Integral Equation Formulations For Scattering Problems

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August 2009

This dissertation is submitted to the Department of Mathematics in partial ful lment of the requirement for the degree of Master of Science.

Abstract

In this thesis we are concerned with the study of scattering of acoustic waves generated by the interaction of an incident wave with an object producing re ected and di racted waves; we study the case of a square in detail. We look at how the wave scattering problem based upon the Helmholtz equation can be reformulated to a lesser dimension size using integral equation formulations. This process may give rise to non unique solutions not inherent to the original problem. To investigate this, we consider two Boundary Element Methods that can be used to nd an approximate solution numerically. The rst, a standard collocation method, illustrates how easily spurious solutions are found. The second, a hybrid Galerkin BEM proposed by Langdon and Chandler-Wilde, illustrates how careful consideration of the desired solution when designing a numerical method can avoid these spurious solutions whilst saving on computational time.

Contents

Chapter 1

Introduction

1.1 Motivation

Scattering problems for acoustic and electromagnetic waves have been the subject of much theoretical and numerical study. There are many applications in the fields of physics, engineering, and geology, including radar and sonar, medical imaging, and geophysical exploration. Direct scattering problems are those which aim to find the scattered field produced by the interaction of a known incident wave with an object, whereas the inverse scattering problem is that of trying to determine the nature of the object or domain of definition, based upon the behaviour of the scattered wave. These problems are often not solvable analytically, and so various numerical techniques have been devised in order to find a good approximation to the true solution, including Finite Element Methods and Boundary Element Methods. However, high wave frequencies can be difficult to approximate numerically, which can pose problems due to the often highly oscillatory nature of the solution. Thus it is advantageous to use a numerical method that approximates these high frequencies well, in addition to demonstrating fast convergence and low storage and computational costs.

1.2 Aims and Outline

We aim first of all to study and explain the theory that allows a problem over an infinite exterior domain to be reformulated into a boundary integral equation over a finite domain, and the problem of non-unique solutions that may arise from this process. We aim to consider methods that achieve a

racy is required, thus avoiding extra computing expense. We then conclude our findings and present some possible research ideas for further expansion around the subject in Chapter 5.

Chapter 2

Background

2.1 Scattering Problems

Direct scattering problems are those which aim to find the scattered field produced by the interaction of a known incident wave with an object, based

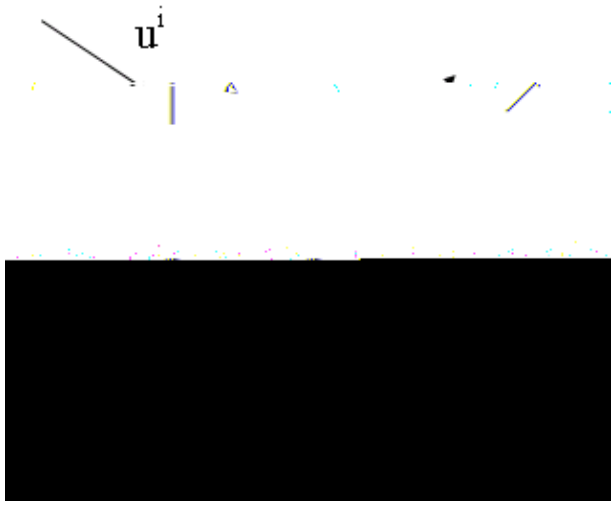


Figure 2.1: Total wave field comprising incident wave u^i and scattered wave u^s

(in two dimensions) and is time harmonic if

$$U = <fe$$

which is satis ed by u^i, u^s and $u = u^i + u^s$. The Helmholtz equation is elliptic, and the use of Green's functions is particularly appropriate for solving this type of partial di erential equation [3]. The fundamental solution of the Helmholtz equation is the Green's function

$$(x; y) = \frac{i}{4} H_0^1(kjx \quad yj) \tag{2.3}$$

for the 2D case, or

$$(x; y) = \frac{e^{ikjx \quad j}}{4 \ jx \quad yj} \tag{2.4}$$

for the 3D case, where H_0^1 denotes the Hankel function of the rst kind of order zero. In both the 2D and 3D case, $(x; y)$ is singular at $x = y$, and $j \ (x; y)j \ ! \ 1$ as $x \ ! \ y$, as explained in [4] and [7]. The Hankel function is also known as a Bessel function of the third kind, so called as it comprises a complex linear combination of Bessel's functions of the rst kind $J \ (x)$, and second kind $Y \ (x)$, resulting in

$$H^1(x) = J \ (x) + iY \ (x):$$

The rst and second kind Bessel functions are so called as they are linearly independent solutions of Bessel's equation

$$x^2y^{00} + xy^0 + (x^2 \quad ^2)y = 0:$$

2.2 Boundary Integral Formulation

The main advantage of using integral equation methods in the solution of boundary value problems is that they allow the problem to be reduced from one involving the whole domain of interest to one involving just the boundary, reducing the dimension of the problem by one. This is especially beni cial for exterior problems where the region of interest is in nite, such as the exterior

2.3 Uniqueness of Solutions

As we have seen in *x*2.2, it is possible to reformulate a problem over an entire domain into one involving just the boundary, using Green's Representation Theorem. Unfortunately, as per [2], due to the reformulation of the problem as a boundary integral equation, there may arise non-unique solutions that were not inherent to the original problem. Although the boundary values of $u(x)$ satisfy the integral equation (2.7), the solution of (2.7) may not be unique. There exist an in nite set of values of k for which the equation has a multiplicity of solutions, which coincide with the `resonant' wavenumbers for a related interior problem.

To explain this we start with two well known theorems of Fredholm integral equations. The rst is a fundamental theorem of integral equations, as explained in [3].

Theorem 2.1 *The Fredholm Alternative*

If the homogeneous equation

$$+ \int_\Gamma K(x;y)\ \ (y)dS_y = 0; \tag{2.8}$$

for boundary and scalar , only possesses the trivial solution $= 0$*, then the inhomogeneous equation*

$$+ \int_\Gamma K(x;y)\ \ (y)dS_y = f \tag{2.9}$$

will have a unique solution, for all square integrable functions f and kernels $K(x;y)$.

Proof. Suppose $_1$ and $_2$ are two linearly independant solutions of (2.9), so we have

$$_1 + \int_\Gamma K(x;y)\ \ _1(y)dS_y = f; \tag{2.10}$$

and

$$\phi_2 + \int_\Gamma K(x;y)\,\phi_2(y)\,dS_y = f. \tag{2.11}$$

Then taking (2.10) from (2.11) results in

$$(\phi_2 - \phi_1) + \int_\Gamma K(x;y)(\phi_2(y) - \phi_1(y))\,dS_y = 0; \tag{2.12}$$

an equation of the form (2.8) for $\phi_2 - \phi_1$. Thus, if there is only the trivial solution to (2.8), we conclude that $\phi_2 = \phi_1$: the solution to (2.9) is unique. ■

As quoted by Burton and Millar in [2], our second theorem:

Theorem 2.2

Consider now the interior problem

$$r^2 v + k^2 v = 0 \quad \text{in} \tag{2.16}$$

and

$$@v=@n = 0 \quad \text{on} \quad . \tag{2.17}$$

This in general only has the solution $v \quad 0$ unless k is one of an in nite set K_1 of discrete resonant eigenvalues for which there exists a non-trivial solution. Similar to (2.5), we have the Green's formulation for v solving (2.16):

$$\frac{1}{2}v(x) + \int_{\Gamma} v(y)\frac{@}{@n_y}(x; y) \qquad (x; y)\frac{@}{@n}v(y)dS_y = 0 \quad (x \; 2 \quad): \tag{2.18}$$

Applying $@v=@n = 0$, we obtain

$$\frac{1}{2}v(x) + \int_{\Gamma} v(y)\frac{@}{@n_y}(x; y)dS_y = 0 \quad (x \; 2 \quad)$$

which is an equation to solve for v in the form of (2.14). If $k \; 2 \; K_1$, there is a non trivial solution v, and hence the solution to (2.15) will be non unique. That is to say, there are other solutions to (2.15) besides $\frac{@u}{@n}$ where u solves (2.2).

