\beta^{1-2\alpha}\sigma^{(n-1)(1-2\alpha)}}{1-2\alpha} + \frac{4M^2\beta^{1-2\alpha}\rho^{-2p}(\sigma^{-1}-1)\sigma^{(n-1)(1-2\alpha)}\theta^{-2\alpha}}{(\rho-1)^2}\frac{\sigma^{(2\alpha-1)l}-1}{\sigma^{2\alpha-1}-1}$$

$$+ \frac{4M^2\rho^{-2p}(\sigma^{-1}-1)\theta^{-1}(n-l-2)}{(\rho-1)^2}$$

$$= \frac{M^2\beta^{1-2\alpha}\sigma^{(n-1)(1-2\alpha)}}{1-2\alpha} + \frac{4M^2\beta^{1-2\alpha}\rho^{-2p}(\sigma^{-1}-1)\sigma^{(n-l-1)(1-2\alpha)}\theta^{-2\alpha}}{(\rho-1)^2}\frac{1-\sigma^{(1-2\alpha)l}}{\sigma^{2\alpha-1}-1}$$

$$+ \frac{4M^2\rho^{-2p}(\sigma^{-1}-1)\theta^{-1}(n-l-2)}{}$$

theory predicts that the res[illegible]und will be non-optimal, so n[illegible] it does not violate the boun[illegible] ideal decaying term is bound[illegible]

$$[illegible]\frac{N}{2\alpha} \frac{-\log\rho}{\log\sigma}$$

and minimising it over σ is [illegible]t to minimising

$$[illegible]\log\sigma$$

which we can diﬀerentiate to [illegible] find the minimum value [illegible]lts in having to solve the follow[illegible]

$$\sigma\log\sigma \left(\frac{1}{1-\sigma} + \frac{[illegible]}{\sqrt{1-\sigma [illegible]}} [illegible] - \frac{[illegible]}{\sqrt{\sigma - \sigma\theta([illegible]}} \right.$$

3 Part II: High Frequency Boundary Element Methods for Scattering by Convex Polygons

3.1 Scattering by Convex Polygons and the Analytic Solution

In the case of a sound-soft smooth shape, when an incident plane wave hits the object there is primarily a reflection and a small amount of diffraction which decays exponentially. However the sharp corners of a polygon cause a strong diffraction, such that the theory predicts u to be unbounded at each corner. In fact it is shown in [steve and simon paper] that the polynomials on each edge satisfy

$$k^{-n}\left|v_{\pm}^{(n)}(s)\right| \leq \begin{cases} C_n(ks)^{-\alpha_{\pm}-n}, & ks \leq 1 \\ C_n(ks)^{-\frac{1}{2}-n}, & ks \geq 1 \end{cases}$$

where

$$\alpha_{\pm} := 1 - \frac{\pi}{a_{\pm}}$$

and $a_{\pm}$ is the external angle at the corner at which $v_{\pm}$ is highly peaked. Because the polygon is convex, this gives $\alpha_{\pm} \in (0, \frac{1}{2})$. Thus the work done in the previous section will give a suitable bound on the error between $v_{\pm}$ and the approximation polynomials. Recalling that on each edge

$$\frac{1}{k}\frac{\partial u}{\partial n}(s) = \Psi(s) + \frac{i}{2}\left(e^{iks}v_{+}(s) + e^{-iks}v_{-}(s)\right)$$

it would suffice to use the maximum value of α determined by the minimum angle in the polygon, and then multiply the bound by the number of sides. In fact, under that assumption we get

$$\varphi - \varphi \quad \leq B \left\| v_+ - v_+ \right\|$$

where B is the number of sides of the polygon.

