The numerical propagation of scaling symmetries of scale-invariant partial differential equations: the S-property for mass-conserving problems

Mike Baines

Department of Mathematics and Statistics,
University of Reading, UK

Abstract

We consider scale-invariant problems governed by nonlinear partial differential equations on finite moving domains that conserve total mass in time.

We show that, under spatial deformations of the domain driven by a local conservation of mass principle, an initial condition that coincides with a self-similar scaling solution is propagated as the self-similar solution for all time, while for spatial deformations generated by the conservation of distributed masses, an initial condition that coincides with the piecewise-linear L^2 projection of the self-similar solution is propagated as the piecewise-linear L^2 projection of the self-similar solution for all time, the latter exhibiting a discrete scaling symmetry.

For more general initial conditions we adapt the proofs to obtain related scale-invariant procedures that possess the S-property, i.e. if the initial condition coincides with a self-similar scaling solution (in an appropriate norm), then it is propagated as the self-similar scaling solution exactly in that norm (modulo a projection error in the L^2

1 Introduction

Many problems governed by partial di erential equations (PDEs) that arise in practical applications possess scaling properties which are in a sense more fundamental than the equations themselves [14]. In apa moving boundary exactly to within rounding error. Also, in [9], velocity-based scale-invariant moving-mesh finite-difference and finite-element schemes based on conservation were shown to propagate self-similar solutions of a fourth-order nonlinear moving boundary problem exactly to within rounding error.

In this paper we generalise these results by proving first that for a general class of scale-invariant mass-conserving PDE problems deformations of the domain generated by the conservation of local (or distributed) masses imply the exact propagation of self-similar solutions (or their L^2 projections) in time. We then adapt the steps of the proof to apply to a class of scale-invariant problems with general initial conditions, resulting in numerical algorithms that possess the S-property, defined as the exact propagation of a scaling symmetry when the initial condition coincides with the self-similar scaling solution in some norm.

The layout of the paper is as follows. In section 2 we prove that, for scale-invariant time-dependent PDE problems that conserve total mass, conservation of local mass implies the propagation of self-similar scaling solutions exactly in time. Then, in section 3 we prove that conservation of distributed (piecewise-linear) masses implies the same property in the case of the L^2 projection of a self-similar scaling solution, thus preserving a discrete scaling symmetry in the L^2 norm.

In section 4 these procedures are extended to general initial conditions, yielding algorithms for a class of first-order-in-time flux-driven PDEs that possess the S-property in some norm.

Finite-difference and finite-element algorithms are presented in section 5 for classes of flux-driven problems, again aiming for the S-property. The finite-difference scheme possesses the S-property in the l^1 norm when a function is interpolated quadratically from adjacent gridpoints, while the finite-element scheme possesses the S-property (in the L^2 norm) but subject to a projection error.

An illustrative example is given in section 6 and the paper summarised in section 7.

We first recall the concepts of scale invariance and similarity.

1.1 Scale invariance

A problem governed by a one-dimensional time-dependent PDE for a scalar function $u(x;t)$ (density) in a moving interval $(a(t);b(t))$ is scale-invariant if it is unaltered under the scalings

$$t \,!\quad t; \quad x \,!\quad x; \quad u \,!\quad u; \tag{1}$$

where is the group parameter. Here ; ; are scaling exponents for the particular PDE, (and $a(t); b(t)$ scale in the same way as x). Without loss of generality we take $= 1$.

Under the transformation (1) the total mass, de ned as

$$(t) = \int_{a(t)}^{b(t)} u(x;t)\, dx; \tag{2}$$

scales as $^{+}$. When the total mass is independent of time $+ \quad = 0$.

Similarity variables (themselves scale-invariant) may be de ned as

$$\frac{x}{t} = \quad ; \qquad \frac{u}{t} =$$

using $+ \quad = 0$. Also de ne

$$_a = \frac{a(t)}{t}; \qquad _b = \frac{b(t)}{t}$$

1.2 Self-similarity

We de ne a self-similar scaling solution to be an ansatz of the form

$$u(x;t) = t \quad ( ); \qquad \text{where} \quad x = t \tag{3}$$

The function ( ) satis es a reduced order di erential equation (see e.g. [8, 11]) in which the partial time derivative of u is

$$@u = t \quad ^{1}$$

2 Propagation of scaling symmetry

2.1 An integral invariant

An invariance property of the self-similar scaling solution (3) is that the local masses between any two coordinates $\hat{x}_1(t)$ and $\hat{x}_2(t)$,

$$\int_{\hat{x}_1(t)}^{\hat{x}_2(t)} u(\ ;t\ )\,d \tag{6}$$

are independent of time for all those $a(t) \quad \hat{x}_1(t) < \hat{x}_2(t) \quad b(t)$ that are proportional to $_a \quad _1 < \ _2 \quad _b$ respectively, by the factor t . The result follows by substituting $= t$ into (6) to obtain the time-invariant quantity

$$\int_{1}^{2} (\ )\,d$$

where $_1 = \hat{x}_1(t)=t$ and $_2 = \hat{x}_2(t)t$.

