An Investigation of the Dynamics of Several Equidistribution Schemes

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Abstract

In order to reduce computation costs in solving Partial Differential quations (PD s) whose solutions possess steep gradients or complex wave interactions, it can be important to use an adaptive computational grid which will move to resolve features of the solution. Many such adaption techniques are based on the principle of equidistribution, where some property of the PD is distributed evenly on the grid. However it is known that some implementations of equidistribution can behave dynamically quite differently from the true solution. Here we study three such techniques, numerically and where possible analytically, both in the context of grid generation, where a grid is fitted to a given function, and as part of an adaptive strategy in solving a time-dependent PD to steady state. In both cases a wealth of dynamical behaviour is uncovered.

In many applications of Computational Fluid Dynamics (CFD), it is necessary the dynamics of three implementations of equidistribution. First, we examine the effect of the equidistribution implementation alone by fitting a grid to known functions. Then we use equidistribution as part of an adaptive technique to solve time-dependent PD s to steady state. We investigate two PD s, one whose solution involves a steep gradient, and one with a boundary layer.

First we formalise what we mean by equidistribution, before we look in detail

$_0$ $_1$

$_{N-1}$ $_N$

$_k$ $_{k+1}$

$_0$ $_1$ $_{N-1}$ $_N$ $_0$ $_N$

x_k x_{k+1} $x_N{=}b$

x_{k-1} x_k $x_0{=}a$

for $k = 1, 2, \ldots, N$, and each value of t.

(There are other equivalent forms of expressing equidistribution, see for example quations (2.6) and (2.30)).

1.2 The Weight Function

The weight, or monitor, function w will reflect some property of the underlying function u, to which the mesh is being fitted. For straightforward equidistribution both w and u will be functions of x, whilst for grid adaption in PD solutions both will be functions of x and t. xamples of monitor functions are

(i) an arclength weight function,

$$w(x,t) = \sqrt{1 + u_x^2(x,t)} \tag{1.2}$$

(ii) a curvature weight function,

(iii) a combination of gradient and curvature,

(iv) truncation error or solution residual of the PD .

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Implement tions

In this section we consider various ways in which the equidistribution process can be implemented.

Grid adaption techniques based on equidistribution are usually applied to time-dependent PD 's by either

(i) Solving quation (1.1) sequentially with the solution of the PD , i.e. one PD solver time-step followed by a regridding step(s), or

(ii) developing a system of Ordinary Differential quations (OD s) for the mesh velocities that are equivalent to quation (1.1) and then solving the resulting coupled system of equations for both mesh location and PD solution simultaneously.

The techniques we consider are all iterative techniques. The first directly approximates quation (1.1), whilst the second and third involve deriving PD s from (1.1).

1

$\frac{1}{2}$ 1 $+\frac{1}{2}$ $+1$

$\overline{2}$ $\frac{1}{2}$ 1 $\overline{2}$ $+\frac{1}{2}$ $+1$

$+1$ 1

$\frac{1}{2}$ $\frac{1}{1}$

$\overline{2}$ $+12$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

where superscripts denote iterates, and fixed end points are $x_0 = 0$ and $x_N = 1$. At each stage of the iteration a tridiagonal system must be solved.

2.2 Ren and Russell Iteration

This is an iterative method for solving a new equidistribution PD derived from quation (1.1). This scheme was derived in [3] as a method for solving a PD , so when considering it as an iterative method for equidistributing a known function, we use the idea of artificial time.

quation (1.1) can be rewritten in an equivalent form as

$$\int_{x_0}^{x_k(t)} w(x,t)dx = \frac{k}{N}\int_{x_0}^{x_N} w(x,t)dx \equiv \frac{k}{N}\theta(t), \tag{2.6}$$

where $\theta(t)$ is a function which Ren and Russell describe as representing the rate at which the mesh moves with time.

Replacing (x,t) in the original PD by a new set of computational coordinates (s,T), defined by

$$s = \frac{1}{\theta}\int_{x_0}^{x} w(\xi,t)d\xi,$$

$$T = t. \tag{2.7}$$

yields a new system to be solved to find the equidistributing mesh. Ren and Russell use this transformation to derive a PD which provides a new formulation of equidistribution. Rearranging the first equation of (2.7) gives

$$s\theta(t) = \int_{x_0}^{x} w(\xi,t)d\xi, \tag{2.8}$$

and assuming $w(\xi,t)$ is a smooth function, they differentiate along lines where

$s(t)$ is constant with respect to time, to obtain

$$s_t\theta + s\dot{\theta} = \int_{x_0}^{x} w_t(\xi, t)d\xi + w\dot{x}, \tag{2.9}$$

i.e.

$$s\dot{\theta} = \int_{x_0}^{x} w_t(\xi, t)d\xi + w\dot{x}. \tag{2.10}$$

Differentiating this with respect to x, gives

$$s_x\dot{\theta} + s\theta_x = w_t(x,t) + \frac{\partial}{\partial x}(w\dot{x}) \tag{2.11}$$

which, since θ is a function of t only, becomes

$$s_x\dot{\theta} = w_t(x,t) + \frac{\partial}{\partial x}(w\dot{x}). \tag{2.12}$$

Finally differentiating quation (2.8) with respect to x and substituting in quation (2.12), Ren and Russell obtain

$$\frac{\partial}{\partial t}w(x,t) + \frac{\partial}{\partial x}(w(x,t)\dot{x}) = \frac{\dot{\theta}(t)}{\theta(t)}w(x,t). \tag{2.13}$$

This is the new equidistribution PD which is studied by Ren and Russell. They look at two different implementations of quation (2.13) and investigate the stability. In the second of these implementations, the one we shall be considering, they introduce the transformation

$$W(x,t) = \frac{w(x,t)}{\theta(t)} = \frac{w(x,t)}{\int_{x_0}^{x_N} w(x,t)dx}, \tag{2.14}$$

where $W(x,t)$ is an "average energy" function.

 quation (2.13) then becomes

$$\frac{\partial}{\partial t}(W(x,t)\theta(t)) + \frac{\partial}{\partial x}(W(x,t)\theta(t)\dot{x}) = \frac{\dot{\theta}(t)}{\theta(t)}W(x,t)\theta(t), \tag{2.15}$$

i.e.

$$\frac{\quad(\quad)}{\quad}\ (\ )+\quad(\quad)^{\cdot}(\ )+\text{—}(\quad(\quad)^{\cdot}(\ ))\ (\ )+\quad(\quad)^{\cdot}\ {}_{x}(\ )=\ ^{\cdot}(\ )\quad(\quad) \qquad (2\ 16)$$

Therefore, since is a function only of , this yields

$$\frac{\quad(\quad)}{\quad}+\text{—}(\quad(\quad)^{\cdot})=0 \qquad (2\ 17)$$

$_{k}\quad _{k+1}\quad ^{n+1}$

$$\frac{\ _{k}^{n+1}\quad _{k}^{n}}{\quad}\quad \frac{\ _{k}^{n+1}\ _{k+1}\quad _{k-1}^{n+1}\ _{k}}{\ _{k+1}\quad _{k}}$$

$$_{k}^{n+1}\quad _{k+1}\quad _{k}\qquad _{k}^{n+1}\ _{k+1}\quad _{k-1}^{n+1}\ _{k}\qquad _{k+1}\quad _{k}\quad _{k}^{n}$$

$_{k-1}\quad _{k}$

$$_{k-1}^{n+1}\quad _{k}\quad _{k-1}\qquad _{k-1}^{n+1}\ _{k}\quad _{k-2}^{n+1}\ _{k-1}\qquad _{k}\quad _{k-1}\quad _{k-1}^{n}$$

$x_{k+1}\qquad x_{k}$

$x_{k}\qquad x_{k-1}$

^{n}

$$_{k}^{n+1}\quad _{k+1}\quad _{k}\qquad _{k}^{n+1}\ _{k+1}\quad _{k-1}^{n+1}\ _{k}$$

$$_{k-1}^{n+1}\quad _{k}\quad _{k-1}\qquad _{k-1}^{n+1}\ _{k}\quad _{k-2}^{n+1}\ _{k-1}$$

$$^{n+1}\quad \frac{w(x,t^{n+1})}{\theta(t^{n+1})}\quad ^{n+1}$$

$_{k}\ _{k+1}\quad _{k-1}\ _{k}\quad _{k-2}\ _{k-1}\quad _{k-1}\quad _{k}\quad _{k-1}\quad _{k}\quad _{k+1}\quad _{k}$

at $=$ $^{n+1}$.

We use the same weight function as before, i.e.

$$(\quad) = \overline{1 + {}^{2}_{x}(\quad)} \qquad (2\ 24)$$

As stated earlier, this will be known exactly when the analytic steady-state solu-

$$_k \qquad \overline{\qquad {}^{2}_{x} \qquad {}_k}$$

$$_k \qquad \overline{\qquad \left(\frac{_{k+1} \quad _k}{_{k+1} \quad _k}\right)^2}$$

$_0$ $_N$ $_0$

$_N$

$_k$

$_k$

$^{n+1}_{k}$ $^{n}_{k}$ $_k$

2.3 Nominal Iteration

This iterative method is derived in [4]. The authors look at the problem from a variational point of view, but also show that this is equivalent to an equidistribution formulation.

In [4] it is noted that the equation for defining a grid with respect to a position dependent weight $w(x,t)$ is given by

$$x'(\xi)w(x(\xi),t) = \text{Constant}. \tag{2.28}$$

This can be seen by considering quation (2.6) and reparameterising with $x = x(\xi)$ where $\xi \in [0,N]$ and $x_k = x(k)$. Hence

$$\int_{x_0}^{x(\xi)} w(x(\xi),t)d\xi = \frac{\xi}{N}\int_{x_0}^{x_N} w(x(\xi),t)d\xi, \tag{2.29}$$

which when differentiated with respect to ξ becomes

$$x'(\xi)w(x(\xi),t) = \frac{1}{N}\int_{x_0}^{x_N} w(x(\xi),t)d\xi = \text{Constant}, \tag{2.30}$$

as above.