As explained in [2], (2.7) will also have the same defect: there are a multiplicity of solutions whenever $k \; 2 \; K_2$, the set of eigenvalues for the interior Dirichlet problem. It is possible when trying to solve (2.6) or (2.7) for $\frac{@u}{@n}$ that a numerical scheme might pick up an eigenfunction of the related interior problem. Nevertheless, the two equations (2.6) and (2.7) always have only one solution in common.

As suggested by [2] and [1], to avoid obtaining an over-determined system of equations, we instead add a multiple i of (2.7) to (2.6), where $2 < n0$,

resulting in

$$\frac{1}{2}\frac{@u(x)}{@n} + \int_{\Gamma} \frac{@\ (x;y)}{@n_x} + i\quad (x;y)\ \frac{@u(y)}{@n} dS_y = \frac{@u^i(x)}{@n} + i\ u^i(x); \quad x\ 2\ : \tag{2.19}$$

This can be written as

$$(I + \ )\frac{@u}{@n} = f \quad x \text{ on} \tag{2.20}$$

where

$$v(x) = 2\int_{\Gamma} \frac{@\ (x;y)}{@n_x} + i\quad (x;y)\ v(y)dS_y \tag{2.21}$$

and

$$f(x) = 2\ \frac{@u^i(x)}{@n} + i\ u^i(x)\ : \tag{2.22}$$

By choosing 6= 0 in (2.19), we can solve to nd a unique $@u=@n$. Generally, solving either (2.19) or, by taking = 0, (2.6) will result in the same solution. However if k is an eigenvalue of the related interior Neumann problem (2.16)-(2.17), (2.6) will have a multiplicity of solutions. To solve (2.19) numerically is more expensive computationally than to solve (2.6) as it requires the evaluation of more Hankel functions, and this is expensive computationally.

The choice of coupling parameter has received much attention in the literature of recent years. As explained in [1], in this case $= k$ is the optimal choice for nding the exterior solution and ensuring the system is well conditioned. Unfortunately, this choice could potentially double the computing time compared to choosing = 0. As can be seen in Table 2.1, the computation time in Matlab for the evaluation of Hankel functions grows at a rate similar to that of the increase in the number to be evaluated. It is therefore of interest to determine whether it is necessary to use (2.19) or whether numerical schemes for (2.6) do in fact converge to the required solution of the exterior problem (2.2).

Table 2.1: Evalution Time of Hankel Functions

Number of Hankel functions	Time (seconds)
1	$3.72 \quad 10^{-4}$
10	$8.72 \quad 10^{-4}$
100	$1.041 \quad 10^{-3}$
1000	$2.5799 \quad 10^{-2}$

2.4 A Speci c Problem

We will consider in this project the scattering of acoustic waves by a convex

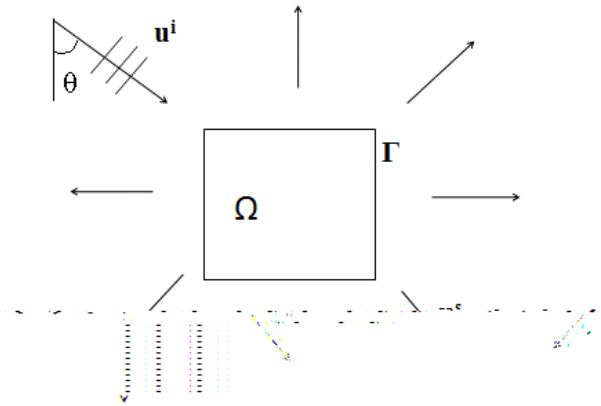


Figure 2.2: Re ection of u^i on a square domain.

downward vertical. The scattered eld must also satisfy the Sommerfeld radiation condition

$$\lim_{r \to 1} r^{1=2} \left(\frac{@u^s}{@r} - iku^s \right) = 0 \qquad (2.26)$$

uniformly in r, where r = x=jxj is a unit vector in the direction of x, and $r = jxj$. This condition ensures u^s is an outgoing wave, so the scattered eld is not re ected back from in nity.

2.5 Eigenvalues of the Interior Neumann Problem

In order to investigate for which values of k the equation (2.20) with = 0

on the interior of the square domain $0 \quad x \quad L$, $0 \quad y \quad L$, with Neumann boundary conditions

$$\frac{@u}{@n} = 0 \quad \text{on} \tag{2.28}$$

on the boundary. We wish to nd the eigenvalues k for which there are non trivial solutions, and the corresponding eigenfunctions. These will be the values of k for which (2.6) and (2.7) will not have a unique solution. These solutions are those which we expect our Boundary Element Methods to pick up when we use values k that are eigenvalues when trying to nd the solution to the exterior Dirichlet problem, that is (2.23) and (2.24) in $<^2 n$.

Using standard separation of variables techniques, we set $u(x;y) = X(x)Y(y)$. From the Helmholtz equation we obtain

$$X^{00}Y + XY^{00} + k^2XY = 0$$

which rearranges to

$$\frac{X^{00}}{X} + k^2 = \quad \frac{Y^{00}}{Y} =$$

where is our separation constant. Rearranging, we obtain

$$X^{00} + (k^2 \quad)X$$

solution $u(x;y)$.

First considering the case

Chapter 3

Solving the Boundary Integral Equation

used to calculate the solution at points in the original solution domain.