3.2 A High Frequency *hp*-version Galerkin Method

The hp-version Galerkin Method (convpolyhp.m) composed by Dr. S. Lang-

$$(1, 6), (2, 3), (3, 2), (6, 1)$$

But to compare errors in the solution, we must realise that as n and p increase, the numerical solution tends to the analytic solution. So to make an estimate of the error we need an approximation space with large n and p such that all other approximation spaces are contained in it. The smallest space which contains all possible spaces with N degrees of freedom is the space (N, N), which has N^2 degrees of freedom and so is a much more accurate result.

For the programming part of this project we will run the code for various subspaces of (8,

3.3 Numerical Results

Relative errors for $k = 2$

	n →							
p	0.96	0.88	0.79	0.74	0.63	0.60	0.54	0.49
↓	0.84	0.70	0.61	0.54	0.49	0.43	0.39	0.35
	0.67	0.49	0.38	0.31	0.26	0.22	0.19	0.17
	0.47	0.26	0.18	0.16	0.18	0.19		
	0.32	0.22	0.18	0.08				
	0.33	0.13	0.09	0.09				
	0.34	0.13	0.09					
	0.34	0.13	0.09					

Relative errors for $k = 3$

	n →							
p	0.97	0.91	0.82	0.76	0.70	0.62	0.59	0.51
↓	0.87	0.72	0.61	0.53	0.48	0.42	0.38	0.33
	0.68	0.49	0.38	0.30	0.26	0.23	0.22	0.22
	0.47	0.26	0.22	0.24	0.27	0.29		
	0.34	0.30	0.25	0.15				
	0.34	0.19	0.16	0.15				
	0.34	0.19	0.16					
	0.34	0.19	0.16					

Relative errors for $k = 4$

	n →							
p	0.99	0.97	0.89	0.81	0.77	0.72	0.68	0.65
↓	0.91	0.79	0.70	0.63	0.58	0.54	0.51	0.47
	0.75	0.60	0.51	0.46	0.42	0.40	0.38	0.38
	0.58	0.42	0.38	0.37	0.38	0.38		
	0.46	0.39	0.36	0.33				
	0.47	0.35	0.33	0.33				
	0.47	0.35	0.33					
	0.47	0.35	0.33					

Relative errors for $k = 5$

	n	→						
p	0.98	0.95	0.88	0.84	0.77	0.73	0.68	0.64
↓	0.91	0.80	0.69	0.62	0.55	0.50	0.46	0.43
	0.74	0.57	0.47	0.39	0.33	0.29	0.26	0.24
	0.54	0.33	0.25	0.22	0.23	0.24		
	0.37	0.24	0.21	0.18				
	0.38	0.20	0.17	0.17				
	0.39	0.20	0.17					
	0.39	0.20	0.17					

Relative errors for $k = 6$

	n	→						
p	1.00	0.97	0.94	0.90	0.88	0.85	0.83	0.80
↓	0.95	0.89	0.84	0.80	0.77	0.75	0.73	0.71
	0.86	0.78	0.73	0.70	0.68	0.67	0.66	0.65
	0.76	0.68	0.65	0.65	0.65	0.65		
	0.69	0.65	0.65	0.64				
	0.69	0.64	0.64	0.64				
	0.69	0.64	0.64					
	0.69	0.64	0.64					

Relative errors for $k = 8$

	n	→						
p	0.99	0.96	0.91	0.86	0.81	0.76	0.72	0.67
↓	0.94	0.82	0.73	0.65	0.58	0.52	0.46	0.42
	0.77	0.59	0.46	0.38	0.31	0.27	0.23	0.21
	0.54	0.31	0.21	0.20	0.21	0.22		
	0.35	0.23	0.19	0.12				
	0.35	0.14	0.10	0.09				
	0.36	0.14	0.10					
	0.36	0.14	0.10					

Relative errors for k = 11

	n	→						
p	1.00	0.96	0.94	0.88	0.85	0.78	0.76	0.70
↓	0.95	0.85	0.75	0.67	0.60	0.54	0.48	0.44
	0.79	0.61	0.48	0.39	0.32	0.26	0.23	0.20
	0.55	0.31	0.21	0.19	0.21	0.22		
	0.35	0.24	0.19	0.09				
	0.34	0.12	0.06	0.05				
	0.35	0.12	0.06					
	0.35	0.12	0.06					