We prove a converse of this property.

2.2 Theorem 1

Theorem 1: Let the density $u(x;t)$ be a strictly positive solution of a time-dependent scale-invariant mass-conserving PDE problem in a moving domain $(a(t);b(t))$.

If

the points $\hat{x}_1(t);\hat{x}_2(t)$ of the domain move in such a way that the local masses

$$\int_{\hat{x}_1(t)}^{\hat{x}_2(t)} u(\ ;t\ )\,d \quad (= c(\hat{x}_1;\hat{x}_2)\,;\text{say}) \tag{7}$$

are constant in time for all $a(t) \quad \hat{x}_1(t) < \hat{x}_2(t) \quad b(t)$,

the initial condition on $u(x;t)$ coincides with a self-similar scaling solution of the form (3) for all x,

then for any moving coordinate $\hat{x}(t)$ the solution $u(\hat{x}(t);t)$ coincides with the self-similar scaling solution (3) in the interval $a(t) \quad \hat{x}(t) \quad b(t)$ for all t, thus preserving a scaling symmetry, and the induced velocity $v(\hat{x}(t);t)$ is the similarity velocity (5)..

at $t = t^0$, for all ^0_a $^0_1 <$ $^0_2$ ^0_b such that

$$^0_1 = \frac{\hat{x}_1(t^0)}{(t^0)}; \quad ^0_2 = \frac{\hat{x}_2(t^0)}{(t^0)}; \quad ^0_a = \frac{a(t^0)}{(t^0)}; \quad ^0_b = \frac{b(t^0)}{(t^0)};$$

Hence from (10)

$$[\quad (t^0) \quad ^{1} \; ^{0} \; (\; ^0) + \quad (\; ^0)v(x; t^0)]^{\,^0_2}_{\,^0_1} = 0 \qquad (11)$$

where $^0 = x=(t^0)$.

In order to solve (8) uniquely for the velocity $v(x; t^0)$ a value is requred at one point. Without loss of generality we take the point at which $v(x; t^0)$ is zero to coincide with the origin of ^{0} . Thus, putting $^0_1 = 0$ and taking $^0_2$ as a general point ^{0} in equation (11), we obtain

$$(\; ^0)f \quad (t^0) \quad ^{1} \; ^{0} + v(x; t^0)g = 0$$

Since $(\; ^0) > 0$ (because $u(x; t^0) > 0$) it follows that

$$v(x; t^0) = \quad (t^0) \quad ^{1} \; ^{0} = \frac{x}{t^0}; \qquad (12)$$

as required.

Remark: Equation (12) does not follow immediately by di erentiating the second of (3) with respect to t since (3) holds only at $t = t^0$. On the other hand, the reduced order equation holds at $t = t^0$ with $@_t u$ given by (4) at $= \; ^0$.

2. In the second part of the proof we show that under a deformation of the domain initiated by the velocity (12) the similarity variable $^1 = x(t^1)=(t^1)$ is equal to ^{0} at time $t^1 = t^0 + h$ to second order in h.

Let $\hat{x}(t)$ be a moving coordinate, coinciding with x at $t = t^0$ and moving with the velocity $v(x; t^0)$ of (12). Given an increment h in time, a Taylor series expansion of $\hat{x}(t)$ at $t^1 = t^0 + h$ yields

$$\hat{x}(t^1) = \hat{x}(t^0) + h\,v(\hat{x}(t^0); t^0) + O(h^2) \qquad (13)$$

Substituting for $v(\hat{x}(t^0); t^0)$ from (12), we obtain

$$\hat{x}(t^1) = \hat{x}(t^0) + h\frac{\hat{x}(t^0)}{t^0} + O(h^2) = \quad 1 + \frac{h}{t^0} \quad \hat{x}(t^0) + O(h^2)$$

$$= \left(1 + \frac{h}{t^0}\right) \hat{x}(t^0) + O(h^2) = \left(\frac{t^1}{t^0}\right) \hat{x}(t^0) + O(h^2);$$

showing that the similarity variable

$$^1 = \frac{\hat{x}(t^1)}{(t^1)} = \frac{\hat{x}(t^0)}{(t^0)} + O(h^2) = \ ^0 + O(h^2) \qquad (14)$$

is invariant to order h^2.