Differentiating the above equation with respect to ξ gives a second order differential equation for $x(\xi)$ which is the same as the equation for the variational problem defining the adaptive grid,

$$x''(\xi) + \frac{\frac{\partial w}{\partial x}(x(\xi),t)}{w(x(\xi),t)}(x'(\xi))^2 = 0, \qquad x(0) = x_0, \quad x(N) = x_N. \tag{2.31}$$

This is the equation which is to be solved in order to equidistribute the grid. The iteration used is called in [4] the nominal iteration. It is

$$(x'')^{n+1} + (\frac{\frac{\partial w}{\partial x}(x(\xi),t)^n}{w(x,t)^n}(x')^n)(x')^{n+1} = 0, \qquad n \geq 0 \tag{2.32}$$

where n is the time level.

As before,

$$w(x,t) = \sqrt{1+u_x^2(x,t)} \tag{2.33}$$

and so

$$\frac{\partial w}{\partial x}(x,t) = u_x(x,t)u_{xx}(x,t)(1+u_x^2(x,t))^{-\frac{1}{2}}. \tag{2.34}$$

Hence, quation (2.32) can be written as

$$x''_{j+1} + \left(\frac{u_x|_j u_{xx}|_j}{(1+u_x^2)_j} x'_j\right) x'_{j+1} = 0, \qquad j \geq 0, \tag{2.35}$$

where j denotes the iterate. If the analytic steady state of the PD is known, or if a known function is being equidistributed, the first and second derivatives will be known exactly. This is the only case considered here, although it is possible to approximate the derivatives using finite differences when they are unknown.

In [4] quation (2.35) the ξ derivatives are discretised using central differences and the equation solved using the existing nodal positions, to find the new nodal positions. Setting

$$\begin{aligned} g(x_j) &= \tfrac{u_x|_j u_{xx}|_j}{(1+u_x^2)_j} x'_j \\ &= \tfrac{u_x|_{k,j} u_{xx}|_{k,j}}{(1+u_x^2)_{k,j}} \left(\tfrac{x_{k+1,j}-x_{k-1,j}}{2\psi}\right), \qquad k = 1,\ldots,N-1, \quad j \geq 0 \end{aligned} \tag{2.36}$$

where $\psi = \frac{x_N - x_0}{N}$, we have

$$\frac{x_{k-1,j+1} - 2x_{k,j+1} + x_{k+1,j+1}}{\psi^2} + g(x_j)\left(\frac{x_{k+1,j+1} - x_{k-1,j+1}}{2\psi}\right) = 0, \tag{2.37}$$

$k = 1,\ldots,N-1$, the nodal positions, $j \geq 0$, the iterates, and $x_0 = 0, x_N = 1$. This can be written as a tridiagonal system, and then solved in the usual way to obtain the grid nodes at the next time level.

As with the Ren and Russell iteration, monotonicity is not ensured using this iterations, so it is possible for nodes to cross.

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Methodology

We compare the dynamics of the three iterative techniques described in the previous chapter when used to equidistribute several test functions. Initially, we examine the dynamics of grid adaption alone by solving the regridding equations (2.4), (2.23), and (2.37) iteratively, using known functions (see Chapter 4).

We then solve the regridding equations and an underlying PD simultaneously, and again examine the dynamics (see Chapter 5).

For the grid adaption alone, with certain functions, it was possible to investigate analytically the case of one free node, and also for the tridiagonal iteration, the case of two free nodes (see Sections 4.3, 4.4, 4.6). Mathematica was used as an aid in this analysis.

When looking at the grid adaption alone, all of the equidistribution methods were used with an odd number, 11, and an even number, 12, of free nodes. The tridiagonal iteration and the Ren and Russell iteration were applied using both the exact and the approximate derivative u_x. The nominal iteration was only solved using the exact derivative, due to a shortage of time.

For all of the test cases used, the "exact" placements of the grid nodes for

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E uidistribution

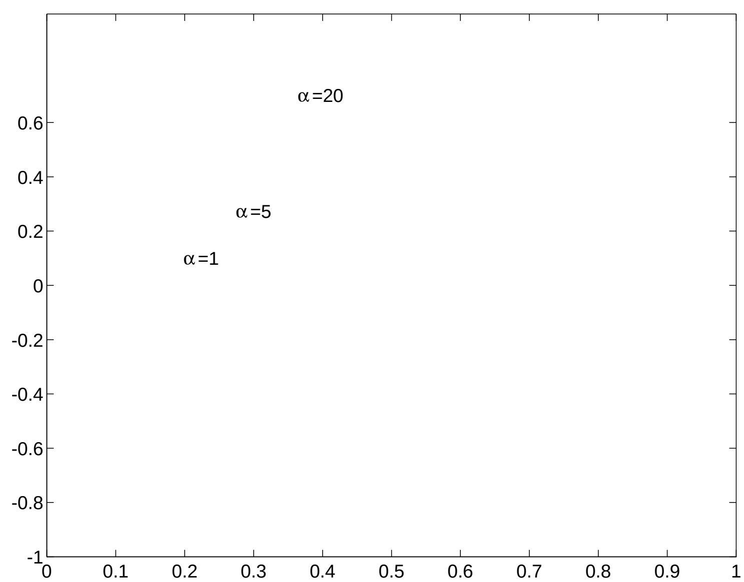
4.1 Test Problems and Exact Results

Four different functions are used as test cases for the equidistribution techniques alone, where $f(x)$ takes the place of $u(x,t)$ in the equations in Chapter 2. They are :-

1. $f(x) = \alpha x(1-x)$: a quadratic function, see Figure 4.1 on the following page. This is a simple symmetric function.

2. $f(x) = \alpha x(\frac{1}{2} - x)(1 - x)$: a cubic function, see Figure 4.2 on the following page. This is a simple unsymmetric function, but whose derivative is symmetric.

3. $f(x) = \frac{1-e^{x/\alpha}}{1-e^{1/\alpha}}$: an exponential function, see Figure 4.3. This is the steady state solution of the linear PD solved Chapter 5 and represents a boundary layer.



α=20
α=5
α=1
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

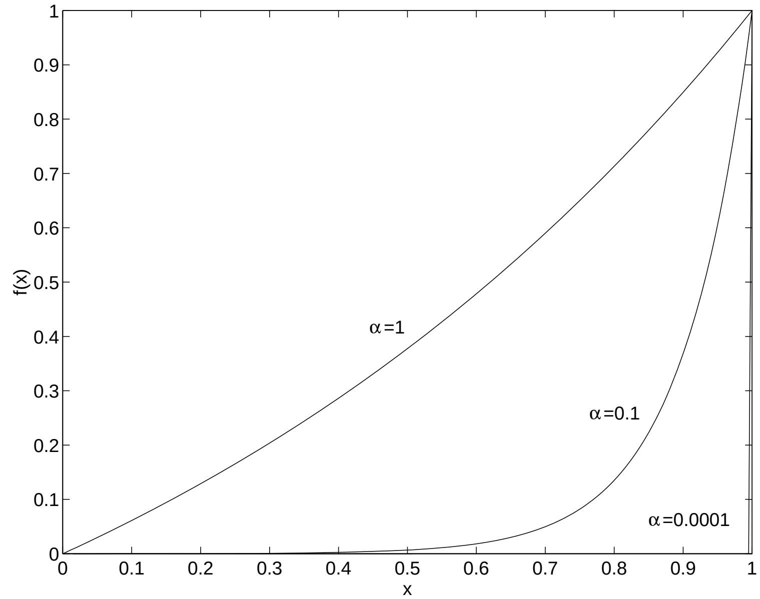


Figure 4.3: xponential Function.

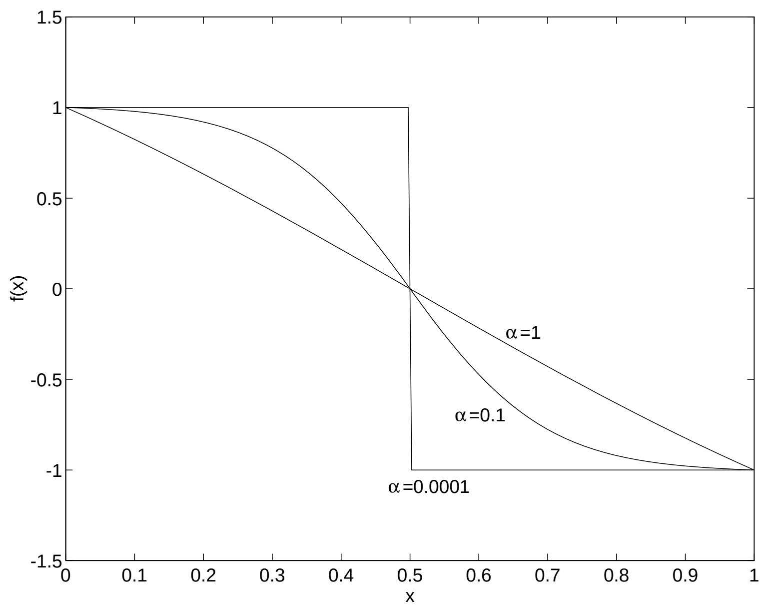


Figure 4.4: Tanh Function.

