Vectorial Variational Problems in L^∞ and Applications to Data Assimilation

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Declaration

This thesis describes the work undertaken at the Department of Mathematics and Statistics of the University of Reading, in fulfillment of the requirements for the degree of Doctor of Philosophy.

The results in Chapter

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Dedication

This thesis is dedicated to my family, friends and everyone else who believed in me.

Abstract

This thesis is a collection of published, submitted and developing papers. Each paper is presented as a chapter of this thesis, in each paper we advance the field of vectorial Calculus of Variations in L^∞. This new progress includes constrained problems, such as the constraint of the Navier-Stokes equations studied in Chapter 2. Additionally the combination of con-

Chapter 4 provides part of the corresponding developing preprint, joint work with Nikos Katzourakis. We consider the problem of minimising the L^∞ norm of a function of the Hessian over a class of maps, subject to a mass constraint involving the L^∞ norm of a function of the gradient and the map itself. We assume zeroth and first order Dirichlet boundary data, corresponding to the “hinged” and the “clamped” cases. By employing the method of L^p approximations, we establish the existence of a special L^∞ minimiser, which solves a divergence PDE system with measure coefficients as parameters. This is a counterpart of the Aronsson-Euler system corresponding to this constrained variational problem.

Contents

Chapter 1

Background and Motivations

In this chapter we review several background concepts that will be assumed throughout the thesis.

1.1 Sobolev spaces

During the early 20th century, there was a substantial development in the theory of differential equations. Specifically, most partial differential equations (PDEs), either linear or nonlinear, cannot be "solved" in the classical sense of writing an explicit formula representing a solution as differentiable as the equation would suggest. This was the beginning of analytic PDE theory, abandoning to a large extent the search for new calculus techniques to represent formulas of solutions.

A related problem, which arose almost simultaneously, is that in general we have to extend our search for solutions to functions of lower regularity. In fact, for the vast majority of PDEs, it is impossible to prove existence of a solution as differentiable as the terms within the equation. Let alone find an explicit formula to describe the solution in terms of elementary functions.

The modern approach to PDEs consists of searching for appropriately defined generalised solutions. Firstly, we ascertain existence, given a specific domain and certain prescribed boundary/initial conditions. The relevant vector spaces to initiate these questions are the Sobolev spaces. Before we can introduce their definition, we must discuss what it means for a function to have a derivative in the weak sense.

Definition 1.1.1. Let $\Omega \subset \mathbb{R}^n$, with $n \in \mathbb{N}$. Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \ldots + \alpha_n = k$. We say that $v \in L^1_{\text{loc}}(\Omega)$ is the α^{th}-weak partial derivative of u, written as

$$\partial^\alpha u = v,$$

provided

$$\int_\Omega u\,\partial^\alpha\varphi\,\mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|}\int_\Omega v\varphi\,\mathrm{d}\mathcal{L}^n,$$

for all test functions $\varphi \in C_c^\infty(\Omega)$. Additionally, our integration is with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$.

Theorem 1.1.2 (Uniqueness of weak derivatives)**.** *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof of Theorem 1.1.2. Let $v, w \in L^1_{\text{loc}}(\Omega)$ such that:

$$\int_\Omega u\,\partial^\alpha\varphi\,\mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|}\int_\Omega v\varphi\,\mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|}\int_\Omega w\varphi\,\mathrm{d}\mathcal{L}^n \quad \forall\varphi \in C_c^\infty(\Omega).$$

Then,

$$\int_\Omega v\varphi\,\mathrm{d}\mathcal{L}^n = \int_\Omega w\varphi\,\mathrm{d}\mathcal{L}^n.$$

Consequently,

$$\int_\Omega (v - w)\varphi\,\mathrm{d}\mathcal{L}^n = 0.$$

Thus, $v - w = 0$ a.e and $v = w$ a.e. Hence, we have uniqueness up to a set of measure zero. □

Let us consider some elementary examples of functions possessing weak derivatives.

Example 1.1.3. *If $u \in C^k(\Omega)$ then its classical partial derivatives are indeed weak partial derivatives for $|\alpha| \leq k$.*

Example 1.1.4. Suppose $n = 1$ with $\Omega = (0, 3)$ and

$$u(x) = \begin{cases} 4x - 6 & \text{if } 0 < x \leq 2, \\ 2 & \text{if } 2 < x < 3. \end{cases}$$

Let

$$v(x) = \begin{cases} 4 & \text{if } 0 < x \leq 2, \\ 0 & \text{if } 2 < x < 3. \end{cases}$$

We intend to show that $u' = v$ in the weak sense. Choose any $\varphi \in C_c^\infty(\Omega)$, we must show that

$$\int_0^3 u\varphi' \, dL = -\int_0^3 v\varphi \, dL.$$

Using additivity and integration by parts, we easily compute

$$\begin{aligned}\int_0^3 u\varphi' \, dL = \int_0^2 u\varphi' \, dL + \int_2^3 u\varphi' \, dL &= \int_0^2 (4x-6)\varphi'(x) \, dL + \int_2^3 2\varphi'(x) \, dL \\ &= \Big[\varphi(x)(4x-6)\Big]_0^2 - \int_0^2 4\varphi(x) \, dL + 2\Big[\varphi(x)\Big]_2^3 \\ &= 2\varphi(2) - \int_0^2 4\varphi(x) \, dL - 2\varphi(2) \\ &= -\int_0^2 4\varphi(x) \, dL = -\int_0^3 v\varphi \, dL,\end{aligned}$$

as required.

Example 1.1.5. The discontinuous function $f : (0,2) \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \le 1, \\ 1 & \text{if } 1 < x < 2, \end{cases}$$

is not weakly differentiable. For any $\varphi \in C_c^\infty(0,2)$, we compute

$$\int_0^2 f\varphi' \, dL = \int_0^1 (0)\varphi' \, dL + \int_1^2 \varphi' \, dL = \Big[\varphi(x)\Big]_1^2 = \varphi(2) - \varphi(1) = -\varphi(1).$$

Consequently, the weak derivative $g = f'$ must satisfy

$$\int_0^2 g\varphi \, dL = \varphi(1),$$

for any $\varphi \in C_c^\infty(0,2)$. Suppose for contradiction and assume there exists a $g \in L^1_{loc}(0,2)$ that satisfies the above. Suppose we have test functions with $\varphi(1) = 0$, then $g\varphi = 0$ a.e. for any $\varphi \in C_c^\infty(0,2)$, so $g = 0$ a.e. dL. This must also hold for test functions l3 or test functions l71sx

Definition 1.1.6. Let $\Omega \subseteq \mathbb{R}^n$ be open and $p \in [1, \infty]$, then we define the Sobolev spaces as follows:

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \text{ for } |\alpha| \leq k \right\},$$

where the derivatives are taken in the weak sense. If $u \in W^{k,p}(\Omega)$ we define its norm to be:

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}, \; 1 \leq p < \infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Remark 1.1.7. An alternative choice of norm is given as follows

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \; 1 \leq p < \infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

These norms are equivalent to the previous choices, in the sense they generate the same topology. However, throughout this thesis we will employ the norms used in Definition 1.1.6, since they significantly simplify our calculations.