3. In the third part of the proof we demonstrate that under the deformation generated by (12) the similarity variable ( ) is also invariant to second order in h. From the conservation property (7)

$$\int_{a(t^1)}^{\hat{x}(t^1)} u(; t^1)\, d = \int_{a(t^0)}^{\hat{x}(t^0)} u(; t^0)\, d \qquad (15)$$

Di erentiating (15) wrt $\hat{x}(t^1)$ we obtain

$$u(\hat{x}(t^1); t^1) = \frac{d\hat{x}(t^0)}{d\hat{x}(t^1)} \frac{d}{d(\hat{x}(t^0); t^0)} \int_{a(t^0)}^{\hat{x}(t^0)} u(; t^0)\, d$$

$$= \frac{d\hat{x}(t^0)}{d\hat{x}(t^1)} u(\hat{x}(t^0); t^0)$$

Thus, due to (14)

$$\frac{u(\hat{x}(t^1); t^1)}{(t^1)} = \frac{u(\hat{x}(t^0); t^0)}{(t^0)} = (^0) + O(h^2);$$

equivalently,

$$(^1) = \frac{u(\hat{x}(t^1); t^1)}{(t^1)} = (^0) + O(h^2); \qquad (16)$$

using (14) again. Thus the similarity variable ( ) of (3) is invariant to order h^2.

4. The fourth part of the proof is concerned with repetition of the rst three parts over a further time step h. t h1h8(.)1 1;TJ -36.533 -27.56(arid85at 10.9091 Tf 6.285 0

where the right hand side of (17) is $O(h)$ rather than $O(h^2)$ since one power of h is lost in the di erentiation with respect to in deriving (10) from (8) using (4). Hence by equations (11) through to (12) with t^0 replaced by t^1,

$$v(x; t^1) = \quad x$$

$$(\ ) = \frac{u(\hat{x}(t);t)}{t} = \frac{u(\hat{x}(t^0);t^0)}{(t^0)} + O(h) = \ (\ ^0) + O(h);$$

In the limit

$$v(\hat{x}(t);t) = \frac{\hat{x}(t)}{t}$$

and

$$= \frac{\hat{x}(t)}{t} = \ ^0; \qquad (\ ) = \frac{u(\hat{x}(t);t)}{t} = \ (\ ^0);$$

for all $t > t^0$. Hence $u(\hat{x}(t);t)$ coincides with the self-similar solution (3), and $v(\hat{x}(t);t)$ concides with the similarity velocity (5) for all $t > t^0$. This completes the proof.

The function $N(\;)$ satisfies a reduced order equation in which the partial time derivative of $U(x;t)$ is

$$\partial U = \quad t \quad {}^{1}N)(\;) + (\quad)t^{\;1}\,\partial N)(\;) = \quad t \quad {}^{1}$$

3.2 Theorem 2

Theorem 2:

If

the nodes of the partition move such that the weighted masses

$$\int_{a(t)}^{b(t)} W_i(\;)\,U(\;;t\;)\,d \qquad (=\;C_i;\;\text{say}); \qquad (27)$$

where $\;=\;=t\;$, are independent of time for all $i = 0;:::;N+1$,

the piecewise-linear weight functions W_i are advected with a piecewise-linear velocity V induced by (27) (NB: velocities that advect piecewise-linear functions W_i exactly must be piecewise-linear.),

the initial condition on $U(x;t)$ coincides with the L^2 projection of a self-similar scaling solution of the form (3) for all $a(t) < x < b(t)$,

then for any moving coordinate $\hat{x}(t)$ the projected solution $U(\hat{x}(t);t)$ coincides with the L^2 projection of the self-similar scaling solution (3) in the domain $a(t) \quad \hat{x}(t) \quad b(t)$ for all t, thus exhibiting a discrete scaling symmetry in the L^2 norm, and $V(\hat{x}(t);t)$ coincides with the similarity velocity (5).

As a preliminary to the proof we obtain the weak form of the differential equation satisfied by a piecewise-linear velocity $V(x;t)$ induced by the invariance of (27).

Lemma 2:
The invariance of (27) together with the advection property of the W_i implies that

$$\int_{a(t)}^{b(t)} W_i(\;)\,\{\partial_t U + \partial(UV)\}\,d \;= 0 \qquad (28)$$

for all $i = 0;:::;N+1$. where $V(x;t)$ is the induced piecewise-linear velocity.

Proof: By the Reynolds Transport Theorem applied to $W(\;)U(x;t)$,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} W_i(\;)\,U(\;;t\;)d \;=\; \int_{a(t)}^{b(t)} W_i(\;;t)\,\{\partial_t U + \partial\,(UV)\}\,d$$

$$+ \int_{a(t)}^{b(t)} U(\; ; t \;) f \, @W_i + V(\; ; t \;) \, @W_i g \, d \; ; \qquad (29)$$

where $V(x; t)$ is any velocity eld consistent with the the boundary velocities.

The advection property of the basis functions W_i gives

$$@W_i + V \, @W_i = 0 \qquad (30)$$

reducing (29) to

$$\frac{d}{dt} \int_{a(t)}^{b(t)} W_i(\;) U(\; ; t \;) d \;\; = \int_{a(t)}^{b(t)} W_i(\; ; t) f \, @U + @ \, (UV) g \, d \qquad (31)$$

where $V(x; t)$ is piecewise-linear.

The Lemma follows from (31) and the time invariance of (27).