Two common boundary element methods are the collocation and Galerkin

For the 2D case with boundary $\Gamma : x \in [a;b]$, to solve the problem

$$u(x) + \int_a^b K(x;y)u(y)\mathrm{d}y = f(x) \tag{3.4}$$

using a simple collocation method we write u as in (3.1), and choose the piecewise constant basis functions

$$\phi_j(x) = \begin{cases} 1 & x \in [x_{j-1};x_j] \\ 0 & \text{elsewhere} \end{cases}; \tag{3.5}$$

where $x_j = a + \frac{j}{M}(b-a)$ for the points on the boundary

$$a = x_0 < x_1 < \ldots < x_M = b.$$

Substituting this into (3.4) it follows that

$$\sum_{j=1}^{M} u_j[\phi_j(x) + \int_a^b K(x;y)\phi_j(y)\mathrm{d}y] = f(x). \tag{3.6}$$

We choose the collocation points s_m to be the mid points of each interval $[x_{j-1};x_j]$ and forcing (3.6) to hold at each of these gives the M equations. The resultant matrix system is

$$[\Phi + K]\mathbf{u} = \mathbf{f} \tag{3.7}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{pmatrix} \tag{3.8}$$

is a vector of unknowns,

$$= \begin{pmatrix} \phi_1(s_1) & \phi_2(s_1) & \cdots & \\ \phi_1(s_2) & \ddots & & \\ \vdots & & \ddots & \vdots \\ & & \cdots & \phi_M(s_M) \end{pmatrix} = I; \tag{3.9}$$

the identity matrix,

$$K = \begin{pmatrix} \int_a^b K(s_1;y)\,\phi_1(y)dy & \int_a^b K(s_1;y)\,\phi_2(y)dy & \cdots & \\ \int_a^b K(s_2;y)\,\phi_1(y)dy & \ddots & & \\ \vdots & & \ddots & \vdots \\ & & \cdots & \int_a^b K(s_M;y)\,\phi_M(y)dy \end{pmatrix}$$

$$= \begin{pmatrix} \int_{x_0}^{x_1} K(s_1;y)dy & \int_{x_1}^{x_2} K(s_1;y)dy & \cdots & \\ \int_{x_0}^{x_1} K(s_2;y)dy & \ddots & & \\ \vdots & & \ddots & \vdots \\ & & \cdots & \int_{x_{M-1}}^{x_M} K(s_M;y)dy \end{pmatrix}; \tag{3.10}$$

and

$$\mathbf{f} = \begin{pmatrix} f(s_1) \\ \vdots \\ f(s_M) \end{pmatrix}. \tag{3.11}$$

For our wave scattering problem (2.20), the kernel is given by

$$K(\mathbf{x};\mathbf{y}) = 2i\ H_0^1(k|\mathbf{x} - \mathbf{y}|) + 2 \quad @$$

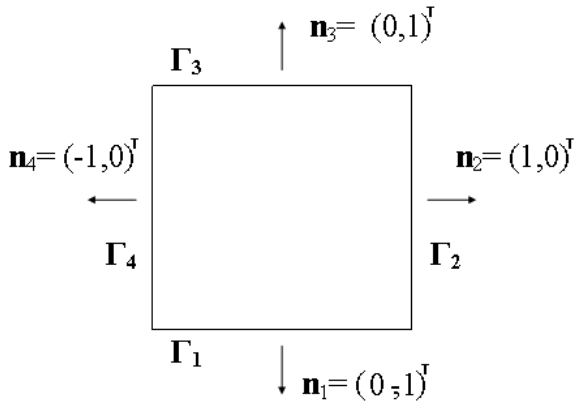


Figure 3.1: Outward normal derivative vectors

For the normal derivative of the incident wave we have,

$$\frac{\partial u^i}{\partial n(\mathbf{x})} = \nabla u^i \cdot \mathbf{n} = \begin{pmatrix} \frac{\partial u^i}{\partial x_1} & n_1 \\ \frac{\partial u^i}{\partial x_2} & n_2 \end{pmatrix} \tag{3.15}$$

where $\mathbf{x} = (x_1, x_2)^T$, so that on Γ_1

$$\frac{\partial u^i}{\partial n} = -\frac{\partial u^i}{\partial x_2} = ik\cos(\theta) e^{ik(x_1 \sin(\theta) - x_2 \cos(\theta))};$$

on Γ_2

$$\frac{\partial u^i}{\partial n} = \frac{\partial u^i}{\partial x_1} = ik\sin(\theta) e^{ik(x_1 \sin(\theta) - x_2 \cos(\theta))};$$

on Γ_3

$$\frac{\partial u^i}{\partial n} = \frac{\partial u^i}{\partial x_2} = -ik\cos(\theta) e^{ik(x_1 \sin(\theta) - x_2 \cos(\theta))};$$

and on Γ_4

$$\frac{\partial u^i}{\partial n} = -\frac{\partial u^i}{\partial x_1} = -ik\sin(\theta) e^{ik(x_1 \sin(\theta) - x_2 \cos(\theta))}.$$

Similarly, for the normal dervivative of the fundamental solution

$$\frac{@\ (x;y)}{@n(x)} = 5\ \ (x;y)\ \ n = \begin{pmatrix} \frac{@\Phi(x;\ )}{@x_1} \\ \frac{@\Phi(x;\ )}{@x_2} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}; \qquad (3.16)$$

where $y = (y_1; y_2)^T$. From [4],

$$\frac{d}{dz}H_0^1(z) = \quad H_1^1(z);$$

so that on $_1$

$$\frac{@}{@n} = \quad \frac{@}{@x_2} = \frac{k(x_2 \quad y_2)}{jx \quad yj}H_1^1(kjx \quad yj);$$

on $_2$

$$\frac{@}{@n} = \frac{@}{@x_1} = \quad \frac{k(x_1 \quad y_1)}{jx \quad yj}H_1^1(kjx \quad yj);$$

on $_3$

$$\frac{@}{@n} = \frac{@}{@x_2} = \quad \frac{k(x_2 \quad y_2)}{jx \quad yj}H_1^1(kjx \quad yj);$$

and on $_4$

$$\frac{@}{@n} = \quad \frac{@}{@x_1} = \frac{k(x_1 \quad y_1)}{jx \quad yj}H_1^1(kjx \quad yj):$$

3.2 Numerical Results

The standard BEM program was run for various values of k, chosen so as to compare the e ectiveness of the method for those k which are eigenvalues of the interior problem, as given by (2.33), and those which are not. Recalling x2.3, we might expect the BEM to work well when k is not an eigenvalue, and to perform poorly when k is an eigenvalue, as in that case the BIE does not have a unique solution. When $= k$ we expect the method to work well in all cases. Each side of the square was split into N collocation points, for $N = 2; 4; 8; 16; 32; 64; 128$. The program was used to test the importance

of the coupling parameter in nding the unique solution to the exterior problem, rather than the spurious solutions related to the interior problem. When computing the L^2 error $jjexact \quad approximatejj_2$ and the relative error $jjexact \quad approximatejj_2=jjexactjj_2$, the exact solution was taken to be that resultant from $= k$, $N = 128$. We would in theory ex1([(N)]8h)-349win

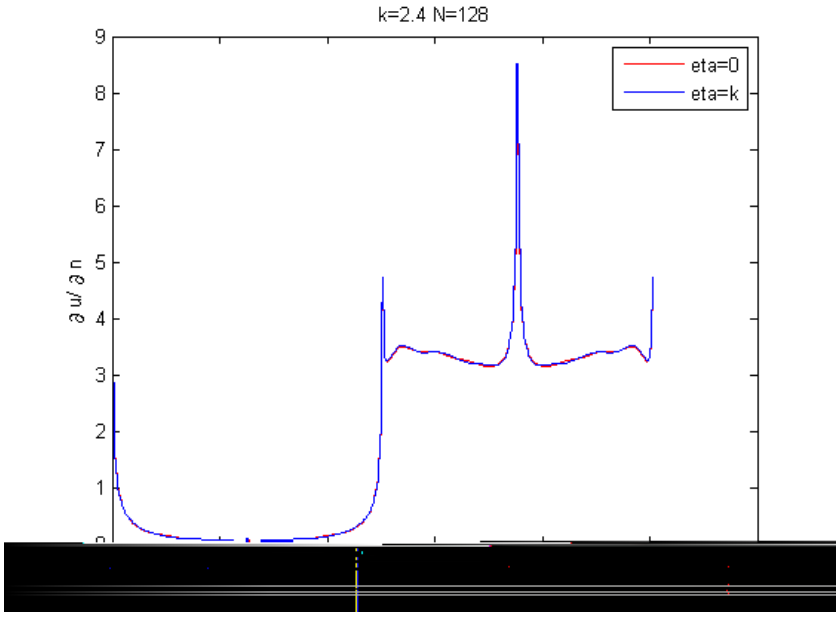


Figure 3.2: k not an eigenvalue

rate regardless of the value of when k is not an eigenvalue. It is also clear that for a desired level of accuracy, as k increases we need to increase N.