Definition 1.1.9. We denote by

$$W_0^{k,p}(\Omega)$$

the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Consequently, $u \quad W_0^{k,p}(\Omega)$ if and only if there exists functions $u_m \quad C_c^\infty(\Omega)$ with $u_m \longrightarrow u$ in $W^{k,p}(\Omega)$. We see the closed subspace $W_0^{k,p}(\Omega)$ as functions within $W^{k,p}(\Omega)$ that exhibit the additional property

$$D^\alpha u = 0 \text{ on } \partial\Omega \text{ for all } |\alpha| \leq k - 1.$$

We must introduce the Trace operator for this expression to make sense, otherwise we have a problem. In the classical setting of $u \quad C(\overline{\Omega})$, u has boundary values in the usual sense. However, there is a substantial issue when we encounter functions in a Sobolev space that are not continuous, or only defined a.e. As $\partial\Omega$ is an n-dimensional Lebesgue null set, there is no clear interpretation for the meaning of "u restricted to $\partial\Omega$".

Theorem 1.1.10 (Trace Theorem). *Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that
(i) $Tu = u|_{\partial\Omega}$ if $u \quad W^{1,p}(\Omega) \cap C(\overline{\Omega})$
and
(ii)

$$Tu \;_{L^p(\partial\Omega)} \leq C_p($$

Theorem 1.1.13 (Poincaré's inequality). *Suppose that* $1 \leq p < \infty$ *and* Ω *is a bounded open set. Then there exits a constant* C *(depending on* Ω *and* p*) such that*

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

for any $u \in W_0^{k,p}(\Omega)$.

Theorem 1.1.14 (Poincaré Wirtinger inequality). *Assume that* $1 \leq p < \infty$ *and* Ω *is a bounded, connected open set with Lipschitz boundary. Then there exits a constant* C*, depending only on* n, p *and* Ω*, such that*

$$\left\| u - \fint_\Omega u \, \mathrm{d}L^n \right\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

for each function $u \in W^{1,p}(\Omega)$.

These results are highly significant, as they allow us to bound the norm of a function, using only the norm of its gradient.

Another useful bound is the Morrey estimate.

Theorem 1.1.15 (Morrey's inequality). *Assume* $n < p \leq \infty$. *Then there exists a constant* C*, depending only on* p *and* n *such that*

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in C^1(\Omega)$*, where*

$$\gamma := 1 - \frac{n}{p}.$$

Thus, if $u \in W^{1,p}(\Omega)$, then u is in fact Hölder continuous of exponent γ. This embedding can actually be made compact. The notion of compact embeddings is used throughout linear and nonlinear functional analysis, it is of the utmost importance within the realm of differential equations.

The second condition means that if $(u_k)_{k=1}^{\infty}$ is a sequence in X with $\sup_k \|u_k\|_X < \infty$, then some subsequence $(u_{k_j})_{j=1}^{\infty} \subset (u_k)_{k=1}^{\infty}$ converges in Y to some limit u:

$$\lim_{j \to \infty} \|u_{k_j} - u\|_Y = 0.$$

Theorem 1.1.17 (Rellich-Kondrachov). *Suppose that Ω is bounded with C^1 boundary. Then, for $p > n$, the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, i.e $W^{1,p}(\Omega) \Subset C(\overline{\Omega})$.*

This result allows us to prove the existence of a uniformly convergent subsequnce, through a $W^{1,p}(\Omega)$ norm bound.

We refrain from discussing this topic any further, as there is a great deal of accessible literature on Sobolev spaces. The reader should consult [1, 21, 42] for a comprehensive exploration. These references also contain the proofs of the results quoted in this section.

1.2 Variational problems

The study of minimisation problems has been undertaken by a variety of mathematicians for diverse intentions. There has been a substantial focus in understanding the relationship between minimality conditions of a functional and the appreciation of PDEs. As there is no general theory for all PDEs, we must exploit the PDE structure where possible. An important collection of such problems are when we can view minimality through a variational approach, this is a corner stone of Calculus of Variations. For instance, suppose we have some potentially nonlinear PDE with the form

$$A[u] = 0, \tag{1.1}$$

where $A[u]$ is a given differential operator and u is the unknown. Equation (1.1) can be characterised as the minimiser of an appropriate energy functional $E[u]$, such that

$$E'[u] = A[u]. \tag{1.2}$$

The practicality of this method is that now we can prove existence of extrema for the energy functional $E[\cdot]$ and consequently the solution of (1.1). This approach provides a much more tractable method than the direct consideration of problem (1.1).

In this thesis we will not explore the Calculus of Variations as a means to study nonlinear PDEs. Neither will we pursue classical problems from the well established field of minimising integral functionals. However, a strong foundation in the study of integral

Calculus of Variations is necessary to examine the problems we face in this thesis. We will recap some of these fundamental ideas in a subsequent subsection. Our interest lie at the heart of minimising constrained vectorial supremal functionals and finding the necessary conditions these minimisers must satisfy. This is the field of vectorial Calculus of Variations in L^∞ and will be the topic of this thesis.

1.3 Literature review

Due to the extensive nature of this branch of mathematics, it is rather challenging to include and produce a completely comprehensive literature review. A substantial quantity of the appropriate literature is reviewed in the introductions of the papers that are presented in this thesis. However, we will briefly outline the most important previous considerations that have inspired the new progress in this thesis.

1.4 Integral Calculus of Variations

We will now recap some rudimentary details, essentially textbook material of integral Calculus of Variations. See [36, 42, 90] for further details.

Let X be a vector space and $E : X \longrightarrow \mathbb{R}$, a real valued continuously differentiable integral functional. Our first natural question of interest concerns the existence of minimisers, this can be investigated through the well established direct method in the Calculus of Variations.

Theorem 1.4.1 (The Direct Method in the Calculus of Variations). *Suppose X is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subseteq X$ be a weakly closed subset of X. Suppose $E : M \longrightarrow \mathbb{R} \cup \{+\infty\}$ is coercive and sequentially weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:*

- $E(u) \longrightarrow \infty$ *as* $\|u\| \longrightarrow \infty, u \in X$.
-

Remark 1.4.2. Notice that the direct method is not only restricted to proving the existence of integral functionals.