We now turn to the proof of Theorem 2.

Proof of Theorem 2

The proof is again in six parts.

1. In the rst part we show that the velocity induced by (27) is the similarity velocity (12).

At time $t = t^0$ the initial condition $U(x; t^0)$ coincides with the L^2 projection $(t^0) \; N(\;^0)$ of the self-similar scaling solution (24), where $^0 = x{=}(t^0)$ and $@U$ is given by (25).

Substituting into (28) at $t = t^0$ we obtain

$$\int_{a(t^0)}^{b(t^0)} W_i(\;) f \quad t \quad {}^1@(\; N \;) + t \quad {}^1@(N \, V(\; ; t \;)) g d \;\; = 0 \qquad (32)$$

at $t = t^0$, where $= \; = (t^0)$, ${}^0_a = a(t){=}t$ and ${}^0_b = b(t){=}t$.

Changing the integration variable from to , equation (32) reduces to

$$\int_{{}^0_a}^{{}^0_b} W_i(\;) f \quad (t^0) \quad {}^1@(\; N(\;)) + \; @(N(\;) V(x; t^0)) g d \;\; = 0 \, ;$$

Let

$$Z(\;) = \quad (t^0) \quad {}^1 \; + V(x; t^0) \qquad (33)$$

so that (32) can be written

$$\int_{{}'_a}^{{}'_b} W_i(\ )\,\partial\{N(\ )Z(\ )\}d\ = 0; \qquad (i = 0;\ldots;N+1) \qquad (34)$$

at $t = t'$.

Expanding $Z(\ )$ as

$$Z(\ ) = \sum_{j=1}^{J} Z_j W_j(\ ;t)$$

equation (34) yields the matrix equation

$$B(N)Z$$

at $t = t^1$ to order h^2 using the conservation law (27) in the form

$$\int_{a(t^1)}^{b(t^1)} W_i(\ ^1)\, U(\ ; t\ ^1)\, d \quad = \int_{a(t^0)}^{b(t^0)} W_i(\ ^0)\, U(\ ; t\ ^0)\, d \qquad (38)$$

$(i = 0; \ldots; N+1)$, where $\ ^1 = \ = (t^1)\ ;\ \ ^0 = \ = (t^0)\ .$

Expanding the piecewise-linear functions $U(\hat{x}(t^1); t^1)$ in terms of the basis functions $W_i(\ ^1)$ as

$$U\ \hat{x}(t^1); t^1 \ = \sum_{j=1}^{J} U_j^1 W_j(\ ^1) \qquad (39)$$

where $\ ^1 = \hat{x}(t^1) = (t^1)$, equation (38) yields the matrix equation

$$\tilde{M}(\underline{\hat{X}}(t^1))\,\underline{U}^1 = \tilde{M}(\underline{\hat{X}}(t^0))\,\underline{U}^0 \qquad (40)$$

where $\underline{\hat{X}} = \{\hat{X}_i\}$, $\underline{U} = \{U_i\}$, and the $\tilde{M}(\underline{\hat{X}}(t))$ are standard piecewise-linear mass matrices, each depending on a vector of the nodal differences $\hat{X}_i(t)\ (= \hat{X}_i(t)\ \ \hat{X}_{i\ 1})(t)$.

By (37) the $\tilde{M}(\underline{\hat{X}}(t^1))$ and $\tilde{M}(\underline{X}(t^0))$ are identical to order h^2 apart from a factor $(t=t^0)$. It follows from equation (40) that $(t^1)\ \underline{U}^1 = (t^0)\ \underline{U}^0$ and hence from (39)

$$\frac{U(x(t^1); t^1)}{(t^1)} = \frac{U(x(t^0); t^0)}{(t^0)} + O(h^2) = N(\ ^0) + O(h^2) \qquad (41)$$

4. The fourth part of the proof is concerned with repetition of the first three parts over a further time interval $(t^1; t^2)$, where $t^2 = t^1 + h$, with $x^0; x^1$ replaced by $x^1; x^2$

We then deduce by the argument from equations (32) to (36) with the super x 0 replaced by 1 that

$$V(x;t^{1}) = \frac{x}{t^{1}} + O(h)$$

Further, by the argument from (37) to (41) with the super xes 0 and 1 replaced by 1 and 2, respectively,

$$^{2} = \frac{b(t^{2})}{(t^{2}}$$

equivalent to

$$V(\hat{x}(t);t) = \frac{\hat{x}(t)}{t};$$

as well as

$$= \quad {}^0; \qquad N(\quad) = N(\quad{}^0);$$

for all $t > t^0$. Thus $U(\hat{x}(t);t)$ coincides with the L_2 projection of the self-similar solution, and $V(\hat{x}(t);t)$ coincides with the similarity velocity at any time $t > t^0$, for all $a(t) \quad \hat{x}(t) \quad b(t)$.

This completes the proof.