In each of these, α is a parameter that can be varied in order to investigate the dynamics of the equidistribution schemes. In the exponential function, as α decreases, the boundary layer becomes narrower. In the tanh function, as α decreases, the front becomes steeper.

For functions 1 and 2, the grid adaption schemes are applied for in the range [0 1 20]. In a few cases the range was extended to [0 1 50]. For functions 3 and 4, the schemes were applied for the range [0 0001 1].

All of the equidistribution methods are used with a skewed initial grid ($_0 = 0$, $_1 = 0\ 01$, $_2 = 2$, . . . , $_N = 1$ where $+1$ is the number of grid points). The quadratic function is symmetric, and the cubic and tanh functions are symmetric in terms of equidistribution of arclength, i.e. when their derivative is squared, and thus for all three functions, symmetrical equidistribution of grid points is expected. However by starting with a skewed grid, it is more likely that non-symmetric solutions will be found, if they exist. The exponential function is

$-$

$^{n+1}$ ^{n} $-$

^{n} $-$ $-$ $-$

Consider a small perturbation ϵ, $||\epsilon||$ 1, from the fixed point $\overline{x}$, i.e.

$$x^n = \overline{x} + \epsilon. \tag{4.1}$$

Substituting this in quation (4.2), and expanding using Taylor Series, we get

$$\begin{aligned} x^{n+1} = \overline{x} + \epsilon &= f(\overline{x} + \epsilon) \qquad (4.2) \\ &= f(\overline{x}) + \epsilon f'(\overline{x}) + \tfrac{\epsilon^2}{2} f''(\overline{x}) + \ldots \quad , \end{aligned}$$

and so

$$\epsilon = \epsilon f'(\overline{x}) + O(\epsilon^2). \tag{4.3}$$

To ensure that the magnitude of the perturbation decreases, we must bound $f'(\overline{x})$ such that

$$|f'(\overline{x})| < 1 \tag{4.4}$$

When this inequality is satisfied the fixed point will be stable.

If the inequality is greater than 1, the fixed point will be unstable, and for the case $|f'(\overline{x})| = 1$, linearised analysis does not give enough information to be able to determine stability.

By knowing the way in which a fixed point becomes unstable it is possible to tell something about its subsequent behaviour. One example, seen in Section 4.3.3, is when the fixed point becomes unstable with $f'(\overline{x}) = -1$. This indicates that the presence of a period doubling bifurcation, i.e. where $\epsilon^{n+1} \approx -\epsilon^n$, a period 2 solution. Alternatively, and this is seen in Sections 4.3.3 and 4.5, if $f'(\overline{x}) = 1$, a transcritical or pitch fork bifurcation will occur. To determine which particular type it is necessary to carry out an analysis of higher order terms (see for example [1]).

It is only possible to apply the analysis described to simple cases using the quadratic and cubic functions, even when using Mathematica as an aid.

We consider the cases of one free node and two free nodes. We also consider the case where the derivative of the function is approximated, as well as the case where the exact derivative is used.

For this function we note first that the weight function using the exact derivative and the weight function using the approximate derivative are identical.

The one free node grid is given by $_0 = 0,\ _1 = \quad ,\ _2 = 1$. Hence $_{\frac{1}{2}} = \frac{X}{2}$ and $_{\frac{3}{2}} = \frac{1+X}{2}$. Consequently, using the exact derivative

$$'(\ ) = \ (1 \quad 2\ ) \qquad (4\ 5)$$

the weight functions are, from quation (1.2),

$$\begin{aligned} _{\frac{1}{2}} &= \overline{1 + \ ^2(1 \qquad)^2} \qquad \text{and} \\ _{\frac{3}{2}} &= \overline{1 + \ ^2 \ ^2} \end{aligned} \qquad (4\ 6)$$

If the approximate derivative given in quation (2.3) is used, then

$$\begin{aligned} _{\frac{1}{2}} &= \overline{1 + (\tfrac{f(x_1)-f(x_0)}{x_1-x_0})^2} = \overline{1 + \ ^2(1 \qquad)^2} \qquad \text{and} \\ _{\frac{3}{2}} &= \overline{1 + (\tfrac{f(x_2)-f(x_1)}{x_2-x_1})^2} = \overline{1 + \ ^2 \ ^2} \end{aligned} \qquad (4\ 7)$$

$^{n} = \quad ^{n+1} = ^{-}$. Hence,

$$\overline{1 + \quad ^{2-2}}(1 \quad ^{-}) = \quad \overline{1 + \quad ^{2}(1 \quad ^{-})^{2-}} \qquad (4\ 8)$$

then squaringW654?NksiaT2Fk)W4thngW654i--W5855sW4k+-WTT6QF?ATW03P105N23ksquar

$2-2 \quad - \quad -2 \quad 2 \quad - \ 2 \ -2$

$-$

$- \quad -$

$$' - \quad \frac{2}{\quad \frac{\alpha^2}{4}}$$

$$\frac{2}{\quad \frac{\alpha^2}{4}}$$

$2 \quad 2 \quad 2$

$- \quad \frac{1}{2}$

0

$1 \quad 1 \quad 2 \quad 2 \quad 3 \quad \frac{1}{2} \quad \frac{x_1}{2} \quad \frac{3}{2} \quad \frac{x_2+x_1}{2} \quad \frac{5}{2} \quad \frac{1+x_2}{2}$

$\frac{1}{2}$ $\overline{\quad 2 \qquad 1^{\,2}}$

$\frac{3}{2}$ $\overline{\quad 2 \qquad 1 \qquad 2^{\;2}}$

$\frac{5}{2}$ $\overline{\quad 2 \; {}^{2}_{2}}$

$\frac{1}{2}$ $\overline{\quad \frac{f(x_1) \quad f(x_0)}{x_1 \quad x_0}^{\;2}}$ $\overline{\quad 2 \qquad 1^{\,2}}$

$\frac{3}{2}$ $\overline{\quad \frac{f(x_2) \quad f(x_1)}{x_2 \quad x_1}^{\;2}}$ $\overline{\quad \frac{\alpha x_2(1 \quad x_2) \quad \alpha x_1(1 \quad x_1)}{x_2 \quad x_1}^{\;2}}$

$\overline{\quad 2 \qquad 2 \qquad 1^{\,2}}$

$\frac{5}{2}$ $\overline{\quad \frac{f(x_3) \quad f(x_2)}{x_3 \quad x_2}^{\;2}}$ $\overline{\quad 2 \; {}^{2}_{2}}$

1 2

$\overline{\quad 2 \qquad \bar{}_1 \qquad \bar{}_2 \quad 2} \; \bar{}_2 \qquad \bar{}_1$ $\overline{\quad 2 \qquad \bar{}_1 \; 2 \bar{}_1}$

$\overline{\quad 2 \bar{}^{2}_{2}}$ $\bar{}_2$ $\overline{\quad 2 \qquad \bar{}_1 \qquad \bar{}_2 \quad 2} \; \bar{}_2 \qquad \bar{}_1$

1 2 $^{n}_{1}$ $^{n+1}_{1}$ $\overline{\;\;}_1$

$^{n}_{2}$ 2^{+1} $\overline{\;\;}_2$

1 2

If we start with a symmetric grid, so that assumption (4.19) will certainly be true,