For values of k that are not eigenvalues of the interior problem, the standard BEM easily converged to the correct solution even for = 0, as shown in Figures 3.2 and 3.4. However, for values of k that are eigenvalues of the interior problem, the BEM clearly has problems nding the unique solution of the exterior problem when = 0, as shown in Figure 3.3. As n and m increase, these eigenvalues become more frequent, and so as k increases it becomes more likely to nd a spurious solution. Figures 3.6, 3.7, and 3.8 show how these eigenvalues start to have an in uence even when k is not an eigenvalue: $k = 12{:}505$ is close to two eigenvalues $k = 12{:}5$ and $k = 12{:}51$, and its approximation with = 0 is less accurate than for smaller k even though it is not actually an eigenvalue, although this may be due to the need for higher N to approximate the increased oscillation well.

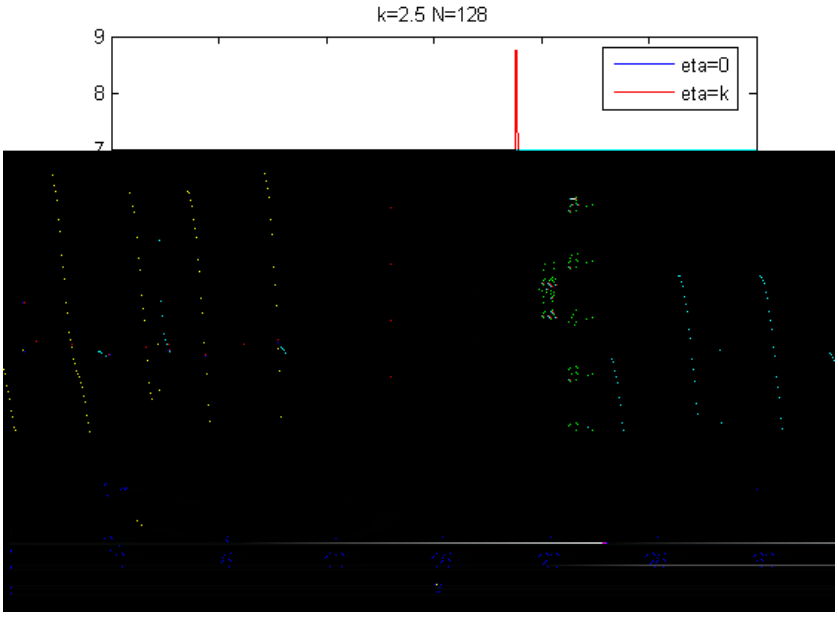


Figure 3.3: k an eigen5 c4690 G 0.s 1 0 0 1 -127.302 -430lueTS

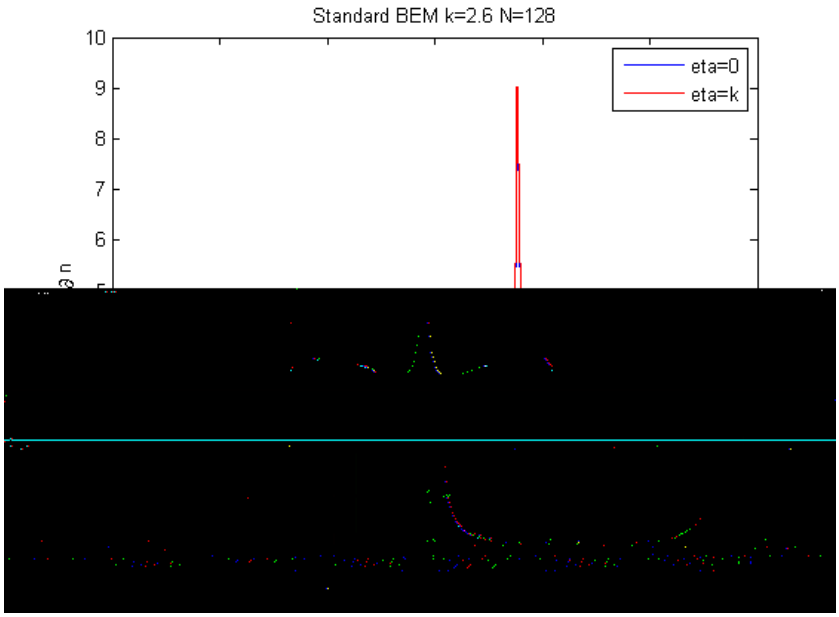


Figure 3.4: k not an eigenvalue

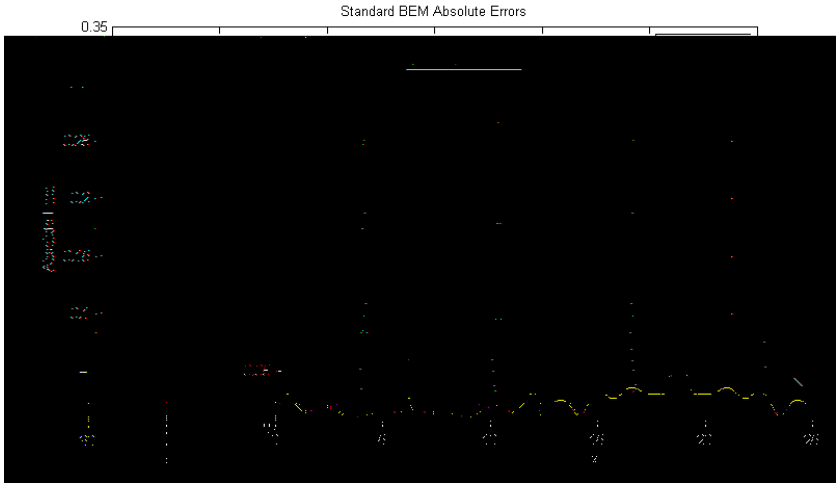


Figure 3.5: Absolute Errors when k is not an eigenvalue, N=128

square's sides. So perhaps in future work more consideration of this should be taken into account when discretising our boundary.