Corollary: The moving nodes $\hat{X}_i$ satisfy

$$\hat{X}_i(t) \quad (t)$$

4.1 Calculation of a general velocity

4.1.1 The analytic case

Suppose that a first-order-in-time scale-invariant PDE for the function $u(x; t)$ is written in the form

$$u_t = Lu; \qquad (a(t) < x < b(t)) \tag{46}$$

where L is a purely spatial operator, with boundary conditions ensuring constant total mass (2).

Then from (8) with $x_1(t) = a(t)$ and $\hat{x}_2(t) = \hat{x}(t)$ and (46), the local conservation of mass principle (7) implies that

$$\int_{a(t)}^{\hat{x}(t)} Lu\, dx + [uv]_{a(t)}^{\hat{x}(t)} = 0 \tag{47}$$

which yields the velocity formula

$$v(\hat{x}(t); t) = \frac{(uv)|_{a(t);t} - \int_{a(t)}^{\hat{x}(t)} Lu\, dx}{u(\hat{x}(t); t)} \tag{48}$$

provided that $u(\hat{x}(t); t) \neq 0$. If $a(t^0)$ is an anchor point at which $v = 0$, the velocity reduces to

$$v(\hat{x}(t); t) = -\frac{\int_{a(t)}^{\hat{x}(t)} Lu\, dx}{u(\hat{x}(t); t)} \tag{49}$$

When u coincides with a self-similar solution the conservation equation (47) reverts to (8) which ensures, as in Theorem 1, that the velocity is the similarity velocity (5).

If the PDE takes the form

$$\partial_t u = Lu = \partial_x f[u]; \tag{50}$$

where $f[u]$ is a flux function depending on u and its space derivatives, the velocity (48) can be written

$$v(\hat{x}(t); t) = \frac{(uv)|_{a(t);t} - [f[u]]_{a(t)}^{\hat{x}(t)}}{u(\hat{x}(t); t)} \tag{51}$$

4.1.2 Piecewise linear L^2 projections

From now on we restrict the argument to proble verned by PDEs of the form (50) with zero net ux boundary co s ensuring that the total mass is constant in time.

De ne the weak form of (50) given by

$$\int_{a(t)}^{b(t)} W_i(\ )\, f\, @W_{i\ \ i} W_{it} ff$$

seeking a solution for b

$$= \frac{d\hat{x}(t^0)}{d\hat{x}(t)} \frac{d}{d\hat{x}(t^0)} \int_{a(t^0)}^{\hat{x}(t^0)} u(\ ; t^{\ 0}) d \quad = \frac{d\hat{x}(t^0)}{d\hat{x}(t)} u(\hat{x}; t^0) \qquad (59)$$

In the event of an initial condition that coincides with a self-similar solution $u(\hat{x}(t^0); t^0) = (\ t^0) \quad (\ ^0)$ so, since $\hat{x}$ is proportonal to t , it follows that $u(\hat{x}(t); t)$ reduces to the self-similar solution (3).

4.3.2 Piecewise-linear L^2 projections

The conservation of distributed mass principle (27) implies that

$$\int_{a(t)}^{b(t)} W_i(\ ) U(\ ; t\ ) d \quad = \int_{a(t^0)}^{b(t^0)} W_i(\ ^0) U(\ ; t^{\ 0}) d \qquad (60)$$

where $= \ = t \ , \ ^0 = \ = (t^0)$.

Expanding $U(\ ; t\ )$ and $U(\ ; t^{\ 0})$ as

$$U(\ ; t\ ) = \sum_j U_j(t) W_j(\ ); \qquad U(\ ; t^{\ 0}) = \sum_j U_j(t^0) W_j(\ ^0);$$

equation (60) yields the matrix equation

$$M(\underline{\hat{X}}(t)) \underline{U}(t) = \underline{C} = M(\underline{\hat{X}}(t^0)) \underline{U}(t^0)) \qquad (61)$$

where $\underline{U}(t) = \{U_i(t)\}$, $\underline{C} = \{C_i\}$, and $M(\hat{X}(t))$ is a standard mass matrix for piecewise-linears in terms of the nodal coordinates $\hat{X}_i(t)$.

When the initial condition coincides with a self-similar solution the components $U_i(t^0)$ are proportional to (t^0) and the $\hat{X}_i(t)$ are proportional to t , so equation (61) leads back to (24).

4.4 Summary

Using local and distributed conservation of mass we have constructed two procedures which propagate a scaling symmetry exactly (modulo a projection error in the L^2 case) for a PDE problem of the form (50) with zero net flux at the boundaries ensuring constant total mass.

In the analytic case the combination of steps (48), (58) and (59) yields a scale-invariant procedure possessing the S-property.

In the piecewise linear L^2 case the combination of steps (66), (58) and (60) gives a scale-invariant procedure possessing the S-property in the L^2 norm modulo the projection error (56).

5 Discrete algorithms

In this section, devoted to discrete methods, we continue to focus on ux-driven PDEs of the form (50), with zero net ux boundary conditions ensuring that the total mass is constant in time.