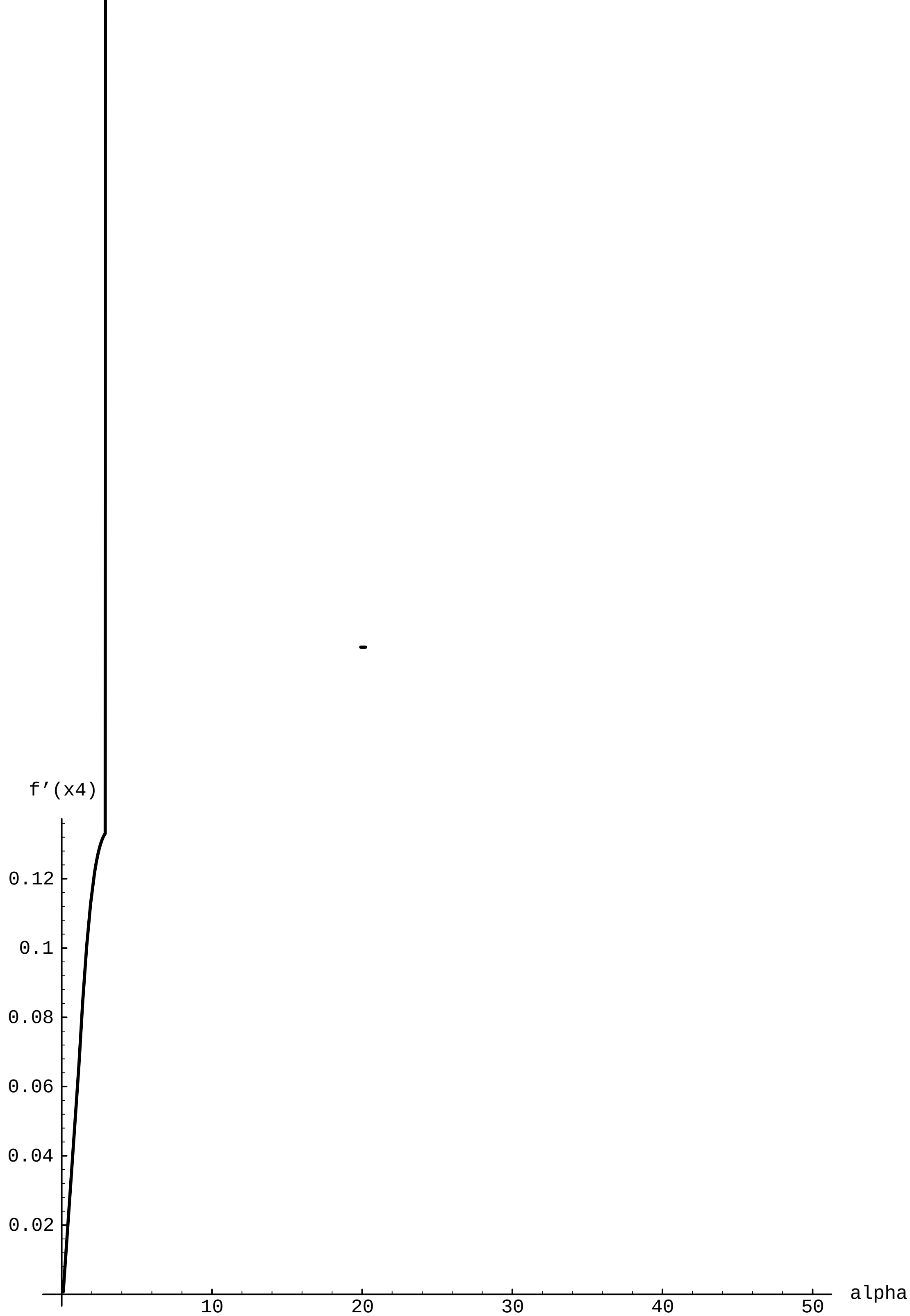
n n
1 2

$n+1$ $n+1$ $n+1$ $n+1$
1 2 1 2

2 2

2

4


f′(x4)
0.12
0.1
0.08
0.06
0.04
0.02
10
20
30
40
50
alpha

$\frac{1}{2}$

$0 \quad 1 \quad 2$

$\frac{1}{2} \quad \frac{X}{2} \quad \frac{3}{2} \quad \frac{1+X}{2}$

$' \quad - \quad 2$

$$\frac{1}{2} \quad \overline{2 \quad \frac{1}{2} \quad \frac{3X}{2} \quad \frac{3X^2}{2} \quad ^2}$$

$$\frac{3}{2} \quad \overline{2 \quad \frac{3X^2}{4} \quad \frac{1}{4} \quad ^2}$$

$$\overline{2 \quad \frac{-2}{\quad} \quad - \quad ^2} \quad - \quad \overline{2 \quad - \quad \frac{-}{\quad} \quad \frac{-2}{\quad} \quad ^2} \quad -$$

$-$

$- \quad \frac{1}{2}$

$$' \quad - \quad \frac{2}{\quad \frac{\alpha^2}{16}}$$

$$\frac{2}{\quad \frac{\alpha^2}{16}}$$

i.e.

$$\frac{256}{5} \quad ^{2} \quad \frac{256}{5} = 51\ 2 \qquad (4\ 25)$$

$2 \qquad - \quad \frac{1}{2}$

$0 \qquad 1 \qquad 2$

$$\frac{1}{2} \quad \overline{\quad \frac{f(x_1)-f(x_0)}{x_1-x_0} \ 2} \quad \overline{\quad 2 \ \frac{1}{2} \quad \frac{3}{2} \quad 2 \ 2}$$

$$\frac{3}{2} \quad \overline{\quad \frac{f(x_2)-f(x_1)}{x_2-x_1} \ 2} \quad \overline{\quad 2 \quad 2 \ \frac{1}{2} \quad 2}$$

$-$

$$\overline{\qquad\qquad} \qquad \overline{\qquad\qquad}$$

$2-2 \ \_ \quad - \ 2 \quad - \quad 2 \ \_ \quad \_- \quad -2 \ 2-$

$-$

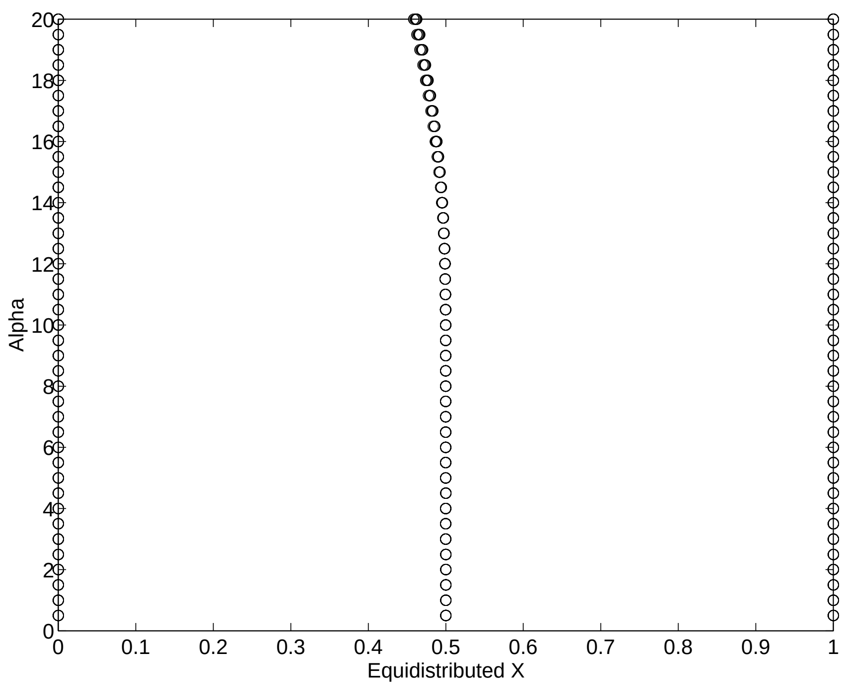
$- \quad \frac{1}{2} \qquad\qquad n+1$

$- \quad \frac{1}{2}$

$' \ \_$

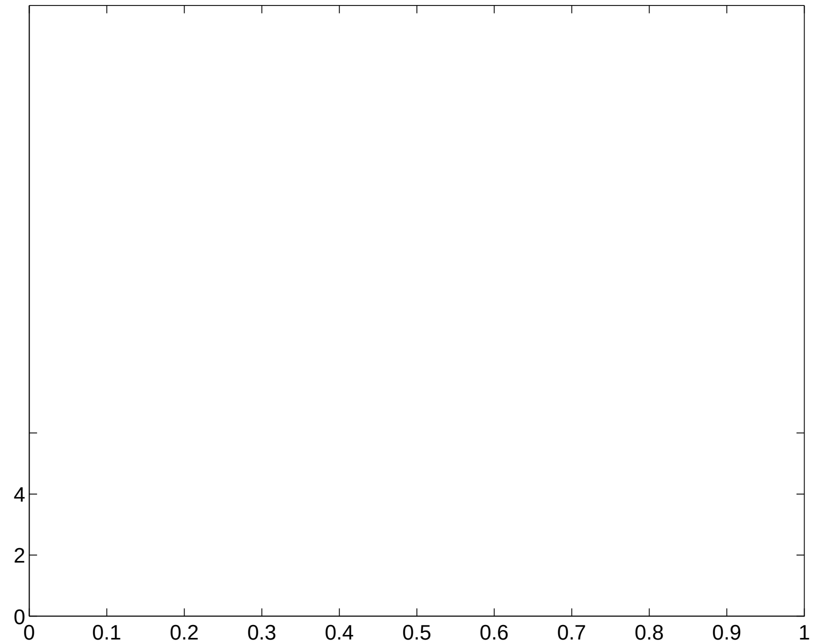
the equations found from (2.4) are too complicated to solve, even when assuming symmetry, and using Mathematica. (see Subsection 4.3.1 using two free nodes.)

$\frac{1}{2}$

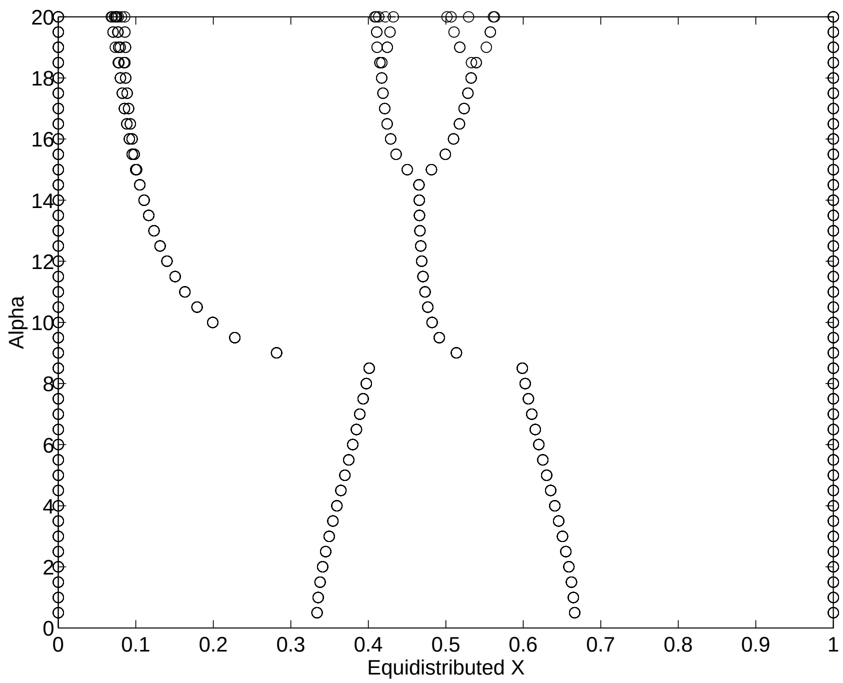


point $= \frac{1}{2}$. We do not know the precise reason for this, and both program and analysis appear correct. There is a possibility that the dependence on the initial

$\frac{1}{2}$



4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Alpha
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Equidistributed X