Once we have established existence of solutions, our next point of inquisition is determining necessary conditions that these minima or maxima must satisfy. These necessary conditions will be in form of PDEs. For vectorial problems these necessary conditions will manifest as a system of PDEs.

If E has local extrema (local minima or maxima) at a point $x_0 \in X$, then

$$E'(x_0) = 0.$$

Under further regularity of E, specifically a C^2 functional, we can deduce that

$$E''(x_0) \geq 0,$$

if x_0 is a local minimum.

Figure 1.1: Local Extrema

We can intuitively visualise lower dimensional problems like the figure above.

Similarly,

$$E''(x_0) \leq 0.$$

if x_0 is a local maximum.

Now let E be a C^1 real valued functional over the bounded open set $\Omega \subset \mathbb{R}^n$. Then for some $u_0 \in \Omega$ and $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$ and for some sufficiently small $\varepsilon_0 > 0$ the function $E(u_0 + \varepsilon\varphi)$ is also continuously differentiable, when $|\varepsilon| < \varepsilon_0$. The first variation is then defined as the derivative of E at point u_0 along the direction of φ for $\varepsilon = 0$. When u_0 is a critical point we conclude that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E(u_0 + \varepsilon\varphi) = 0. \tag{1.3}$$

We can visualise an elementary situation as follows.

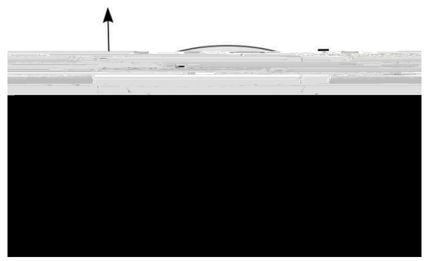

Figure 1.2: Directional Derivative

Consider the functional E defined as above, where L $C^1(\Omega\times\mathrm{R}^N\times\mathrm{R}^{N\times n})$ is the Lagrangian

$$\left.\frac{\mathrm{d}}{\mathrm{d}}\right|_{\varepsilon=0} E(u_0 + \varepsilon\varphi) = \int_\Omega \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} L =$$

This can be rewritten in the following index notation

$$\sum_i \mathrm{D}_i \Big(L_{P_{ij}}(\cdot, u, \mathrm{D}u) \Big) + L_{\eta_j}(\cdot, u, \mathrm{D}u) = 0, \quad j = 1, \dots, N \text{ in } \Omega.$$

For example, consider the p-Dirichlet integral functional

$$E_p(u) := \int_\Omega |\mathrm{D}u|^p \, \mathrm{d}\mathcal{L}^n, \quad u \in W^{1,p}(\Omega; \mathbb{R}^N).$$

The corresponding Euler-Lagrange equations are given by the renowned p-Laplacian

$$\Delta_p u := \mathrm{Div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0 \text{ in } \Omega. \tag{1.5}$$

Note that for any $P \in \mathbb{R}^{N\times n}$, the notation $|P|$ denotes its Euclidean (Frobenius) norm:

$$|P| = \Bigg(\sum_{i=1}^{N} \sum_{j=1}^{N} (P_{ij})^2 \Bigg)^{\frac{1}{2}}.$$

1.5 Calculus of Variations in L^∞

Calculus of Variations in L^∞ has a reasonably short history, with the first developments being made by Gunnar Arronsson in the 1960s. He considered L^∞ variational problems in the scalar case [4]-[9]. The evolution of vectorial problems did not begin till much later, with Nikos Katzourakis initiating its growth in the 2010s. In this thesis we will study constrained vectorial problems, only a very small quantity of previous literature existed at the commencement of this project [65, 66]. There has already been subst0 0 rgo3-43552 Tf

As mentioned, in the classical setting of integral functionals, where

$$E(u) = \int_{\Omega} L(\cdot, u, Du)\, d\mathcal{L}^n,$$

A standard difficulty, when dealing with these types of problems, is the complexity of the PDE system given in (1.8). As previously mentioned, these systems do not possess some motivational ideas into why we consider the problem, specifically what is variational data assimilation and how this could support weather prediction. Subsequently, we can pose our research question as a constrained supremal minimisation problem. Once we have established the theoretical foundations, introduced appropriate vector spaces and devised an admissible class of functions, we start to inspect some fundamental questions. The first is clearly existence of minimisers, indeed our initial theorem in this chapter. Once existence has been ascertained, we can pursue PDE conditions that these minimisers satisfy, this is the contents of our second and third theorems. It turns out that our L^∞ minimisers solve a divergence PDE system involving measure coefficients. This is a divergence form counterpart of the corresponding non-divergence Aronsson-Euler systems that have been previously mentioned. Given that measures are present in our equation, we also investigate some of their properties in our third result.

Chapter 3 presents the joint paper with Nikos Katzourakis. This paper was accepted to the journal Advances in Calculus of Variations in March 2023. Here we investigate a more abstract problem: The minimisation of a general quasiconvex first order L^∞ functional that is constrained by two quantities. Specifically, the sublevel set of another supremal functional and the zero set of a nonlinear operator.

The chapter begins as before, by assembling an outline of the problem. Given the anatomy of the research, the same natural questions must be examined. Thus, our first result provides existence of minimisers through utilisation of the direct method, subsequently constructing the connection between minimisers of the L^p and L^∞ problem. Our next step involves exploiting the generalised Kuhn-Tucker theory to discover equations that the constrained minimisers satisfy. The final result is rather challenging to prove, we can not pass to the limit as easily as we did the previous chapter. The issue is we have products that converge in a weak sense and we can not use duality to overcome it. Due to the specificity of the problem, we can bypass the comprehensive machinery of Young measures and employ the theory of Hutchinson's measure function pairs. This allows us to pass to the limit and produce the desired PDE condition. However, this still requires a substantial body of work. Throughout this project, we must impose ever increasing restrictions upon the nonlinear operator Q. The final section illustrates the variety of problems still available to us, despite the initial limitations of assumptions in our previous results. For instance, examples of potential operators include those expressing pointwise, unilateral, integral isoperimetric, elliptic quasilinear differential, Jacobian and null Lagrangian constraints.

In Chapter 4 we illustrate a component of the developing preprint paper, joint work with Nikos Katzourakis. The complete paper was submitted to the journal Proceedings of the Royal Society of Edinburgh, in March 2023. In this final piece of research, we examine an extension of the previously existing first order problem [67]. Specifically, allowing the functional in question to depend on Hessians as opposed to gradients. Additionally, the constraint depends on the gradient and the function itself. Following an analogous line of inquiry, we determine PDE conditions for constrained minimisers, utilising our knowledge of the approximating problems.