5.1 Semi-discrete velocities

5.1.1 A pointwise approach

From (49) with a zero net ux condition at $a(t)$, the semi-discrete velocity $v_(t)$ at position $\hat{x}(t)$ is given by

$$v(\hat{x}(t);t) = \frac{f\,[u]_{a(t)}^{\hat{x}(t)}}{u(\hat{x}(t);t)} \qquad (62)$$

Pointwise, a semi-discrete velocity may be de ned in terms of the semi-discrete solution $u_i(t)$ by sampling (62) at mesh points $\hat{x}_i(t)$, giving

$$v_i(t) = \frac{[f\,[u]]_{a(t)}^{\hat{x}_i(t)}}{u_i(t)} \qquad (63)$$

5.1.2 The piecewise-linear L^2 case

In the piecewise-linear L^2 case a semi-discrete velocity $V(x;t)$ may be determined in terms of $U(t)$ from (55) omitting the projection error (56), i.e.

$$\int_{a(t)}^{b(t)} (\partial W)\,(UV + f\,[U])\,d \quad = 0; \qquad (64)$$

which is already discrete in space.

Since $V(x;t)$ is piecewise-linear it can be expanded as

$$V(x;t) = \sum_{j=0}^{N+1} V_j(t)\,W_j(\ )$$

where $= x{=}t$. From (64),

$$\int_a^b (\partial W_i)\,U(\ ;t\ )$$

$$\int_a^b (\partial W_i) \qquad f\,[U(t2t7\ 7.9701-\ 109910.90984\ Tf\ 70.848\ 5\text{-}28(e)]T$$

5.2.2 A finite-element approach

In the finite element approximation the data representation is piecewise-linear, thus the functions $W(x; t^n)$, $V(x; t^n)$, $X(x; t^n)$, and $U(x; t^n)$ are all piecewise-linear.

At time $t = t^n$ equation (65) can be written as the matrix equation

$$B(U^n)\underline{V}^n = \underline{\hat{b}}^n \tag{69}$$

where $B(U^n)$ is the matrix with entries

$$\int_{a^n}^{\hat{b}^n} (@W_i(\ ))\, U(\ ; t^n)\, W_j(\ )\, d$$

where $= \ = (t^n)$, and $\underline{V}^n = \{V_i^n\}$, $\underline{\hat{b}}^n = \{\hat{b}_i^n\}$ in which $\hat{b}_i^n$ from (65) is

$$\hat{b}_i^n = \int_{a^n}^{\hat{b}^n} (@W_i^n)\, f\,[U(\ ; t^n)]\, d \tag{70}$$

We now consider discretisation in time.

5.3 Time stepping

When the nodal velocities are not similarity velocities, as in the case of general initial conditions, the time evolution (58) from t^0 to t is not exact. Nevertheless, (58) can still be used as one step of a first-order-in-time explicit scheme from t^n to t^{n+1} ($= t^n + h$), where h is the time step, having the property that it is exact in the case of similarity.

We therefore use the first-order scheme

$$x_i^{n+1} = \left(1 + {}^{1}h\,\frac{v_i^n}{\hat{x}_i^n}\right)\hat{x}_i^n \tag{71}$$

where x_i^n and v_i^n are the nodal positions and nodal velocities, respectively, having the property that the x_i^{n+1} are exact in the case of self-similarity. In the finite-element algorithm the nodal positions $\hat{X}_j^n$ are updated using (71) in the form

$$\hat{X}_i^{n+1} = \left(1 + {}^{1}h\,\frac{V_i^n}{\hat{X}_i^n}\right)\hat{X}_i^n \tag{72}$$

where V_i^n is the nodal velocity.

5.4 Solution retrieval

It remains to retrieve the approximate solutions u_i^{n+1} or U_i^{n+1} at the forward time $t = t^{n+1}$.

5.4.1 Finite-di erence solution retrieval

In the nite-di erence algorithm for mass-conserving problems of the form (67) equation (59) may be discretised over a time step from t^n to t^{n+1} as

$$u_i^{n+1} = \frac{\Delta x_i^n}{\Delta x_i^{n+1}} u_i^n$$

where Δx_i is a spatial di erence approximating dx. We use a centred where $M(\underline{\hat{X}}(t))$ is a mass matrix and $\underline{U} = \{U_i\}$.

In the case of similarity the exactness of the nodal positions and the invariance of the distributed mass-fractions (27) ensure that the L^2 projection property of the finite element solution is maintained in time.

5.5 Algorithms

We now summarise these algorithms.