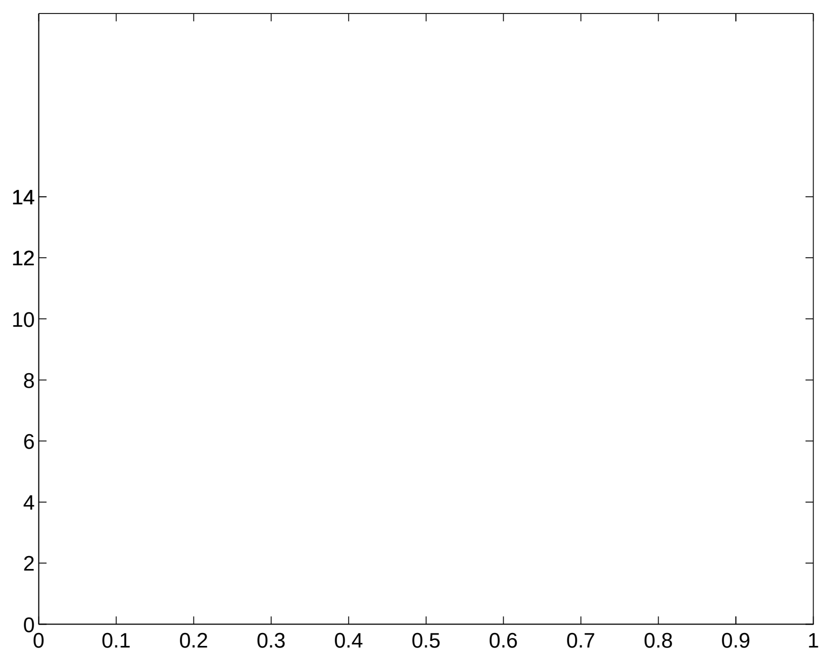
With 11 and 12 free nodes, using the exact and the approximate derivatives, the cubic function appears to converge to a spurious solution (incorrect when

−1.7

−1.9

−1.4

−2.2 −1.4 −2.2



14
12
10
8
6
4
2
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

so, for stability we require

$$\left| \frac{\alpha^2}{4\left(1+\frac{\alpha^2}{4}\right)} \right| < 1, \tag{4.33}$$

i.e.

$$-4-\alpha^2 < \alpha^2 < \alpha^2 + 4. \tag{4.34}$$

Clearly, the fixed point is unconditionally stable.

This is the same stability condition as was found for the tridiagonal iteration and the quadratic function, for the fixed point $\overline{x} = \frac{1}{2}$. The derivatives, $f'(x^n)$, are not the same from these two cases, but when the fixed point $\overline{x} = \frac{1}{2}$ is substituted, the two cases give identical results.

One free node - Exact derivative

With the one free node grid, $x_0 = 0$, $x_1 = X$, $x_2 = 1$, we get, from quation (2.24) using the exact derivative, $f'(x) = \alpha(1-2x)$,

$$\begin{aligned} w_0 &= \sqrt{1+\alpha^2}, \qquad \text{and} \\ w_1 &= \sqrt{1+\alpha^2(1-2X)^2}. \end{aligned} \tag{4.35}$$

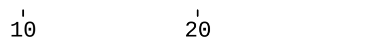
Substituting these into quation(2.23), gives

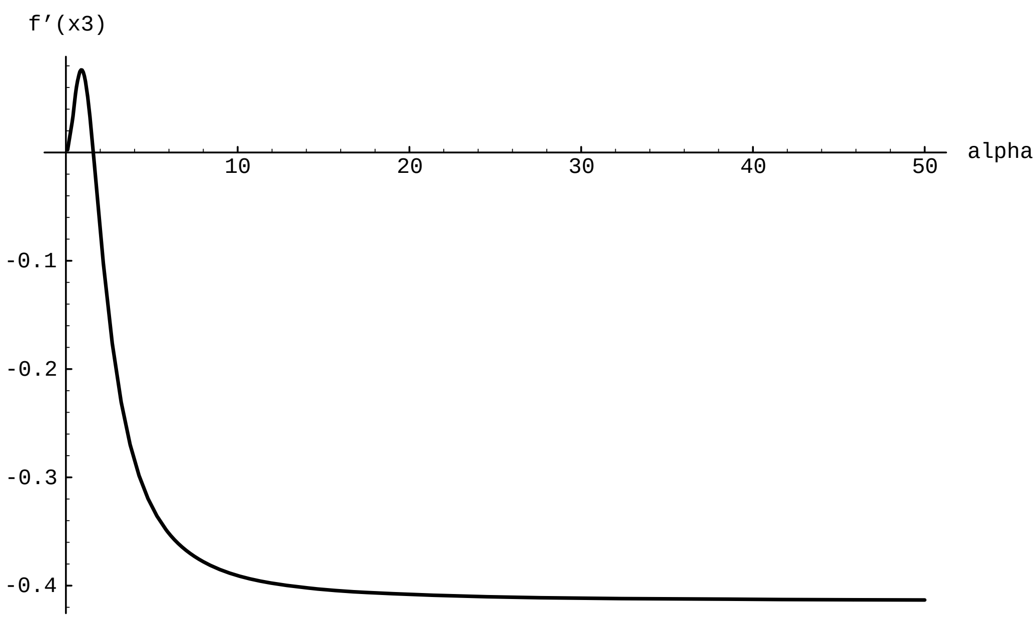
$$-2\Delta t\dot{X}\sqrt{1+\alpha^2} = \sqrt{1+\alpha^2}X^n - \sqrt{1+\alpha^2(1-2X^n)^2}(1-X^n), \tag{4.36}$$

and using quation (2.27), we substitute for $\dot{X}$, and get

$$-2(X^{n+1}-X^n)\sqrt{1+\alpha^2} = \sqrt{1+\alpha^2}X^n - \sqrt{1+\alpha^2(1-2X^n)^2}(1-X^n). \tag{4.37}$$

Looking for the steady states of this equation, we find four roots. The first two are complex and so cannot be attained. The next root is always greater than 1, and if attained would cause grid tangling. The last root is in the region $[0,1]$,


10
20


f′(x3)
alpha
10
20
30
40
50
-0.1
-0.2
-0.3
-0.4

$\frac{1}{2}$

$0 \quad 1 \quad 2$

$0 \quad \overline{\frac{f(x_1)-f(x_0)}{x_1-x_0} \quad 2} \quad \overline{2 \quad \frac{1}{2} \quad 2}$

$1 \quad \overline{\frac{f(x_2)-f(x_1)}{x_2-x_1} \quad 2} \quad \overline{2 \quad 2 \quad \frac{1}{2} \quad 2}$

$n+1 \quad n \quad \overline{2 \quad \frac{1}{2} \quad n \quad n \quad 2}$

$\overline{2 \quad \frac{1}{2} \quad n \quad n \quad 2} \quad n \quad \overline{2 \quad n \quad 2 \quad \frac{1}{2} \quad n \quad 2} \quad n$

$- \quad n \quad n+1$

$- \quad \frac{1}{2}$

$' \quad -$

$' \quad n$

As has been shown in the analysis, when the uler equation (2.27) is substituted in quation (2.23), the Δ term will cancel out.

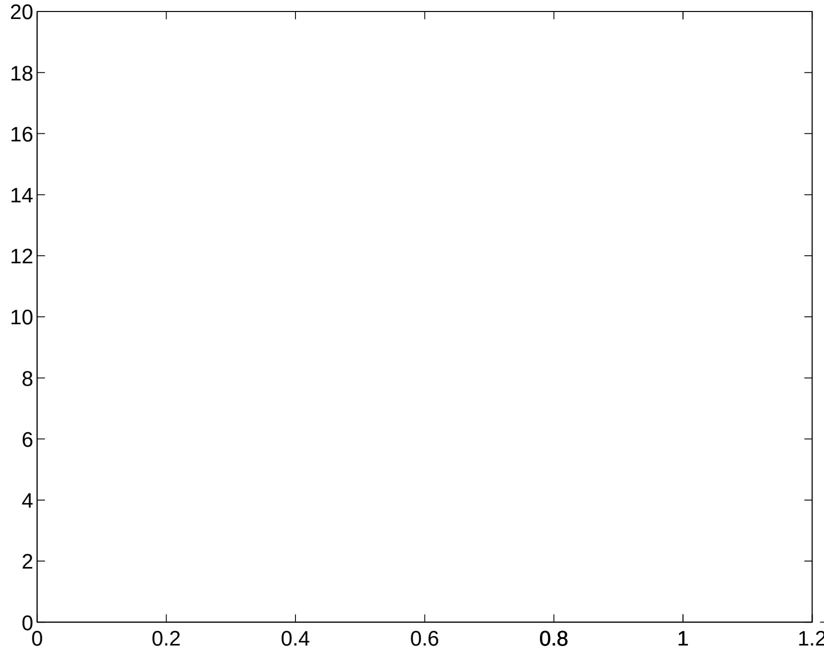
We find that, with one exception, the results obtained for the quadratic and cubic functions with one free node, behave as predicted by the analysis. The exception is the quadratic function using an approximate derivative. This behaves in the same unexplained way as the tridiagonal iteration and the quadratic function using the exact derivative. (see Figure 4.7 in Section 4.3.3) As in that case, if the starting grid is skewed the other way, the equidistribution solution moves in that direction, and if a symmetric starting grid is used, the solution stays at $= \frac{1}{2}$. Clearly, the behaviour is dependent on the starting grid.

Using the exact derivative, we find for both the quadratic and cubic functions, using an odd and an even number of free nodes, that the method breaks down when reaches approximately 23. This is due to the matrix becoming singular, (this is the matrix holding the values on the right-hand side of quation (2.23)). A plot of the condition number of the matrix against shows the condition number steadily increasing an increases.