Relative errors for k = 16

	n	→						
p	1.00	0.97	0.95	0.90	0.88	0.82	0.80	0.74
↓	0.97	0.88	0.79	0.71	0.64	0.58	0.52	0.48
	0.82	0.64	0.51	0.41	0.33	0.28	0.24	0.21
	0.57	0.33	0.22	0.20	0.22	0.23		
	0.35	0.25	0.20	0.08				
	0.34	0.11	0.05	0.02				
	0.35	0.11	0.05					
	0.35	0.11	0.05					

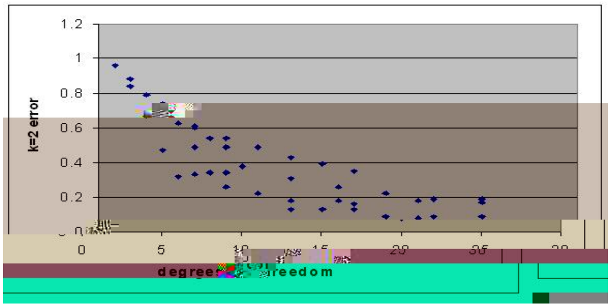


Figure 3.1: Scatter plot demonstrating exponential convergence of error with degrees of freedom

3.4 Conclusions

Firstly, the results for $k = 1$ appear to be anomalous, as the error appears to increase with the degrees of freedom rather than decrease. It is likely a side-effect of choosing such a low parameter for the convpolyhp.m program, which was intended for high frequencies. As a result there will be no further comment on these results.

The graphs of degrees of freedom versus error indeed show an exponential convergence. Finding an equation for the graphs was not done because they were merely meant to confirm what has been theorised many times before. In fact the worst error values for each degree of freedom show the same convergence, though they have left out the cases when n or p are equal to 1, and the surface plots indicate that those have the highest relative error.

The tables of numbers reveal a curious result. It appears the optimum ratio of p to n, that is the fastest decay slope on the surface plots, is not $n < (p + 1)$ after all. It appears to be the other way round (i.e. values below the diagonal are less than those above the diagonal), and the best results for the degrees of freedom versus error plots show the best ratio is somewhere around 0.7.

Perhaps the reason for this is that the results of section 2.2 do not apply to the functions $v_{\pm}(s)$, because the results assume that the best approximation on the first mesh cell is zero. Clearly that is not the case for such a singular function, meaning the first term in the error bound is incorrect. Given that a large amount of algebra resulted from that choice of approximation, it is unfortunate that it is not useful in this case (although the results will be very strong for a function which is best approximated by zero near the origin).

Nevertheless the theory suggested an avenue to investigate, and though the results were incorrect the preferred results were found i.e. exponential convergence with respect to degrees of freedom and an ideal ratio of n to p.

Another odd artifact of the results is that some wavenumbers have significantly lower errors for given values compared to others, particularly for $k = 6$ and $k = 4$. This is likely to do with a physical complication due to the relation of the wavelength to the obstacle which makes the solution converge more slowly, e.g. the polynomials $v_{\pm}$ may be significantly more oscillatory than for other wavenumbers.

$x_i =$

of $v_{\pm}$ we should take $\sigma \quad (\sigma_{opt}, 0.2032)$ rather than $\sigma = 0.15$ as in the convpolyhp.m program and as recommended in [4].

4 References

1. A Galerkin Boundary Element Method for High Frequency Scattering by Convex Polygons [S.N. Chandler-Wilde and S. Langdon]

2. Boundary Element Methods for Acoustics [S.N. Chandler-Wilde and S. Langdon]

3. Notes on hp-FEM [J.M. Melenk]

4. hp-FEM Lecture Course [J.M. Melenk]