In Chapter 5 we discuss the conclusions and future work.

Appendix A provides the derivation of a bound stated in Chapter 2.

Appendix B contains a simple computational proof of the modified Hölder inequality utilised in Chapter 2.

Chapter 2

Vectorial Variational Problems in L^∞ Constrained by the Navier Stokes Equations

2.1 Introduction and main results

Let Ω $\mathbb{R}^n$ be an open boundd [(R)]TJ/F44 729 11.950TJ/F44 729 5J84 7cs

A problem of interest in the geosciences, in particular in data assimilation for atmospheric flows in relation to weather forecasting (see e.g. [22, 23, 24]), can be formulated as follows: find solutions (u, p) to (2.1) such that, in an appropriate sense,

$$\begin{cases} y \approx 0, \\ \mathrm{Q}(\cdot, \cdot, u, \quad u, \mathrm{p}) - q \approx 0, \end{cases} \tag{2.3}$$

where $q : \Omega_T \longrightarrow \mathbb{R}^N$ is a vector of given measurable “data” arising from some specific measurements, taken through the “observation operator” Q of (2.2). In (2.1) and (2.3), y represents an error in the measurements which forces the Navier-Stokes equations to be satisfied only approximately for solenoidal (divergence-free) vector fields. Namely, we are looking for solutions to (2.1) such that simultaneously the error y vanishes, and also the measurements q match the prediction of the solution (u, p) through the observation operator (2.2). In application, Q is typically some component (e.g. linear projection or nonlinear submersion) of the atmospheric flow that we can observe. Unfortunately, the data fitting problem (2.3) is severely ill-posed; an exact solution may well not exist, and even if it does, it may not be unique.

In this paper, inspired by the methodology of data assimilation, especially variational data assimilation in continuous time (for relevant works we refer e.g. to [18, 25, 39, 47, 48, 75, 77, 86]), we seek to minimise the misfit functional

$$(u, \mathrm{p}, y) \rightarrow (1 - \lambda)\big\| \mathrm{Q}(\cdot, \cdot, u, \quad u, \mathrm{p}) - q \big\| + \lambda \big\| y \big\|$$

over all (u, p, y) admise[3]

argument, the L^∞ norm is not additive but only sub-additive. Further, one would also need estimates for (2.1) in appropriate subspaces of L^∞ for weakly differentiable functions, which, to the best of our knowledge, do not exist even for linear strongly elliptic systems (see e.g. [52]). Even then, if one somehow solves the L^∞ minimisation problem (by using, for instance, the direct method of the Calculus of Variations as in [36], under the appropriate quasiconvexity assumptions for $|Q - q| + |y|$ as in [17]), the analogue of the Euler-Lagrange equations for the L^∞ problem cannot be derived directly by perturbation/sensitivity methods due to the lack of smoothness of the L^∞ norm.

In this paper, to overcome the difficulties described above, we follow the methodology of the relatively new field of Calculus of Variations in L^∞ (see e.g. [34, 61] for a general introduction to the scalar-valued theory), and in particular the ideas from [64, 65, 66, 68] involving higher order and vectorial problems, as well as problems involving PDE-constraints, which have only recently started being investigated. To this end, we follow the approach of solving the desired L^∞ variational problem by solving respective approximating L^p variational problems for all p, and obtain appropriate compactness estimates which allow to pass to the limit as $p \to \infty$. The case of finite $p > 2$ studied herein is also of independent interest, especially for numerical discretisation schemes in L^∞ (see e.g. [70, 71]), but in this paper we treat it mostly as an approximation device to solve efficiently the L^∞ problem. The idea of this approach is based on the observation that, for a fixed essentially bounded

Then, for any $p \in (1,\infty)$, we define the L^p misfit $\mathrm{E}_p : \mathfrak{X}^p(\Omega_T) \longrightarrow \mathbb{R}$ by setting

$$\mathrm{E}_p\big(u,\mathrm{p},y\big) := (1-\lambda)\Big\|\mathrm{K}(\cdot,u,\mathrm{D}u,\mathrm{p})\Big\|_{\dot{L}^p(\Omega_T)} + \lambda\|y\|_{\dot{L}^p(\Omega_T)}. \tag{2.6}$$

We note that in (2.6) and subsequently, the dotted $\dot{L}^p$ quantities are regularisations of the respective norms at the origin, obtained by regularising the Euclidean norm in the respective target space:

$$\|h\|_{\dot{L}^p(\Omega_T)} := \Big\||h|_{(p)}\Big\|_{L^p(\Omega_T)}, \qquad |\cdot|_{(p)} := \sqrt{|\cdot|^2 + p^{-2}}. \tag{2.7}$$

Further, since we will only be dealing with finite measures, we will always be using the normalised L^p norms in which we replace the integral over the domain with the respective average, for example for $L^p(\Omega_T)$ with the $(n+1)$-Lebesgue measure, the norm will be

$$\|h\|_{L^p(\Omega_T)} := \bigg(\,⨍_{\Omega_T} |h|^p \,\mathrm{d}\mathcal{L}^{n+1}\bigg)^{1/p}.$$

The admissible minimisation class $\mathfrak{X}^p(\Omega_T)$ over which E_p is considered, is defined as follows:

$$\mathfrak{X}^p(\Omega_T) := \Big\{(u,\mathrm{p},y) \in \mathcal{W}^p(\Omega_T) \,:\, (u,\mathrm{p},y) \text{ satisfies weakly (2.1)}\Big\}, \tag{2.8}$$

where

$$\mathcal{W}^p(\Omega_T) := W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}(\Omega_T) \times L^p(\Omega_T;\mathbb{R}^n). \tag{2.9}$$

The rather complicated functional spaces appearing in (2.9) are defined as follows. The space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ consists of solenoidal maps which are $W^{2,p}$ in space and $W^{1,p}$ in time, and also laterally vanishing on $\partial\Omega \times (0,T)$:

$$\left\{ W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) := L^p\Big((0,T); W^{2,p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big) \dots); \right.$$

The associated norms in these spaces are the expected ones, namely

$$\begin{cases} \|v\|_{W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)} := \|v\|_{L^p(\Omega_T)} + \|v\|_{L^p(\Omega_T)} + \|\mathrm{D}^2 v\|_{L^p(\Omega_T)}, \\ \|g\|_{W^{1,0;p}(\Omega_T)} := \|g\|_{L^p(\Omega_T)} + \|\mathrm{D}g\|_{L^p(\Omega_T)}. \end{cases} \tag{2.12}$$

Note also that the divergence-free condition for u in (2.1) has now been incorporated in the functional space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)$. Finally, the L^∞ misfit $\mathrm{E}_{\mathrm{mis}(}$

Assumption (2.15), albeit restrictive, is compatible with situations of interest in weather forecasting (see e.g. [22, 23, 24]). Our first main result concerns the existence of E_p-minimisers in $\mathfrak{X}^p(\Omega_T)$, the existence of E_∞-minimisers in $\mathfrak{X}^\infty(\Omega_T)$ and the approximability of the latter by the former as $p \to \infty$.