Using the approximate derivative, the quadratic function with 11 free nodes to have a transcritical bifurcation at $= 16\ 5$, see Figure 4.14 (again Figure 4.10 shows the "exact" solution), since the solution remains unchanged if the starting grid is skewed the other way. With 12 free nodes, convergence is obtained for 16, but is unconverged above.

As expected, the results with the exponential function are not significantly different if there is an odd or an even number of free nodes. Using the exact derivative the solution converges correctly for $10^{-1.1}$. For $10^{-1.8}$ $10^{-1.1}$, the solution is unconverged, and for $10^{-1.8}$ the solution converges incorrectly into equally spaced parallel tracks. With the approximate derivative, the method will not work for $10^{-1.9}$, due to the matrix becoming singular. For greater than this, the solution converged correctly.

For the tanh function using the approximate derivative, there is not a great difference between the results for 11 and 12 free nodes. The solution converged correctly for approximately 10^{-2}, and below this the solution is unconverged or there is a period 2 solution. In this region, takes values in the range $[\ 1\ 2\ 5]$. Using the exact derivative, there is again not much difference between 11 and 12 free nodes in both cases the solution has converged correctly for approximately $10^{-0.9}$, and in the region $10^{-2.2}$ $10^{-1.2}$, is unconverged. With 11 free nodes, for $10^{-2.2}$, there is a period 2 solution. With 12 free nodes and $10^{-2.2}$ the solution has converged into equally spaced parallel tracks.


20
18
16
14
12
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1
1.2

0 1 2

$\frac{1}{2}$

+1

$\frac{1}{2}$ 2 2 2 $\frac{1}{2}$ $\frac{1}{2}$

2

+1

2 2

+1 111++126(66(+16(+126(222

$\frac{1}{2}$ 1112 2

+1

We find that

$$f'(\frac{1}{2}) = \frac{\alpha^2}{2}, \tag{4.47}$$

and so for stability we require

$$\left| \frac{\alpha^2}{2} \right| < 1, \tag{4.48}$$

i.e.

$$\alpha^2 < 2. \tag{4.49}$$

Hence the fixed point $\overline{x} = \frac{1}{2}$ will be stable for $\alpha^2 < 2$, and will be unstable elsewhere.

For the other two roots, their stability conditions are identical, see Figure 4.15 for a plot of the derivative with one of the roots substituted against α. The derivative becomes equal to one at $\alpha = \sqrt{2}$, so these two fixed points are stable for $\alpha > \sqrt{2}$.

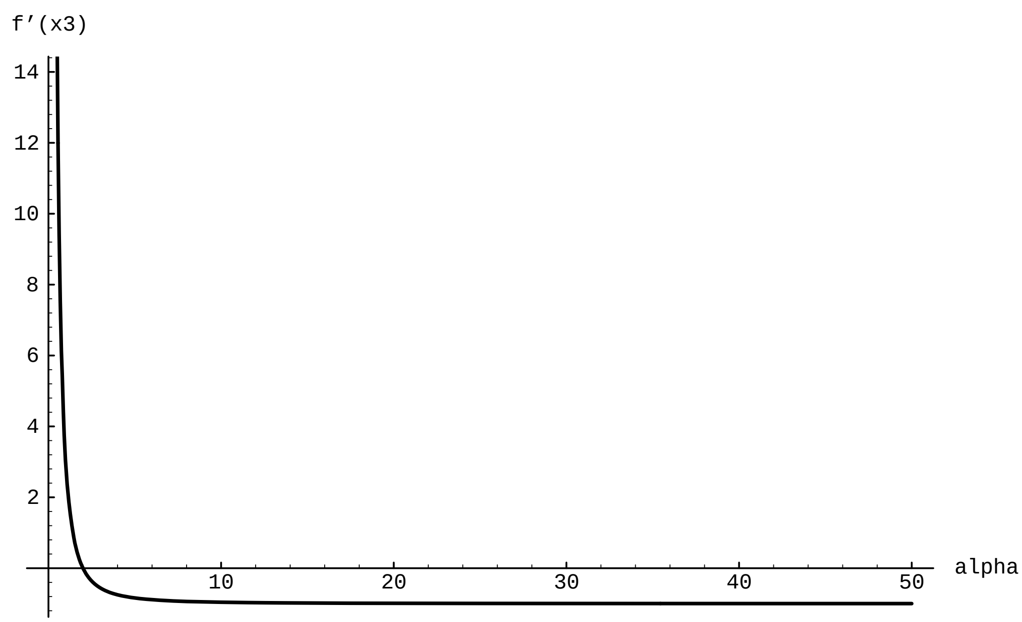


Figure 4.15: Analytic stability condition for the Nominal iteration, quadratic function, exact derivative, 1 free node.

$\frac{1}{}$

$0 \quad 1 \quad 2$

$' \quad - \quad 2$

$''$

$\frac{1}{2}$

$$n+1 \quad \frac{\frac{1}{2} \quad n \quad n \; 2 \quad n}{\frac{1}{2} \quad n \quad n \; 2 \; 2}$$

$- \quad n \quad n+1$

$- \quad \frac{1}{2}$

$2 \quad 3 \quad - \quad \overline{\qquad \overline{\overline{\quad}\, 2}}$

$4 \quad 5 \quad - \quad \overline{\qquad \overline{\overline{\quad}\, 2}}$

$-$

$- \quad \frac{1}{2}$

$$\frac{2}{\quad 2}$$

2

$2 \quad 3$

$2 \quad ' \quad 2$

$\frac{32+32\sqrt{10}}{3}$

$-$

The fixed points x_4 and x_5 are unstable for all α where the roots are real.

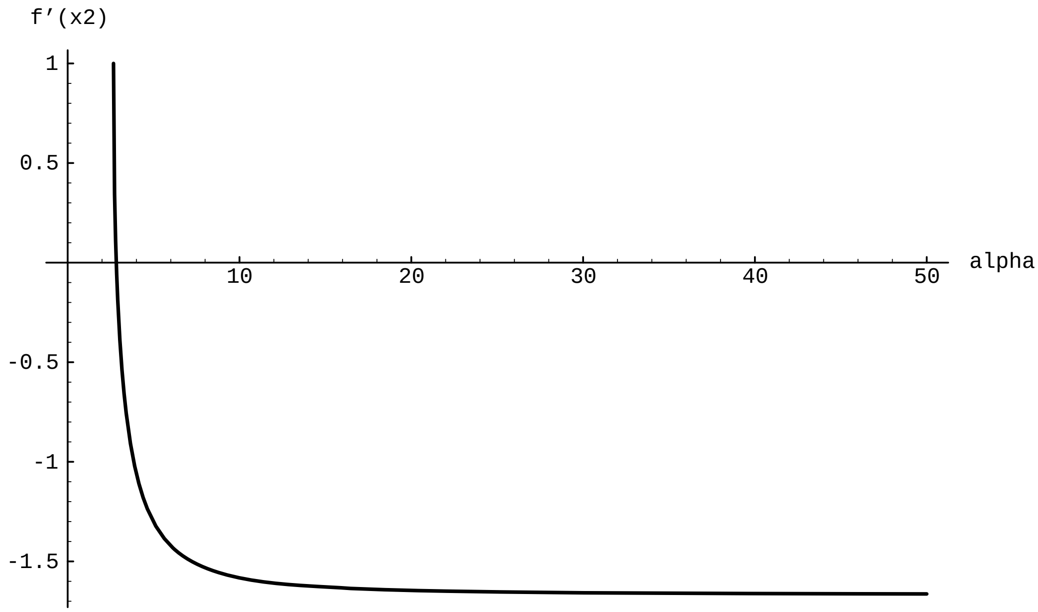


Figure 4.16: Analytic stability condition for the Nominal iteration, cubic function, exact derivative, 1 free node.

We only implement the nominal iteration with the exact derivative. We find that with one free node the results are as predicted by analysis. With the quadratic function, the equidistribution solution for $\overline{2}$ is $= \frac{1}{2}$. At $= \overline{2}$ there is a pitchfork bifurcation where there are two stable steady states, and an unstable steady state between them. Which of the two stable states is obtained depends on the starting grid.

We note that the results for all the functions were very similar for both an odd and an even number of free nodes.