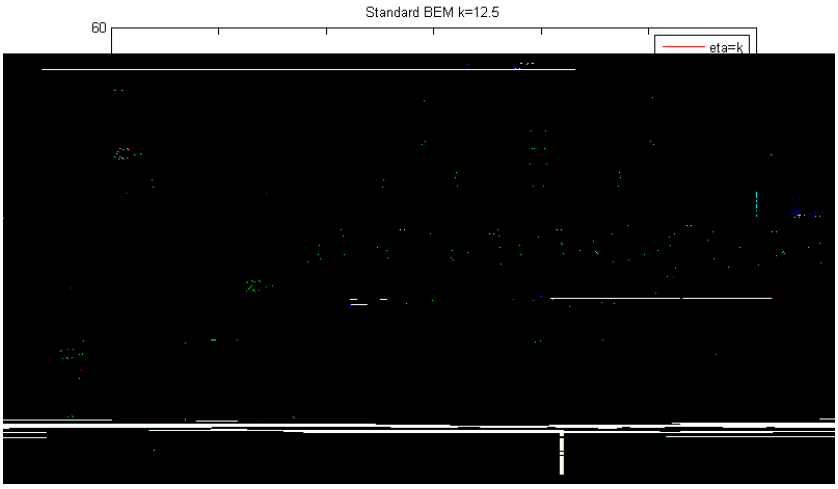


Figure 3.6: k an Eigen-value, N=128

Figure 3.7: k not an Eigen-value

Table 3.2: Relative L2 errors for the case that k is an eigenvalue.

		= 0			= k		
k	N	$\|\|$error$\|\|_2$	$\|\|$Relative error$\|\|_2$	EOC	$\|\|$error$\|\|_2$	$\|\|$Relative error$\|\|_2$	EOC
2.5	2	4.2036 10^{1}	2.1337 10^{0}	2.1866	9.2925 10^{0}	4.7169 10^{1}	0.6261
	4	1.9136 10^{2}	9.7132 10^{0}	-4.3380	1.4342 10^{1}	7.2802 10^{1}	-2.2385
	8	9.4617 10^{0}	4.8027 10^{1}	0.2625	3.0392 10^{0}	1.5427 10^{1}	-0.8842
	16	1.1349 10^{1}	5.7610 10^{1}	0.1127	1.6466 10^{0}	8.3579 10^{2}	-0.7975
	32	1.2272 10^{1}	6.2292 10^{1}	0.0477	9.4733 10^{1}	4.8086 10^{2}	-0.8555
	64	1.2684 10^{1}	6.4385 10^{1}	0.0196	5.2354 10^{1}	2.6575 10^{2}	
	128	1.2857 10^{1}	6.5265 10^{1}				
5	2	1.0968 10^{1}	5.8400 10^{1}	3.4245	2.1035 10^{1}	1.1200 10^{0}	0.1140
	4	1.1776 10^{2}	6.2702 10^{0}	1.4589	2.2764 10^{1}	1.2121 10^{0}	1.7471
	8	3.2373 10^{2}	1.7237 10^{1}	-3.3416	7.6415 10^{1}	4.0688 10^{0}	-4.8844
	16	3.1936 10^{1}	1.7004 10^{0}	0.6605	2.5873 10^{0}	1.3776 10^{1}	-0.4640
	32	5.0477 10^{1}	2.6877 10^{0}	0.2371	1.8756 10^{0}	9.9870 10^{2}	-0.5724
	64	5.9493 10^{1}	3.1677 10^{0}	0.0903	1.2614 10^{0}	6.7161 10^{2}	
	128	6.3333 10^{1}	3.3722 10^{0}				
12.5	2	3.5982 10^{1}	6.1848 10^{1}	-1.2645	1.6611 10^{1}	2.8553 10^{1}	1.3987
	4	1.4977 10^{1}	2.5744 10^{1}	1.8373	4.3798 10^{1}	7.5283 10^{1}	-0.6760
	8	5.3522 10^{1}	9.1996 10^{1}	-0.5991	2.7414 10^{1}	4.7121 10^{1}	0.4003
	16	3.5334 10^{1}	6.0734 10^{1}	0.3944	3.6182 10^{1}	6.2191 10^{1}	-1.5862
	32	4.6442 10^{1}	7.9828 10^{1}	0.8961	1.2050 10^{1}	2.0712 10^{1}	-1.3584
	64	8.6429 10^{1}	1.4856 10^{0}	0.3941	4.6996 10^{0}	8.0780 10^{2}	
	128	1.1358 10^{2}	1.9523 10^{0}				
12.51	2	4.1890 10^{1}	7.1720 10^{1}	-1.7994	1.5079 10^{1}	2.5817 10^{1}	1.4406
	4	1.2035 10^{1}	2.0605 10^{1}	2.1400	4.0929 10^{1}	7.0074 10^{1}	-0.9416
	8	5.3044 10^{1}	9.0817 10^{1}	0.0040	2.1310 10^{1}	3.6485 10^{1}	0.9251
	16	5.3189 10^{1}	9.1066 10^{1}	-0.6646	4.0464 10^{1}	6.9278 10^{1}	-1.7559
	32	3.3556 10^{1}	5.7451 10^{1}	-1.5223	1.1981 10^{1}	2.0512 10^{1}	-1.3627
	64	1.1682 10^{1}	2.0001 10^{1}	-0.5549	4.6586 10^{0}	7.9760 10^{2}	
	128	7.9519 10^{0}	1.3615 10^{1}				