Theorem 2.1.1 (E_∞-minimisers, E_p-minimisers & convergence as $p \to \infty$). *Suppose that (2.5) and (2.15) hold true. Then, for any $p \in (n+2, \infty]$, the functional E_p (given by (2.6) for $p < \infty$ and by (2.13) for $p = \infty$) has a constrained minimiser (u_p, p_p, y_p) in the admissible class $\mathfrak{X}^p(\Omega_T)$:*

$$\mathrm{E}_p\big(u_p, \mathrm{p}_p, y_p\big) = \inf\Big\{\mathrm{E}_p\big(u, \mathrm{p}, y\big) : \big(u, \mathrm{p}, y\big) \in \mathfrak{X}^p(\Omega_T)\Big\}. \tag{2.16}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that the sequence of respective E_{p_j}-minimisers $(u_{p_j}, \mathrm{p}_{p_j}, y_{p_j})$ satisfies $(u_p, \mathrm{p}_p, y_p) \rightharpoonup (u_\infty, \mathrm{p}_\infty, y_\infty)$ in $W^q(\Omega_T)$ for any $q \in (1, \infty)$, as $p_j \to \infty$. Additionally,

$$\left\{\begin{array}{lll} u_p \rightharpoonup u_\infty, & \text{in } W^{2,1;q}_{L,\sigma}(\Omega_T; \mathbb{R}^n), \\ u_p \longrightarrow u_\infty, & \text{in } C\big(\overline{\Omega_T}; \mathbb{R}^n\big), \\ \mathrm{D}u_p \longrightarrow \mathrm{D}u_\infty, & \text{in } C\big(\overline{\Omega_T}; \mathbb{R}^{n\times n}\big), \\ \mathrm{p}_p \rightharpoonup \mathrm{p}_\infty, & \text{in } W^{1,0;q}_{\#}(\Omega_T; \mathbb{R}^n), \\ y_p \rightharpoonup y_\infty, & \text{in } L^q(\Omega_T), \end{array}\right. \tag{2.17}$$

for any $q \in (1, \infty)$, and also

$$\mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \longrightarrow \mathrm{E}_\infty(u_\infty, \mathrm{p}_\infty, y_\infty) \tag{2.18}$$

as $p_j \to \infty$.

Given the existence of constrained minimisers established by Theorem 2.1.1 above, the next natural question concerns the existence of necessary conditions in the form of PDEs governing the constrained minimisers. We first consider the case of $p < \infty$. Unsurprisingly, the PDE constraint of (2.1) used in defining (2.8) gives rise to a generalised Lagrange multiplier in the Euler-Lagrange equations, obtained by utilising well-known results on the Kuhn-Tucker theory from [94]. Interestingly, however, the incorporation of the solenoidality constraint into the functional space (recall (2.10)), allows us to have only one generalised multiplier corresponding only to the parabolic system in (2.1), instead of two.

To state our second main result, we first need to introduce some notation. For any $M \in \mathbb{N}$ and $p \in (1,\infty)$, we define the operator

$$\mathfrak{M}_p \;:\; L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M),$$

where $p' := p/(p-1)$, by setting

$$\mathfrak{M}_p(V) := \frac{|V|_{(p)}^{p-2}\, V}{\Big(\|V\|_{\dot{L}^p(\Omega_T)}\Big)^{p-1}}. \tag{2.19}$$

Here $|\cdot|_{(p)}$ is the regularisation of the Euclidean norm of $\mathbb{R}^M$, as defined in (2.7). By Hölder's inequality it is immediate to verify that (for the normalised L^p norm) we actually have

$$\Big\|\mathfrak{M}_p(V)\Big\|_{\dot{L}^{p'}(\Omega_T)} \leq 1,$$

and therefore $\mathfrak{M}_p$ is valued in the unit ball of $L^{p'}(\Omega_T;\mathbb{R}^M)$. Further, for brevity we will use the notation

$$\left\{\begin{aligned} \mathrm{K}[u,\mathrm{P}] &:= \mathrm{K}\big(\cdot,\cdot,u,\mathrm{D}u,\mathrm{P}\big),\\ \mathrm{K}_\eta[u,\mathrm{P}] &:= \mathrm{K}_\eta\big(\cdot,\cdot,u,\mathrm{D}u,\mathrm{P}\big),\\ \mathrm{K}_{(A,a)}[u,\mathrm{P}] &:= \mathrm{K}_{(A,a)}\big(\cdot,\cdot,u,\mathrm{D}u,\mathrm{P}\big),\\ \mathrm{K}_r[u,\mathrm{P}] &:= \mathrm{K}_r\big(\cdot,\cdot,u,\mathrm{D}u,\mathrm{P}\big), \end{aligned}\right. \tag{2.20}$$

for K and its partial derivatives K_η, $\mathrm{K}_{(A,a)}$, K_r with respect to the arguments for u, $\mathrm{D}u$ and P respectively.

Theorem 2.1.2 (Variational Equations in L^p). *Suppose that* (2.5) *and* (2.15) *hold true. Then, for any $p \in (n+2,\infty)$, there exists a Lagrange multiplier*

$$\Psi_p \in \Big(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^* \tag{2.21}$$

associated with the constrained minimisation problem (2.16)*, such that the minimising triplet $(u_p,\mathrm{P}_p,y_p) \in \mathfrak{X}^p(\Omega_T)$ satisfies the relations*

$$\left\{\begin{aligned} &(1-\lambda)\int_{\Omega_T}\Big(\mathrm{K}_\eta[u_p,\mathrm{P}_p]\cdot u \;+\; \mathrm{K}_{(A,a)}[u_p,\mathrm{P}_p] : \mathrm{D}u\Big)\cdot\mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{P}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1}\\ &\quad = -\lambda\int_{\Omega_T}\Big(\partial_t u - \nu\Delta u + (u\cdot\mathrm{D})u_p + (u_p\cdot\mathrm{D})u \end{aligned}\right.$$

$$(1-\lambda)\int_{\Omega_T} \mathrm{K}_r[u_p,\mathrm{p}_p]\,\mathrm{p}\cdot\mathfrak{M}_p\big(\mathrm{K}[u_p,\mathrm{p}_p]\big)\,\mathrm{d}\mathcal{L}^{n+1} = -\lambda\int_{\Omega_T} \mathrm{Dp}\cdot\mathfrak{M}_p(y_p)\,\mathrm{d}\mathcal{L}^{n+1} \tag{2.23}$$

for all test mappings

$$(u,\mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)\times W^{1,0;p}(\Omega_T),$$

where the operators K, K_η, $\mathrm{K}_{(A,a)}$, K_r *are given by* (2.20).