5.5.1 The finite-difference algorithm

A scale-invariant finite-difference algorithm for scale-invariant mass-conserving PDE problems of the form (67) (where is the scaling power for x) with zero net flux boundary conditions is as follows:

Algorithm 1

Given nodes x_i^0 and nodal values u_i^0 sampled from an initial condition at time t

Algorithm 2

Given nodes X_i^0 and U^0, the L^2 projection of the initial condition $u(x;t^0)$, at time t^0, then at each time t^n t^0,

1. Obtain the piecewise-linear velocity V^n from (69)
2. Advance the nodes X_i^n to X_i^{n+1} using (72)
3. Retrieve U_i^{n+1} using (75)

The algorithm is scale-invariant with the same scaling invariants as the PDE problem and possesses the S-property in the L^2 norm, modulo the projection error (56). A small enough time step is required for the time step to be stable.

Boundary conditions on U can be imposed in step 3 but care is required that the family of test functions $W_i(x;t)$ remains a partition of unity (see e.g. [19, 27]).

It is known that the matrix in the reduced form of equation (69) is awkward to invert numerically since the entries oscillate in sign and the matrix $B(U^n)$ is poorly conditioned

A similar algorithm appears in the literature [1, 2, 3, 4, 9, 27] although the time step there is always the explicit Euler scheme rather than that of (71) and the velocity is obtained indirectly through a velocity potential rather than from (69), avoiding the ill-conditioning of the matrix $B(U^n)$.

6 Numerical illustrations

6.1 A nonlinear PDE problem

We illustrate the behaviour of the errors in the nite element and nite di erence algorithms for the example of a nonlinear di usion problem governed by the porous medium equation PDE

$$u_t = \partial\{u^2(\partial u)\} = \partial\{u\partial(u^2/2)\}; \qquad (a(t) < x < b(t)); \qquad (76)$$

(in which $f[u] = u^2\partial u$ and $p(u) = \partial(u^2/2)$), where $u = 0$ on the free boundaries $a(t); b(t)$ (so zero net mass ux), which is mass-conserving and scale-invariant with $= 1/4$.

The initial time is $t^0 = 1$ and the initial domain is $(\ 1 < x < \ 1)$. We consider the two initial conditions,

$$(a) \qquad u(x;1) = \ ^1$$

6.2.2 Case (b)

In the more general case (b), with N ranging from 10 to 80 the errors in both the relative l^1 norm of the solution and the relative boundary position when compared the solution for 160 nodes (taken to be a very accurate solution) are shown in Table 1. The time step taken to avoid instability is $h = 1/N^2$.

N	t	Relative error $e_N(u)$	Relative error $e_N(x)$
10	0.01	1.2×10^{-2}	2.6×10^{-3}
20	0.0025	5.5×10^{-3}	9.0×10^{-4}
40	0.000625	2.4×10^{-3}	3.0×10^{-4}
80	0.00015625	8.7×10^{-4}	7.3×10^{-5}

Table 1: Relative errors $e_N(u)$ in the l^1 norm of u and $e_N(x)$ in the boundary position, at $t = 2$, when compared with the solution for 160 nodes (taken as a very accurate solution) for the PME (76) when the initial condition is (78).

6.3 Finite elements

In the finite-element algorithm of section 5.5.2 the velocity V^n is given by (69) where in this case b_i^n is defined from (70) by

$$b_i^n = \int_{a^n}^{b^n} (\partial W_i)^n (U^2 U_\partial)^n \, d\partial \, ; \tag{79}$$

omitting the projection error (56). Since the functions W_i; U are piecewise-linear, the integrand is piecewise quadratic and the integration in (79) can be carried out exactly using a composite Simpson's Rule.

(chosen to ensure stability) the relative L^2 norm of the solution is approximately $0{:}008$ and the relative l^1 norm of the boundary $0{:}0004$. If the exact velocity is used instead of the velocity computed from (69) the errors reduce to the level of rounding error.

Comparative results are given only for the initial condition case (a) where, with N ranging from 10 to 80, errors are shown in Table 2. The time step taken to avoid instability is $h = 1{=}N^2$.

N	h	Relative error e_N	Relative error X_N
10	0:01	$1{:}3 \quad 10^{2}$	$1{:}8 \quad 10^{3}$
20	0:0025	$8{:}0 \quad 10^{3}$	$5{:}0 \quad 10^{4}$
40	0:000625	$4{:}3 \quad 10^{3}$	$2{:}5 \quad 10^{4}$
80	0:00015625	$2{:}2 \quad 10^{3}$	$8{:}7 \quad 10^{5}$
160	0:0000390625	$1{:}1 \quad 10^{3}$	$3{:}1 \quad 10^{5}$

Table 2: Table of relative errors e_N in the L^2 norm of U, and $e_N(X)$ in the absolute value of the boundary position, at $= 2$, in the case of initial data (a) for the nite-element algorithm.

7 Summary

In this paper we have studied the invariance of scaling symmetry in the evolution of one-dimensional time-dependent scale-invariant mass-conserving PDE problems. It was shown that, under local conservation of mass, initial conditions that coincide with self-similar solutions are propagated exactly in time, while under distributed conservation of mass, piecewise linear L^2 projections of initial conditions that coincide with the piecewise linear L^2 projections of self-similar solutions are propagated as piecewise linear L^2 projections exactly in time.