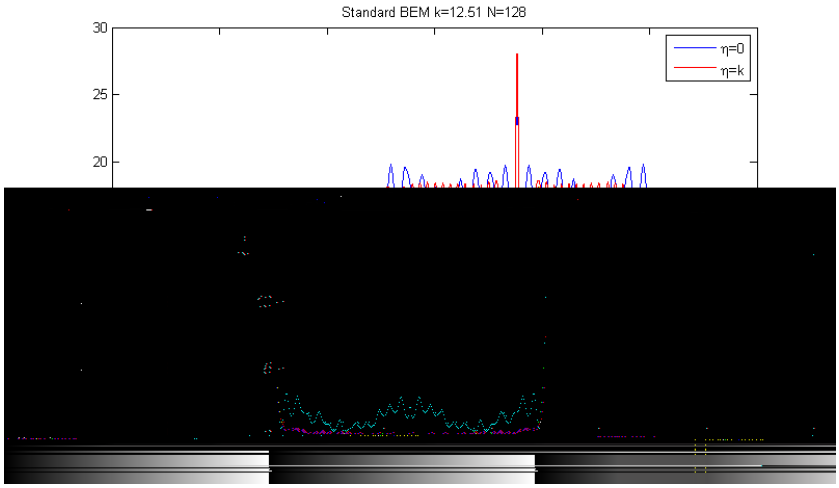


Figure 3.8: k an Eigen-value

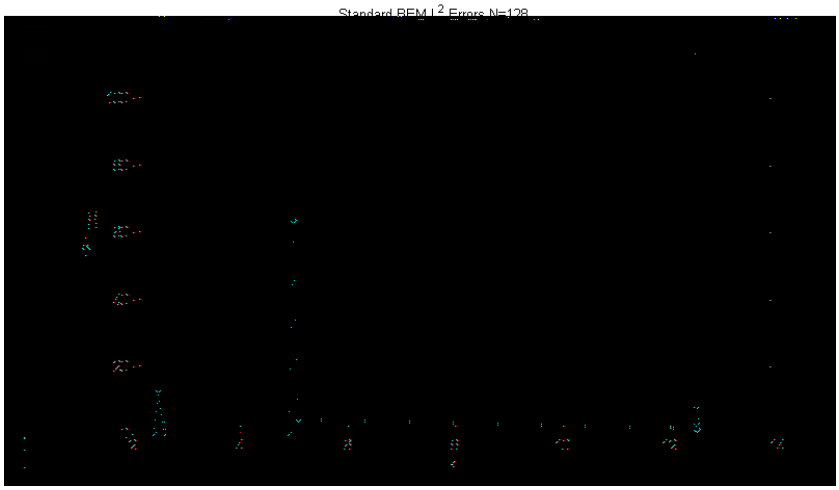


Figure 3.9: L^2 Errors against k

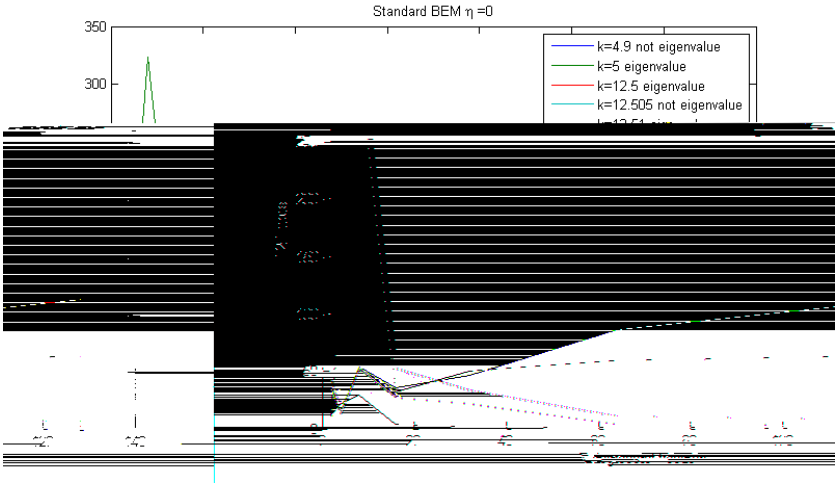


Figure 3.10: L^2 Errors against degrees of freedom, for various k

Chapter 4

Is the Coupled Layer Formulation Necessary?

The theory from [2], supported by results in Chapter 3, tells us that solving the Helmholtz equation (2.2) by reducing it to a boundary integral formulation (2.20) may produce spurious solutions w8 562.195 T,1(p)h562.19re75(e(ro)-2)-312.19a50

The corners usually cause di racted waves to illuminate the shadow side strongly, as they travel along the polygon's sides. As k increases, the leading order behavior on the illuminated sides $_3$ and $_4$ is made up of the incident plane wave and a known re ected wave, whereas these are zero on shadow sides $_1$ and $_2$. The method approximates the normal derivative solution by separating it into the leading order behaviour and a linear combination of the products of piecewise polynomials and plane waves travelling parallel

which when considering interior eigenvalues $k = \frac{p}{L}\sqrt{n^2 + m^2} = \frac{1}{2}\sqrt{n^2 + m^2}$ as in (2.33), is tuned to approximate $\cos(ks) = \cos(\frac{1}{2}\sqrt{n^2 + m^2}s)$ or $\sin(ks) = \sin(\frac{1}{2}\sqrt{n^2 + m^2}s)$. Thus these eigenfunctions will oscillate at different frequencies to e^{iks} if $m, n \neq 0$, and in this case they will not be well approximated by the basis functions. It is only for the case when $m = 0$ or $n = 0$, and k is an integer, that the eigenfunctions may oscillate at the same rate as the solution to the exterior problem.

The functions $v(s)$ are approximated by piecewise polynomials on a graded mesh, specially designed with subintervals spaced so as to equidistribute the approximation error. Since the exterior solution is highly peaked near the corners of the square, the mesh is suitably refined, with larger elements away from the corners and a higher concentration of mesh points around the corners. Thus this should avoid approximating the interior eigenfunctions, even in the case that $m = 0$ or $n = 0$, as there are few mesh points away from the corners. The interval of width one wavelength from the centre is split into N mesh points, and an algorithm then puts $O(\log N)$ points on the rest of the side.

4.2 Numerical Results

The hybrid BEM program was run for $N = 2, 4, 8, 16, 32, 64$ for some of the same values of k as run for the standard BEM in §3.2, as well as for some larger values of k. This enables us to compare whether the hybrid BEM is more effective at finding the exterior solution for those k which are eigenvalues to the interior problem, as given by (2.33) for the cases $\eta = 0$ in which case the standard BEM fails, and for the case $\eta = k$. The program was used to test the importance of the coupling parameter η in finding the unique solution to the exterior problem, rather than the spurious solutions related to the interior problem. When computing errors and relative errors

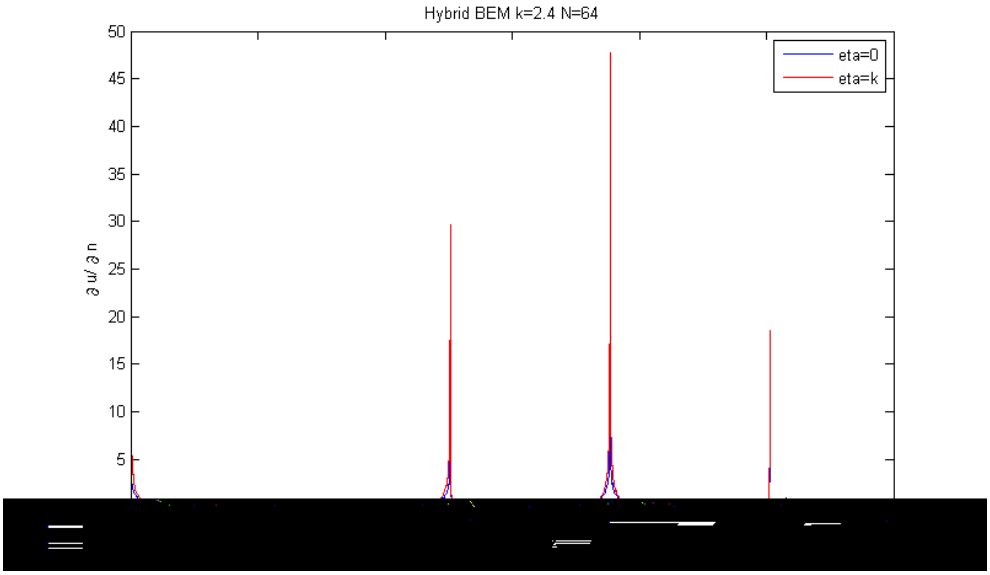


Figure 4.1: k not an Eigenvalue

4.7, the exact solution was taken to be that of $= k$, $N = 64$, and the approximate solution to be that of $= 0$, $N = 64$. The graded mesh seems to have been e ective in most cases of removing the peaks in error around the corners of the square.

An interesting case arises when considering low frequencies. As shown in Figure 4.8 the numerical solutions from the hybrid BEM seem to converge to a solution using $= k$ for high N, or using $= 0$ up to a certain number of degrees of freedom, so we can assume this to be the a good approximation to the true solution. However, it seems that for $= 0$ for $N \; ! \; 1$ the solution starts to diverge again. This is most likely because when N is large, there are enough of the piecewise polynomials that approximate the non-oscillatory functions $v(s)$ to approximate the di erent wavelengths of the interior eigenfunctions. The unusual results for $k = 2{:}5$, such as the positive EOC, may be explained by this. Figure 4.3 also suggests that the L^2 error may be starting to increase again as N increases for $k = 2{:}5$.