Now we consider the case of $p=\infty$. For this extreme case, which is obtained by an appropriate passage to limits as $p\to\infty$ in Theorem 2.1.2, we need to assume additionally that the operator $\mathrm{K}[u,\mathrm{p}]$ does not depend on $(\partial_t u,\mathrm{p})$, hence in this case we will symbolise

$$\begin{cases} \mathrm{K}[u] := \mathrm{K}\big(\cdot,\cdot,u,\mathrm{D}u\big), \\ \mathrm{K}_\eta[u] := \mathrm{K}_\eta\big(\cdot,\cdot,u,\mathrm{D}u\big), \\ \mathrm{K}_A[u] := \mathrm{K}_A\big(\cdot,\cdot,u,\mathrm{D}u\big), \end{cases} \tag{2.24}$$

for K and its partial derivatives K_η, K_A with respect to the arguments for u, $\mathrm{D}u$ respectively, all of which will also need to be assumed to be continuous. We note that, when $p=\infty$, there is no direct analogue of the divergence structure Euler-Lagrange equations. Instead, one of the central points of Calculus of Variations in L^∞ is that Aronsson-Euler PDE systems may be derived, under appropriate (stringent) assumptions. Even in the unconstrained case, these PDE systems are always non-divergence and even fully nonlinear and with discontinuous coefficients (see e.g. [12, 13, 35, 63, 70]). The case of L^∞ problems involving only first order derivative of scalar-valued functions is nowadays a well established field which originated from the work of Aronsson in the 1960s [4, 5], today largely interconnected to the theory of Viscosity Solutions to nonlinear elliptic PDE (for a general pedagogical introduction see e.g. [34, 61]). However, vectorial and higher L^∞ variational problems involving constraints, have only recently been explored (see [65, 66], but also the relevant earlier contributions [10, 11, 15]). For several interesting developments on L^∞ variational problems we refer the interested reader to [14, 16, 19, 20, 27, 39, 49, 76, 80, 81, 84].

In this paper, motivated by recent progress on higher order and on constrained L^∞ variational problems made in [68] by the second author jointly with Moser and by the second author in [65, 66] (inspired by earlier contributions by Moser and Schwetlick deployed in a geometric setting in [79]), we follow a slightly different approach which does not lead an Aronsson-Euler type system; instead, it leads to a *divergence structure* PDE system. However, there is a toll to be paid, as the divergence PDEs arising as necessarystem. Howeve

Euler-Lagrange equations before letting $p \to \infty$, which is different from the scaling used to (formally) derive the Aronsson-Euler equations as $p \to \infty$.

In the light of the above comments, our final main result concerns the satisfaction of necessary PDE conditions for the PDE-constrained minimisers in L^∞ constructed in Theorem 2.1.1, and reads as follows.

Theorem 2.1.3 (Variational Equations in L^∞). *Suppose that* (2.5) *and* (2.15) *hold true, and that additionally* K *does not depend on* $(\partial_t u, \mathrm{p})$ *with* $\mathrm{K}, \mathrm{K}_\eta, \mathrm{K}_A$ *in* (2.24) *being continuous on* $\overline{\Omega_T} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$. *Then, there exists a linear functional*

$$\Psi_\infty \in \bigcap_{r>n+2} \Big(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^* \tag{2.25}$$

which is a Lagrange multiplier associated with the constrained minimisation problem (2.16) *for* $p=\infty$. *There also exist vector measures*

$$\Sigma_\infty \in \mathcal{M}\big(\overline{\Omega_T};\mathbb{R}^N\big), \quad \sigma_\infty \in \mathcal{M}\big(\overline{\Omega_T};\mathbb{R}^n\big) \tag{2.26}$$

such that the minimising triplet $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \mathfrak{X}^\infty(\Omega_T)$ *satisfies the relations*

$$\begin{cases} (1-\lambda)\displaystyle\int_{\overline{\Omega_T}} \Big(\mathrm{K}_\eta[u_\infty]\cdot u + \mathrm{K}_A[u_\infty] : \mathrm{D}u\Big)\cdot \mathrm{d}\Sigma_\infty \\ \quad = -\lambda \displaystyle\int_{\overline{\Omega_T}} \Big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_\infty + (u_\infty\cdot\mathrm{D})u\Big)\cdot \mathrm{d}\sigma_\infty + \Big\langle \Psi_\infty, u(\cdot,0)\Big\rangle, \end{cases} \tag{2.27}$$

$$\int_{\overline{\Omega_T}} \mathrm{D}\mathrm{p}\cdot \mathrm{d}\sigma_\infty = 0, \tag{2.28}$$

for all test mappings

$$(u,\mathrm{p}) \in \Big(W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \cap C^2\big(\overline{\Omega_T};\mathbb{R}^n\big)\Big) \times \Big(W^{1,0;\infty}(\Omega_T) \cap C^1\big(\overline{\Omega_T}\big)\Big).$$

Further, the multiplier Ψ_∞ *and the measures* $\Sigma_\infty, \sigma_\infty$ *can be approximated as follows:*

$$\begin{cases} \Psi_p \longrightarrow \Psi_\infty, & \text{in } \Big(W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^*, \text{ for all } r>n+2, \\ \Sigma_p \longrightarrow \Sigma_\infty, & \text{in } \mathcal{M}\big(\overline{\Omega_T};\mathbb{R}^N\big), \\ \sigma_p \longrightarrow \sigma_\infty, & \text{in } \mathcal{M}\big(\overline{\Omega_T};\mathbb{R}^n\big), \end{cases} \tag{2.29}$$

along a subsequence $p_j \to \infty$, *where*

$$\begin{cases} \Sigma_p := \mathfrak{M}_p\big(\mathrm{K}[u_p]\big)\mathcal{L}^{n+1}\llcorner_{\Omega_T}, \\ \sigma_p := \mathfrak{M}_p(y_p)\mathcal{L}^{n+1}\llcorner_{\Omega_T}. \end{cases} \tag{2.30}$$