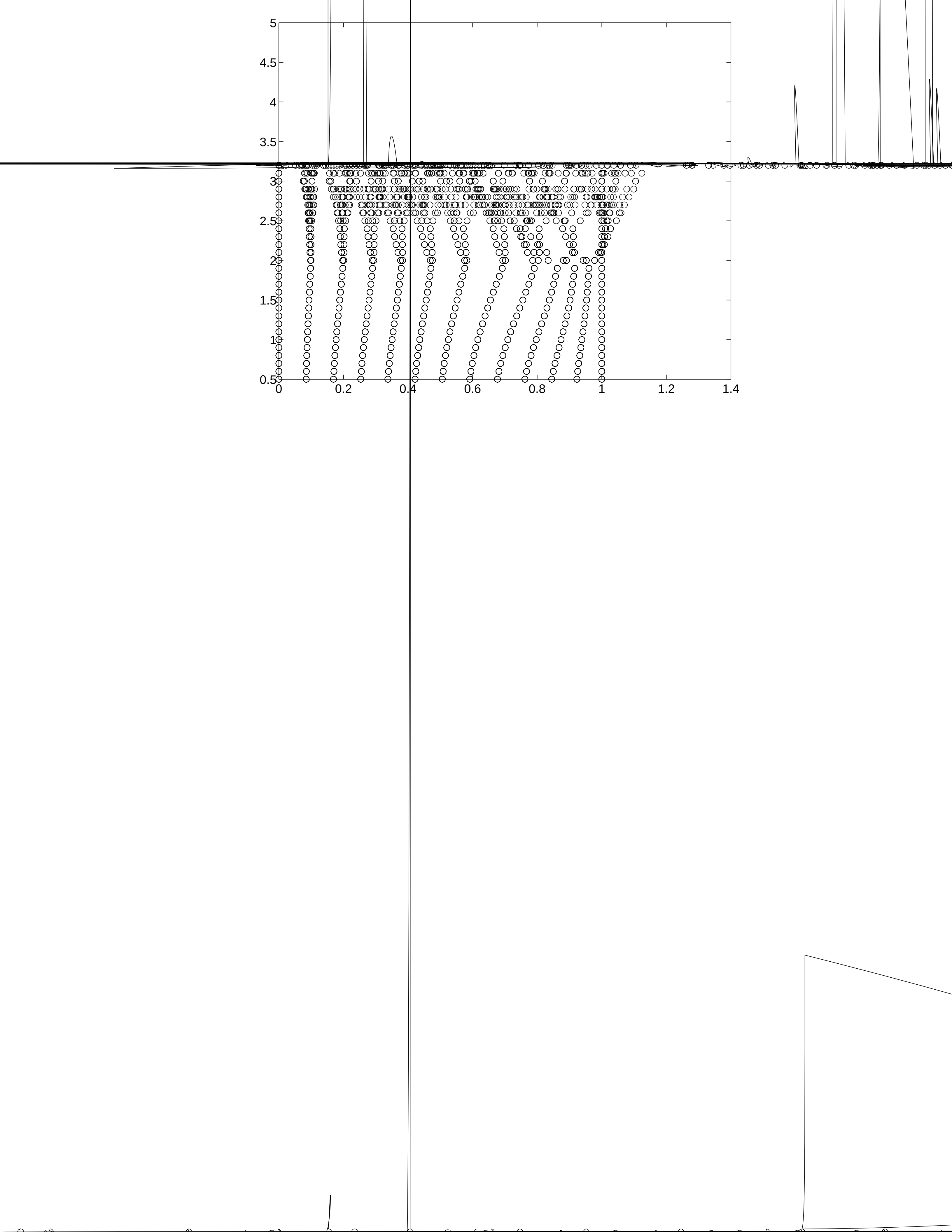
The quadratic function converged incorrectly for 4 2, but is unconverged for 4 2.

The cubic function converged incorrectly for 2. Then, as increased, it had a period 2 solution, and then was unconverged. It converged again for 3 7 4 2. See Figure 4.17.

The exponential function appears to converge correctly for $10^{-0.4}$. For 10^{-2} $10^{-0.5}$, the solution is unconverged and for 10^{-2} the solution has converged into equally spaced parallel tracks.

The tanh function shows convergence and unconvergence in the same pattern

-0.6



5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Ch pter 5

DE Solution with Grid Ad ption

5.1 Test Problems

The test problems used in this section are the linear form of the convection-diffusion equation

$$u_t + u_x = \epsilon u_{xx} \tag{5.1}$$

with boundary conditions $u(0,t) = 0$, and $u(1,t) = 1$.

and the viscous Burger's equation

$$u_t + (\frac{1}{2}u^2)_x = \epsilon u_{xx} \tag{5.2}$$

with boundary conditions $u(0,t) = -1$, and $u(1,t) = -1$.

The steady state solutions are, for the linear case, the exponential function, and for the non-linear case, the tanh function (see Section 4.1). ϵ is substituted for α in these functions.

Following along the lines of [5], we use a method of lines approach to solve the PD with central spatial differences,

$$u_{xx\ j} = \frac{\frac{u_{j+1}-u_j}{x_{j+1}-x_j} \quad \frac{u_j-u_{j-1}}{x_j-x_{j-1}}}{\frac{1}{2}(x_{j+1} \quad x_j)}, \tag{5.3}$$

and either upwind,

$$u_{x\ j} = \frac{f(u)_j \quad f(u)_{j-1}}{x_j \quad x_{j-1}}, \tag{5.4}$$

or central differences

$$u_{x\ j} = \frac{f(u)_{j+1} \quad f(u)_{j-1}}{x_{j+1} \quad x_{j-1}}, \tag{5.5}$$

for the convective term. The resulting system of ordinary differential equations (OD s) for $\frac{du_k}{dt}$ is then solved using linearised implicit (backward) uler, due to the small grid sizes that will occur, which would severely restrict time-steps for explicit methods.

The equation for implicit uler is

$$^{n+1} = \ ^{n} + \Delta t F(\ ^{n+1}), \tag{5.6}$$

where $\dot{\ } = F(\ )$, which is then linearised to give

$$^{n+1} = \ ^{n} + \Delta t(F(\ ^{n}) + J(\ ^{n})(\ ^{n+1} \quad \ ^{n}) + \ldots) \tag{5.7}$$

where $J(\ ^{n}) = \frac{\partial F}{\partial u}$.

Neglecting higher order terms, we get

$$(I \quad \Delta t J(\ ^{n})) \ ^{n+1} = (I \quad \Delta t J(\ ^{n})) \ ^{n} + \Delta t F(\ ^{n}), \tag{5.8}$$

which can be rewritten as

$$^{n+1} = \ ^{n} + \Delta \ (\quad \Delta \quad (\ ^{n}))^{-1} \quad (\ ^{n}) \qquad (5\ 9)$$

solutions, and so are not included here.

5.2 Results - Linear Problem

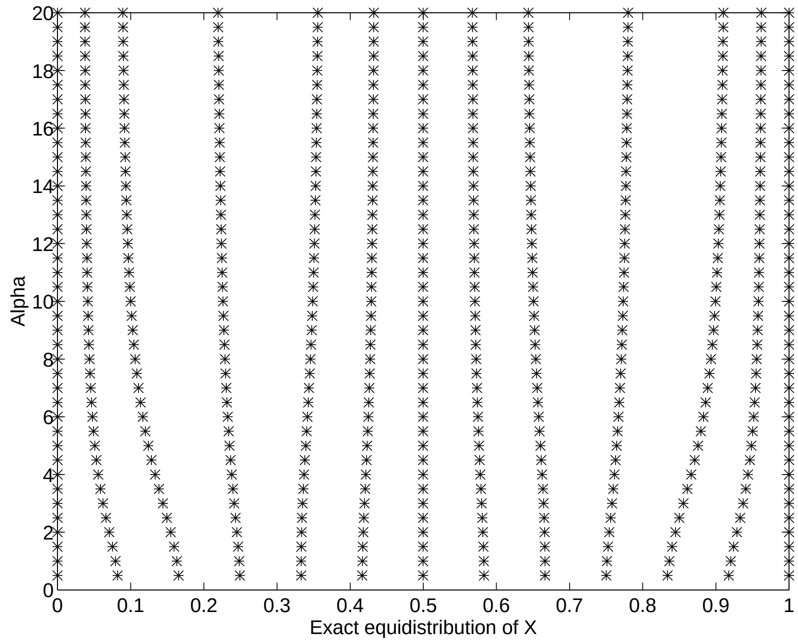


Figure 5.1: " xact" nodal positions, using fine quadrature, of the steady state solution of the linear problem.

5.2.1 Tridiagonal Iteration

For all cases, when $\Delta t = 0.001$ the solution was not quite converged, and it appeared that if it had converged the solution would be incorrect, see Figure 5.2. For this reason, all of the rest of the results in this subsection are with $\Delta t = 1$, $\Delta t = 0.1$, and $\Delta t = 0.01$. Using central differences for the convective term, and both with and without interpolation, the method performs better as Δt decreases. For $\Delta t = 1$, the solution only converges for $\alpha \geq 10^{-2.2}$ (see Figure 5.3), whereas for $\Delta t = 0.01$ the solution converges correctly for all values of α.

Using the upwind scheme for the convective term, both with and without interpolation, the solution converges correctly for all three values of Δ .

Again, for all cases when $\Delta \ = 0\ 001$, the solution was not quite converged and did not appear correct, similar to Figure 5.2 with the tridiagonal iteration. Hence, the following results are all for $\Delta \ = 1$, $\Delta \ = 0\ 1$, and$\Delta \ = 0\ 01$.