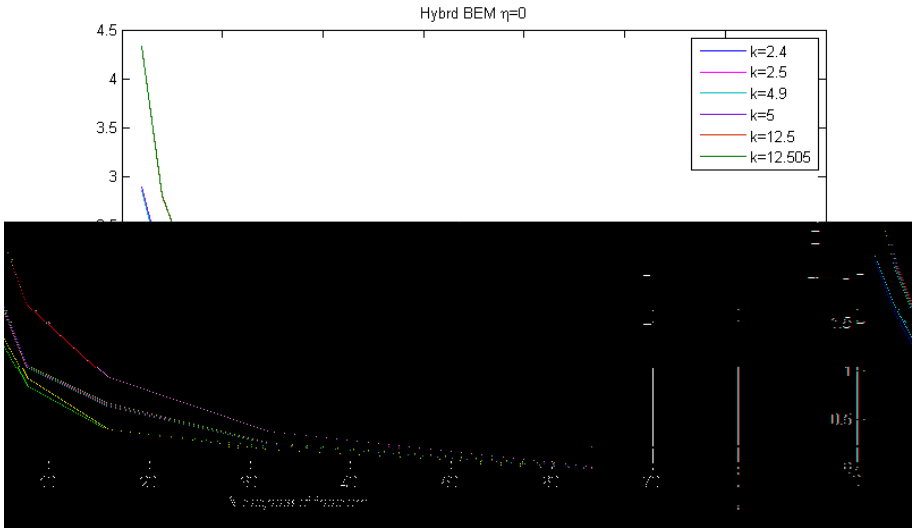


Figure 4.3: L2 Errors against degrees of freedom, for various k

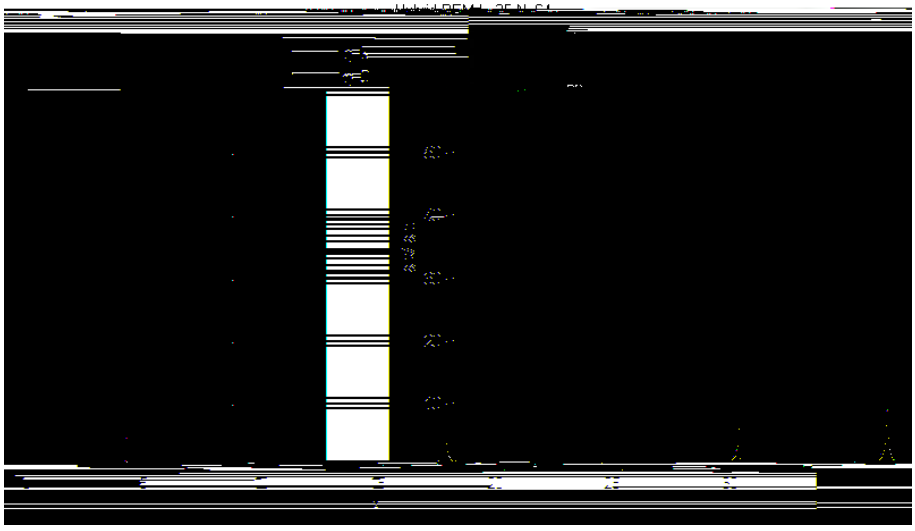
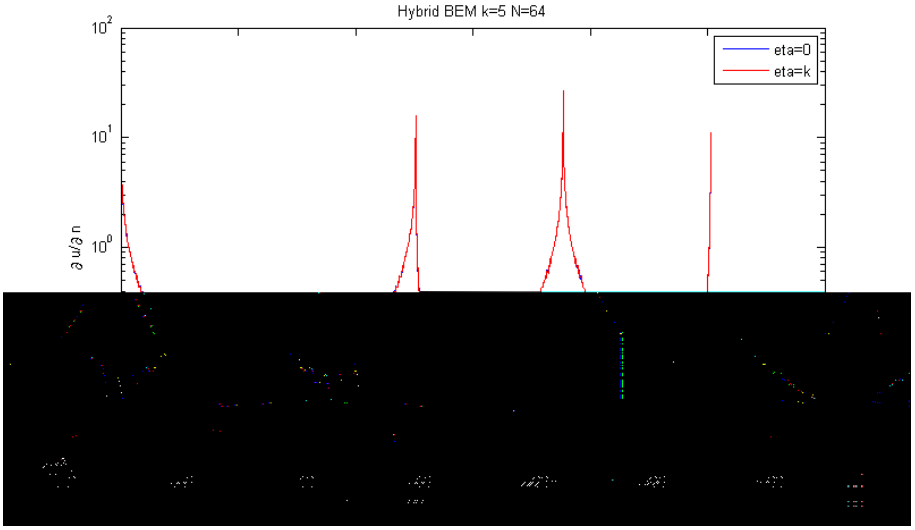