Finally, Σ_∞ concentrates on the set whereon $|K[u_\infty]|$ is maximised over $\overline{\Omega_T}$

$$\Sigma_\infty\left(\left\{\left|K[u_\infty]\right| < \left\|K[u_\infty]\right\|_{L^\infty(\Omega_T)}\right\}\right) = 0, \tag{2.31}$$

and σ_∞ asymptotically concentrates on the set whereon $|y_\infty|$ is approximately maximised over $\overline{\Omega_T}$, in the sense that for any $\varepsilon > 0$ small,

$$\lim_{p\to\infty} \sigma_p\left(\left\{|y_p| < \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\right\}\right) = 0. \tag{2.32}$$

Even though the weak interpretation of the equations (2.22)-(2.23) is relatively obvious, this is not the case for (2.27)-(2.28) despite having a simpler form. The reason is that the limiting measures $(\Sigma_\infty, \sigma_\infty)$ are not product measures on $\overline{\Omega_T} = \overline{\Omega} \times [0, T]$ in order to use the Fubini theorem, therefore due to the temporal dependence, (2.28) cannot be simply interpreted as "div$(\sigma_\infty) = 0$". Similar arguments can be made for (2.27) as well. Since this point is not utilised any further in this paper, we only provide a brief discussion in the next section.

We conclude this introduction with some remarks regarding the organisation of this paper. This introduction is followed by Section 2.2, in which we discuss some preliminaries and also establish some basic estimates which are utilised subsequently to establish our main results. In Section 2.3 we prove Theorem 2.1.1 by establishing the existence of constrained minimisers for all p including $p = \infty$, as well as the convergence of minimiser of the former problems to those of the latter. In Section 2.4 we prove Theorem 2.1.2, deriving the necessary PDE conditions which constrained minimisers in L^p satisfy. Finally, in Section 2.5 prove Theorem 2.1.3, deriving the necessary PDE conditions that constrained minimisers in L^∞ satisfy, as well as the additional properties that the measures arising in these PDEs satisfy. A key ingredient here is that we establish appropriate weak* compactness for the Lagrange multipliers arising in the L^p problems in order to pass to the limit as $p \to \infty$.

2.2 Preliminaries and the main estimates

We begin by recording for later use the following modified Hölder inequality for the dotted $\dot{L}^p$ regularised "norms" definedving a simpler which can be very easily confirmed by a direct computation. Next, we continue with a brief discussion regarding the weak interpretation of the equations (2.27)-(2.28). As already noted in the introduction, since $(\Sigma_\infty, \sigma_\infty)$ are generally neither product measures or absolutely continuous with respect to the $(n+1)$-Lebesgue measure on $\overline{\Omega_T} = \overline{\Omega} \times [0, T]$, one needs to use the disintegration "slicing" theorem for Young measures in order to express them appropriately, as follows. Since σ_∞ is a vector measure in $M(\overline{\Omega_T}; \mathbb{R}^n)$, by the Radon-Nikodym theorem, we may decompose

$$\sigma_\infty = \frac{d\sigma_\infty}{d\,|\sigma_\infty|}\,|\sigma_\infty|,$$

where $|\sigma_\infty| \in M(\overline{\Omega_T})$ is the scalar total variation measure and $d\sigma_\infty/d\,|\sigma_\infty|$ is the vector-valued Radon-Nikodym derivative of σ_∞ with respect to $|\sigma_\infty|$. Fix any $h \in L^1(\overline{\Omega_T}, |\sigma_\infty|)$. By the disintegration "slicing" theorem for Young measures (see se.g. [44, Theorem 3.2, p. 179]), we have the representation formula

$$\int_{\overline{\Omega_T}} h\,d\,|\sigma_\infty| = \int_{[0,T]} \left(\int_{\overline{\Omega}} h(x,t)\,d\,|\sigma_\infty|_t(x) \right) d\,|\sigma_\infty|^o(t)$$

where the measure $|\sigma_\infty|^o \in M([0,T])$ and the family of measures $(|\sigma_\infty|_t)_{t \in [0,T]} \subseteq M(\overline{\Omega})$ are defined as follows:

$$|\sigma_\infty|^o := |\sigma_\infty|\left(\overline{\Omega} \times \cdot\right), \qquad |\sigma_\infty|_t(A) := \frac{d\,|\sigma_\infty|\left(A \times \cdot\right)}{d\,|\sigma_\infty|\left(\overline{\Omega} \times \cdot\right)}(t), \ \text{ for } A \subseteq \overline{\Omega} \text{ Borel.}$$

Namely, $|\sigma_\infty|^o$ is one of the marginals of σ_∞ and for $|\sigma_\infty|^o$-a.e. $t \in [0, T]$, the measure $|\sigma_\infty|_t$ evaluated at A is defined as the Radon-Nikodym derivative of the measure $|\sigma_\infty|(A \times \cdot)$ with respect to $|\sigma_\infty|(\overline{\Omega} \times \cdot)$ at the point $t \in [0, T]$. Then, in view of (2.28), by choosing p in the form $\mathrm{p}(x,t) = \pi(x)\tau(t)$, we have

$$
\begin{aligned}
0 &= \int_{\overline{\Omega_T}} \mathrm{Dp} \cdot d\sigma_\infty \\
&= \int_{\overline{\Omega_T}} \left(\mathrm{Dp} \cdot \frac{d\sigma_\infty}{d\,|\sigma_\infty|} \right) d\,|\sigma_\infty| \\
&= \int_{[0,T]} \left(\int_{\overline{\Omega}} \left(\mathrm{Dp} \cdot \frac{d\sigma_\infty}{d\,|\sigma_\infty|} \right)(x,t)\,d\,|\sigma_\infty|_t(x) \right) d\,|\sigma_\infty|^o(t) \\
&= \int_{[0,T]} \left(\int_{\overline{\Omega}} \left(\mathrm{D}\pi(x) \cdot \frac{d\sigma_\infty}{d\,|\sigma_\infty|}(x,t) \right) d\,|\sigma_\infty|_t(x) \right) k\sigma_\infty\ \sigma_\infty
\end{aligned}
$$

The arbitrariness of τ implies that for $|\sigma_\infty^o|$-a.e. $t \in [0, T]$, we have

$$\int_{\overline{\Omega}} \left\langle D\pi(x) \cdot \frac{d\sigma_\infty}{d|\sigma_\infty|}(x, t) \right\rangle d|\sigma_\infty|_t(x) = 0.$$

When restricting our attention to those test function for which $\pi|_{\partial\Omega} \equiv 0$, we obtain the next weak interpretation of (2.28):

div

Since $2\theta > 1 + n/p$, by the standard Sobolev embedding theorem for fractional spaces (e.g. [38, Theorem 8.2], we have that $W^{2\theta,p}(\Omega)$ is continuously embedded in the space $C^{1,\alpha}(\overline{\Omega})$, where $0 < \alpha \leq 2\theta - 1 - n/p$. The conclusion ensues. □