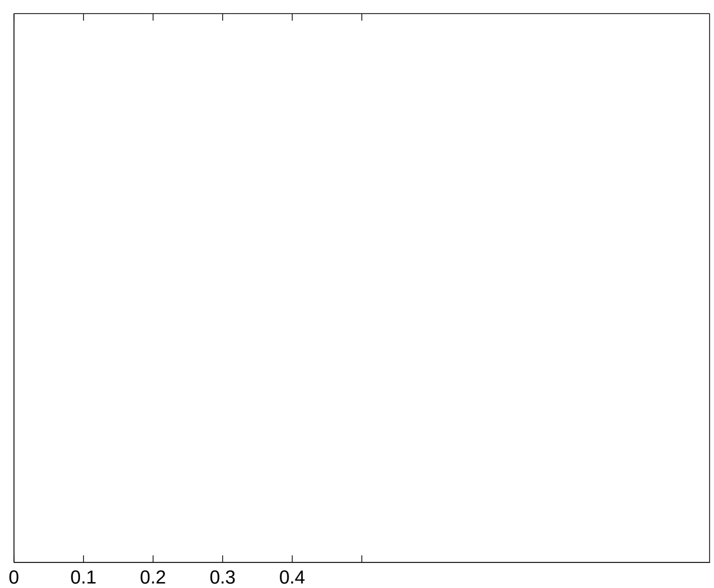
−1.8

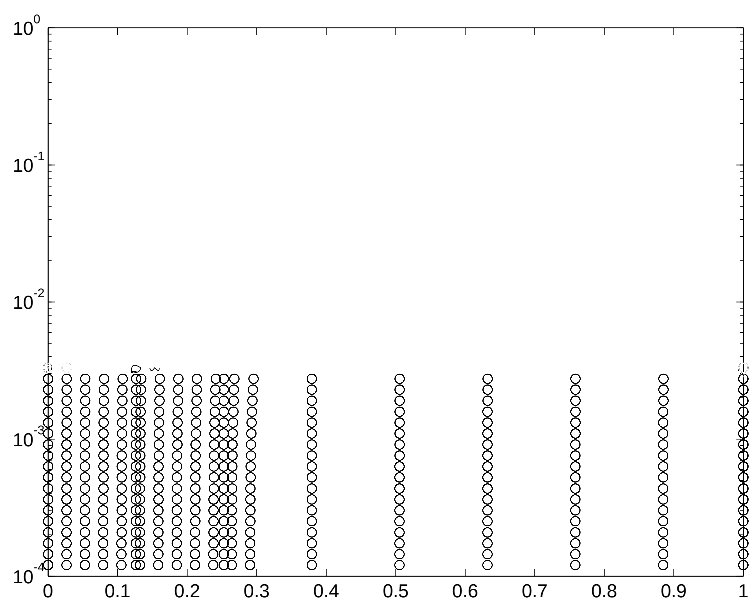
−2.2 −1.5

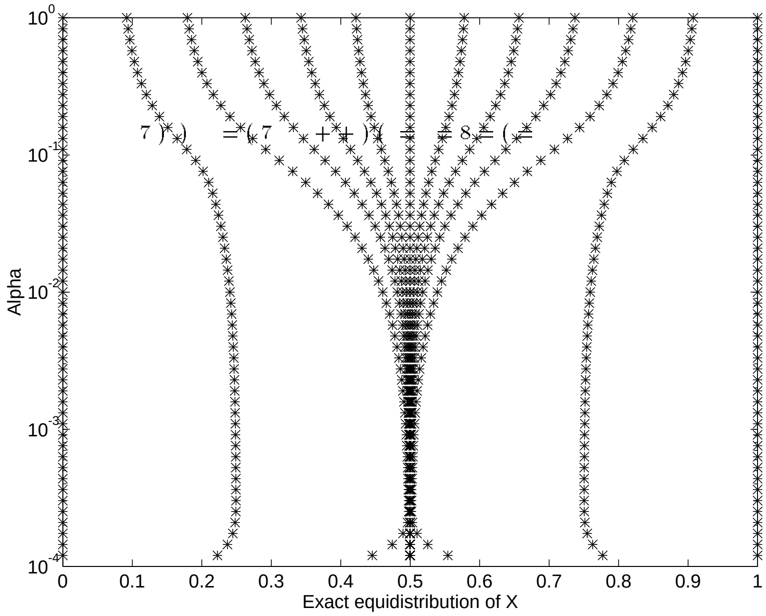
−2.6

−2.6 −1.6

−1.3 −3


0
0.1
0.2
0.3
0.4


10^0
10^-1
10^-2
10^-3
10^-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Alpha
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Exact equidistribution of X

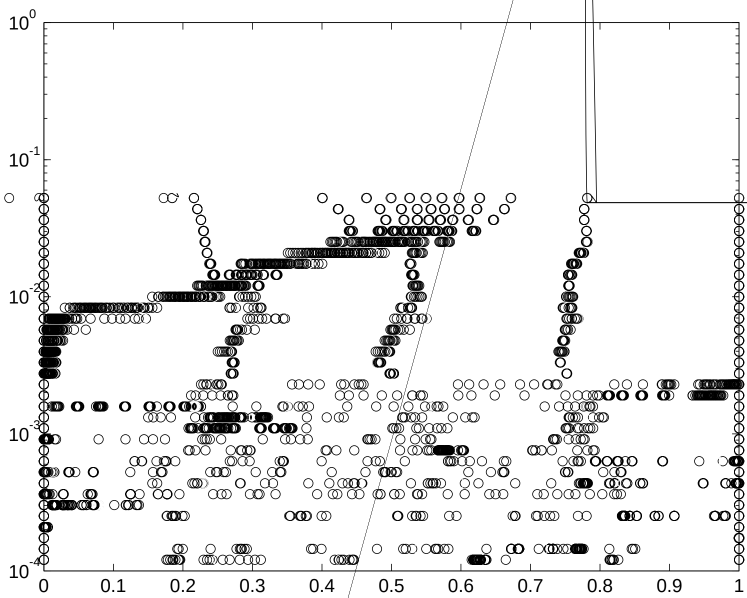
0 9 [illegible]

the same, and with the symmetric grid the tracks have not quite converged for $\alpha \leq 10^{-1.4}$.

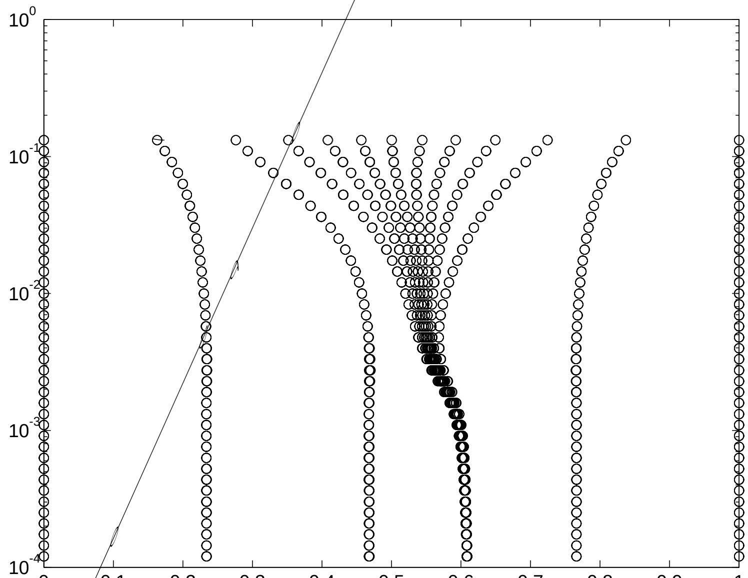
When interpolation is performed after each step the method improves considerably and does not fail for any values of α. There are now negligible differences between starting with the skewed or the symmetric initial grid. For the smallest time-step, $\Delta t = 0.001$, the solution has nearly converged into almost parallel tracks. For the other three time-steps, the method improves slightly as Δt decreases; for $\Delta t = 1$ the method only converges for $\alpha \geq 10^{-1.3}$, whereas for $\Delta t = 0.01$ the method is converged for $\alpha \geq 10^{-1.6}$.

Using the 1st order upwind scheme for the convective term, and no interpolation, the method improves as Δt increases. For $\Delta t = 1$ and $\Delta t = 0.1$ the method converges correctly for the full range of ϵ, with either starting grid. For $\Delta t = 0.01$ the method converges completely with the symmetric starting grid, but with the initial grid skewed to the left, the solution moves in from the right (see Figure 5.9). With $\Delta t = 0.001$, the results are similar but for some values of α the solution is slightly unconverged.

When interpolation is used, the results become much worse. For $\Delta t = 0.001$ the solution has nearly converged to almost parallel tracks (as when using central differences and interpolation). For $\Delta t = 0.01$, $\Delta t = 0.1$, and $\Delta t = 1$ the method improves as Δt decreases, but still is never converged for $\epsilon > 10^{-2}$. There are no significant differences between the results with the different starting grids.



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



10^0
10^-1
10^-2
10^-3
10^-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Ch pter 6

Summ ry

We have investigated three different mesh equidistribution schemes, the tridiagonal iteration, the Ren and Russell iteration, and the nominal iteration. We first looked at the dynamics of these methods when equidistributing a grid to one of four known functions, all of which are dependent on a parameter α. These functions consisted of, a quadratic and a cubic, for which some analysis was possible, and two more complicated functions which were the steady state solutions of the PD s studied later. In these two cases the parameter α ranged over several orders of magnitude, small values resulting in steep features.

Looking at the equidistribution alone, we found that the tridiagonal and Ren and Russell iterations far out-performed the nominal iteration. It should be noted that computations using the nominal iteration and approximate derivatives were not performed. The improvement in the solutions when the first two iterations were used with the approximate as opposed to the exact derivatives suggest that the performance of the nominal iteration might therefore be improved. However, it was clear when comparing all three methods using exact derivatives that the nominal iteration was worst.

Spurious dynamics were evident in all three methods. However, this was far less prevalent for the tridiagonal iteration on the simple quadratic and cubic functions. For the other two functions, both the tridiagonal and the Ren and Russell iterations did not appear to exhibit spurious dynamics. However both did give incorrect solutions for small values of the parameter α. In general results did not appear to be heavily dependent on whether there were an odd or an even number of free nodes, nor in general on the initial grid. However, it was noted that in some instances different initial grids could produce different final node placements. This indicates that in some situations the basins of attraction of the solutions have been badly distorted by spurious dynamics.

Next, we used the two more successful iterations, the tridiagonal and the Ren and Russell, coupled with a PD solver to adaptively solve two time-dependent PD s to steady state. The first PD was the linear form of the convection-diffusion equation, which has a boundary layer steady state solution. The second was the viscous Burger's equation, the boundary conditions giving a steep front solution.

Spatial derivatives were discretised using upwind or centred finite differences, whilst an implicit scheme was used for time-stepping in order to avoid over-restrictive stability conditions. Since we were only interested in steady state solutions, no complicated correction of solution values due to grid movement was performed. In particular only linear interpolation or no correction at all were used.

For the linear PD , the tridiagonal iteration outperformed the Ren and Rus-

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