Figure 4.4: k is an Eigenvalue


Hybrid BEM k=5 N=64
eta=0
eta=k
10^2
10^1
10^0
∂ u/∂ n

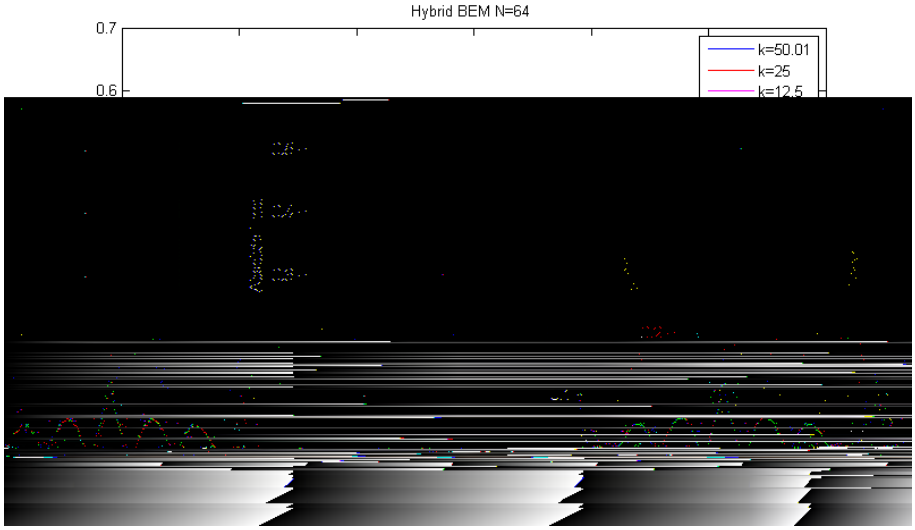


Figure 4.7: Absolute Errors for various k

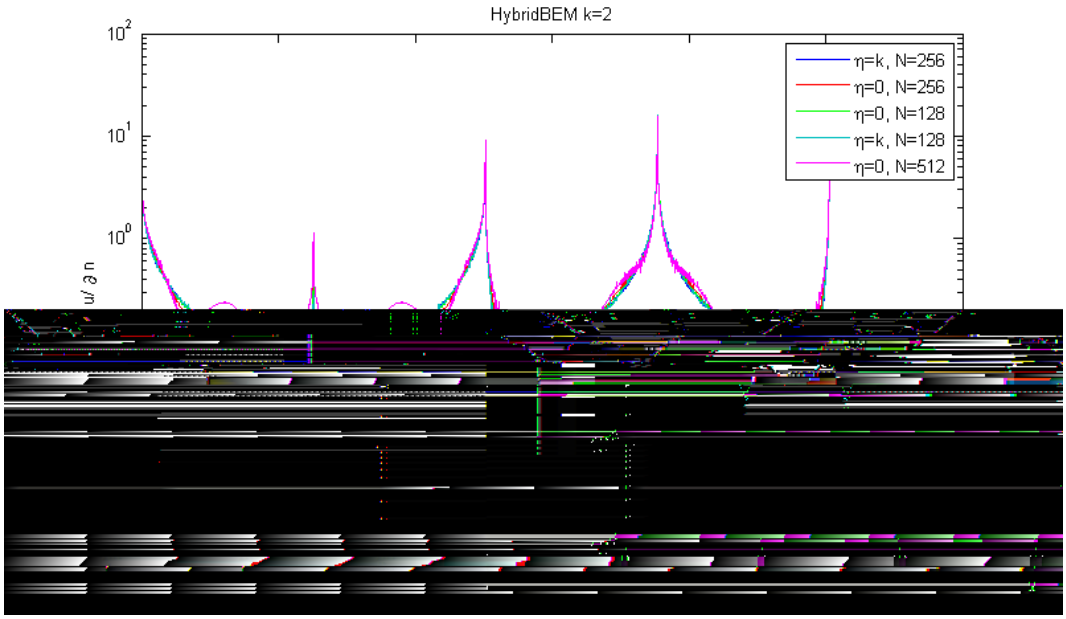


Figure 4.8: Errors arising for a low wavelength $k = 2$ and high N

Table 4.2: Errors and relative L^2 errors for the case that k is not an eigenvalue.

		= 0			= k		
k	N	$\|\|$error$\|\|_2$	$\|\|$Relative error$\|\|_2$	EOC	$\|\|$error$\|\|_2$	$\|\|$Relative error$\|\|_2$	EOC
2.4	2	2.1897 10^0	6.1962 10^{-1}	-0.39756	2.5422 10^0	7.1934 10^{-1}	-0.60605
	4	1.6623 10^0	4.7038 10^{-1}	-0.84118	1.6702 10^0	4.7260 10^{-1}	-0.85249
	8	9.2791 10^{-1}	2.6256 10^{-1}	-1.2059	9.2498 10^{-1}	2.6174 10^{-1}	-1.2118
	16	4.0222 10^{-1}	1.1382 10^{-1}	-1.1091	3.9934 10^{-1}	1.1300 10^{-1}	-1.0852
	32	1.8647 10^{-1}	5.2765 10^{-2}	-5.6384	1.8823 10^{-1}	5.3261 10^{-2}	
	64	3.7437 10^{-3}	1.0593 10^{-3}				
4.9	2	2.8498 10^0	5.9675 10^{-1}	-0.4813	5.6672 10^0	1.1867 10^0	-1.4335
	4	2.0415 10^0	4.2748 10^{-1}	-0.9695	2.0982 10^0	4.3936 10^{-1}	-0.6961
	8	1.0426 10^0	2.1831 10^{-1}	-0.6860	1.2951 10^0	2.7119 10^{-1}	-1.0150
	16	6.4804 10^{-1}	1.3570 10^{-1}	-1.2852	6.4085 10^{-1}	1.3419 10^{-1}	-1.2879
	32	2.6590 10^{-1}	5.5679 10^{-2}	-5.1130	2.6245 10^{-1}	5.4956 10^{-2}	
	64	7.6835 10^{-3}	1.6089 10^{-3}				
12.505	2	4.3281 10^0	5.8922 10^{-1}	-0.6270	7.5541 10^0	1.0284 10^0	-1.3328
	4	2.8026 10^0	3.8154 10^{-1}	-0.7289	2.9990 10^0	4.0828 10^{-1}	-0.5790
	8	1.6910 10^0	2.3021 10^{-1}	-0.8461	2.0077 10^0	2.7332 10^{-1}	-1.0595
	16	9.4064 10^{-1}	1.2806 10^{-1}	-1.3141	9.6327 10^{-1}	1.3114 10^{-1}	-1.0735
	32	3.7831 10^{-1}	5.1502 10^{-2}	-4.6524	4.5771 10^{-1}	6.2312 10^{-2}	
	64	1.5043 10^{-2}					

Table 4.3: Processing time in seconds, $N = 64$

k	$= 0$	$= k$
2.4	1459.13949	1457.918595
2.5	1443.757474	1550.433721
4.9	1721.390138	1709.063943
5	1752.560662	1747.952968
12.5	2459.256118	2445.934317
12.505	2455.168543	2491.280168
25	3802.594433	3813.400389
50.01	6924.080258	

Chapter 5

Conclusions and Future Work

5.1 Summary

The main aim of this project was to consider the theory that allows a problem over an in nite exterior domain to be reformulated into a boundary integral formulation over a nite domain. We looked at reformulation using Green's Representation Theorem, and explained the problems of non-uniqueness that arise from this process. Studying in detail the case of a square convex polygon, we showed how these spurious solutions arise. We also aimed to consider methods nding a good approximate solution numerically, even at high wave frequencies, whilst minimising computational expense and avoiding spurious solutions.

We looked at two numerical methods: a standard collocation Boundary Element Method and a hybrid Galerkin BEM as proposed by Langdon and Chandler-Wilde in [1]. By comparing the results of these two methods, it is clear how careful consideration of our desired solution and possible spurious solutions can be used to design a numerical method which avoids the extra computational expense that the theory suggests we will require. The

standard BEM demonstrated how the spurious solutions which relate to the interior problem are easily found. Even when using wavenumbers k that are not eigenvalues of the interior problem, by setting the coupling parameter $= 0$ to avoid the coupled formulation, thus saving computational expense, we still found signi cant errors. The hybrid BEM incorporated oscillatory basis functions with a graded mesh, based upon detailed study of the exterior solution. These considered improvements did avoid spurious solutions being found even when k was an interior eigenvalue and the coupled formulation was not used, provided a very high level of accuracy was not required. However, the main advantage of using $= 0$ rather than the coupled formulation was expected to be a save in computational time and expense: what resulted in either case turned out to be very similar.

5.2 Further Work

There is a fair amount of scope for further work on this subject. The results we have so far are fairly accurate for lower frequency waves, however the solutions we considered to be exact were those resultant of relatively low N. It would be interesting to allow the hybrid BEM code to run for higher values of N and lower wavenumbers k, and to expand futher into what e ect their relationship has on errors. With a deeper understanding of the hybrid code, it may also be relevant to do some error analysis on the numerical results we have produced. It would also have been appropriate to consider how well the hybrid BEM approximates even higher frequencies that would have been considered here, computational time and storage space permitting. As we have found the hybrid BEM to be e ective even when using $= 0$, it would be worthwhile to edit the code to make it more cost e ective time wise. To do this the code should no longer compute all functions and then multiply them by $= 0$, instead these functions could be removed altogether.

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