The steps in the proof were then adapted for general initial conditions in the case of rst-order-in-time ux-driven problems, with the aim of obtaining a general procedure that possesses the S-property, i.e. exact propagation of a self-similar solution or its L^2 projection. A deformation velocity was constructed and used to move the nodes via a symmetry-preserving scheme. The solution was then post-processed algebraically from the Lagrangian form of the conservation.

A nite-di erence algorithm based on this procedure was con-

l^1 norm when the velocity is calculated by a special interpolation. A piecewise-linear nite-element algorithm was also described possessing the S-property in the L^2 norm, but subject to a projection error. Numerical illustrations verifying these results were shown for a nonlinear porous medium equation problem with a constant total mass, exhibiting results in accordance with the theory and showing the levels of accuracy in the propagation of relative errors for a non self-similar initial condition.

The S-property can be regarded as a yardstick for con dence in numerical schemes in the case of nonlinear scale-invariant problems, similar to the way in which standard schemes on xed grids for linear problems based on Taylor series expansions are constructed so as to be exact for polynomial solutions of given degree.

One outcome of this paper is the scale-invariant nite-di erence Algorithm 1, for mass-conserving PDE problems of the form (67), possessing the S-property in the l^1 norm, when the initial condition is sampled from a self-similar solution at the nodes and the velocity is interpolated in a particular way. (The corresponding scale-invariant nite-di erence Algorithm 2 does not achieve the same accuracy (in the L^2 norm) due to a projection error.) Comparisons with self-similar solutions are a favourite testing ground for numerical schemes: in this paper Algorithm 1 propagates the solution at the nodes exactly, thus preserving a discrete scaling symmetry.

oft.desc.54sri117J Td [()-4Nlems,

[3] M.J.Baines, M.E.Hubbard, P.K.Jimack and R. Mahmood, A Moving-Mesh Finite Element Method and its Application to the Numerical Solution of Phase-Change Problems. Communications in Computational Physics, 6 (2009), 595-62.

[4] M.J.Baines, M.E.Hubbard and P.K.Jimack. Velocity-based Moving-mesh Methods for Nonlinear Partial Di erential Equations. CiCP, 10 (2011), 509-576.

[5] M.J.Baines, T.E.Lee, S.Langdon, and M. J. Tindall A moving mesh approach for modelling avascular tumour growth.Applied Numerical Mathematics, 72, (2013), 99114.

[6] Explicit time stepping for moving meshes, J. Math Study, 48 (2015), 93-105.

[7] K.W.Blake, Moving mesh methods for nonlinear partial di erential equations, PhD thesis, Department of Mathematics, University of Reading, UK, 2001.

[8] G.I.Barenblatt, Intermediate Asymptotics (1996), Scaling (2003), CUP.

[9] N.Bird, Fourth order nonlinear di usion. PhD thesis, University of Reading, UK (2015).

[10] B.Bonan, M.J.Baines, N.K.Nichols and D.Partridge, A moving-point approach to model shallow ice sheets: a study case with radially symmetrical ice sheets, The Cryosphere, 10, 1-14, doi: 10.5194/tc-10-1-2016, 2016.

[11] C.J. Budd, G.J. Collins, W.Z. Huang and R.D. Russell, Self-similar numerical solutions of the porous medium equation using moving mesh methods,Phil. Trans. Roy. Soc. A 357,1754 (1999).

[12] C.J.Budd and M.D.Piggott. The Geometric Integration of Scale Invariant Ordinary and Partial Di erential Equations. J. Comp. Appl. Math.,128 (2001), 399-422

[13] C. J. Budd, W. Huang and R. D. Russell, Adaptivity with moving grids, Acta Numerica, Volume 18, pp 111-241 2009.

[14] C.J.Budd and M.D.Piggott, Geometric integration and its applications. Foundations of Computational Mathematics, Handbook of Numerical Analysis XI, ed. P.G.Ciarlet and F.Cucker, Elsevier,pp. 35{139, 2003.

[15] C.J.Budd and J.F.Williams, Moving mesh generation using the parabolic MongeAmpre equation. SIAM Journal on Scienti c Computing, 31 (5), pp. 3438-3465, 2009.

[16] W.Cao, W.Huang, and R. D. Russell. A Moving Mesh Method Based on the Geometric Conservation Law,SIAM Journal of Scienti c Computation ,24 (2002), 118-142.

[17] W.Cao, W.Huang, and R.Russell. Approaches for Generating Moving Adaptive Meshes : Location versus Velocity,Applied Numerical Mathematics,47 (2003), 121-138.

[18] N.N.Carlson and K Miller, Design and application of a gradient-weighted moving nite element code, I: in one dimension, II: in

[28] B.V.Wells, A moving mesh finite element method for the numerical solution of partial differential equations and systems, PhD thesis, Department of Mathematics, University of Reading, UK, 2005.