Remark 2.2.2. Let us now record for later use the following simple inclusion of space (which is in fact a continuous embedding):

$$C^{0,\alpha}\left([0,T]; C^{0,\alpha}(\overline{\Omega})\right) \subset C^{0,\alpha}\left(\overline{\Omega_T}\right).$$

Indeed, for any $h \in C^{0,\alpha}\left([0,T]; C^{0,\alpha}(\overline{\Omega})\right)$, we compute

$$\begin{aligned}\left|h(t_1,x_1) - h(t_2,x_2)\right| &\leq |h(t_1,x_1) - h(t_2,x_1)| + |h(t_2,x_1) - h(t_2,x_2)| \\ &\leq \|h(t_1,\cdot) - h(t_2,\cdot)\|_{C(\overline{\Omega})} + \|h(t_2,\cdot)\|_{C^{0,\alpha}(\overline{\Omega})}|x_1 - x_2|^\alpha \\ &\leq \left(|t_1 - t_2|^\alpha + |x_1 - x_2|^\alpha\right)\|h\|_{C^{0,\alpha}([0,T}\end{aligned}$$

ΘW878 Td [(ΩΘW87:)]TJ/Fpe [(2)]TJ/F79 11.9552 satisfies that $(u_0, 0, y_0)$

assumption (2.5)(f) yields that

$$(\partial_t u, \mathrm{p}) \to \left\| \mathrm{K}\left(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p}\right)\right\|^p_{\dot{L}^p(\Omega_T)}$$

is also convex. By standard results in the Calculus of Variations (see e.g. [36]) it follows that E_p is weakly lower semicontinuous in $W^p(\Omega_T)$. Since the convex combination of p-th roots of two weakly lower semicontinuous functionals is indeed a weakly lower semicontinuous functional. By the bounds obtained in Lemma 2.2.3, it follows that $\mathfrak{X}^p(\Omega_T)$ is weakly closed in $W^p(\Omega_T)$. Furthermore, E_p is weakly lower semicontinuous in $\mathfrak{X}^p(\Omega_T)$. Hence, E_p attains its infimum at some $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$.

Consider now the family of minimisers $(u_p, \mathrm{p}_p, y_p)_{p>n+2}$. For any $(u, \mathrm{p}, y) \in \mathfrak{X}^\infty(\Omega_T)$ and any $q \leq p$, minimality and the Hölder inequality for the dotted $\dot{L}^p$ functionals yield

$$\mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \leq \mathrm{E}_p(u, \mathrm{p}, y) \leq \mathrm{E}_\infty(u, \mathrm{p}, y) + p^{-1}.$$

By choosing $(u, \mathrm{p}, y) = (u_0, 0, y_0)$, by Lemma 2.2.3 and a standard diagonal argument, we have that the family of minimisers is weakly precompact in $W^q(\Omega_T)$ for all $q \in (n+2, \infty)$. Further, by Lemma 2.2.1 and Remark 2.2.2, $W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T; \mathrm{R}^n)$ is compactly embedded in $C^{0,\alpha}\left([0,T]; C^{1,\alpha}(\overline{\Omega}; \mathrm{R}^n)\right)$. Hence, for any sequence of indices $p_j \to \infty$, there exists $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \cap_{q \in (n+2,\infty)} W^q(\Omega_T)$ and a subsequence denoted again as $(p_j)_1^\infty$ such that (2.17) holds true. Additionally, due to these modes of convergence, it follows that $(u_\infty, \mathrm{p}_\infty, y_\infty)$ solves (2.1), therefore in fact $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \mathfrak{X}^\infty(\Omega_T)$.1 , p 1 (Ω for any $(u, \mathrm{p}, y) \quad \mathfrak{X}^{\infty}(\Omega_T)$. The above inequality establishes on the one hand that $(u_\infty, \mathrm{p}_\infty, y_\infty)$ minimises E_∞ over $\mathfrak{X}^{\infty}(\Omega_T)$, and on the other hand by choosing $(u, \mathrm{p}, y) := (u_\infty, \mathrm{p}_\infty, y_\infty)$ that (2.18) holds true. Hence, Theorem 2.1.1 has been established. □

2.4 The equations for L^p PDE-constrained minimisers

In this section we establish the proof of Theorem 2.1.2

where the operator $\mathfrak{M}_p : L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M)$ (for $M \in \{N, n\}$) is given by (2.19) and we have used the notation introduced in (2.20). Next, we note that the mapping G which incorporates the PDE constraint is also Fréchet differentiable and it can be easily confirmed that its derivative

$$\mathrm{dG} \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow B\Big(\mathcal{W}^p(\Omega_T), L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big),$$

$$(\mathrm{dG})_{(\overline{u},\overline{\mathrm{p}},\overline{y})}(u,\mathrm{p},y) \ = \ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathrm{G}\Big(\overline{u}+\varepsilon u, \overline{\mathrm{p}}+\varepsilon \mathrm{p}, \overline{y}+\varepsilon y\Big)$$

is given by the formula

$$\Big(\mathrm{dG}\Big)_{(\overline{u},\overline{\mathrm{p}},\overline{y})}(u,\mathrm{p},y) \ = \ \begin{pmatrix} \partial_t u - \nu\Delta u + (u\cdot \mathrm{D})\overline{u} + (\overline{u}\cdot\mathrm{D})u + \mathrm{Dp} - y \\ u(\cdot,0) \end{pmatrix}.$$

We now claim that the differential

$$(\mathrm{dG})_{(\overline{u},\overline{\mathrm{p}},\overline{y})} \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

is a surjective map, for any $(\overline{u},\overline{\mathrm{p}},\overline{y}) \in \mathcal{W}^p(\Omega)$. This is equivalent to the statement that for any $p > n+2$, the linearised Navier-Stokes problem

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot \mathrm{D})\overline{u} + (\overline{u}\cdot\mathrm{D})u + \mathrm{Dp} \ = \ F, & \text{in } \Omega_T, \\ \mathrm{div}\, u \ = \ 0, & \text{in } \Omega_T, \\ u(\cdot,0) \ = \ v, & \text{on } \Omega, \\ u \ = \ 0, & \text{on } \partial\Omega\times(0,T), \end{cases}$$

has a solution $(u,\mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}(\Omega_T)$, for any $\overline{u} \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ and any data

$$(F,v) \in L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n).$$

This is indeed the case, and it is a consequence of a classical result of Solonnikov [87, Th. 4.2] for $n=3$ and of Giga-Sohr [54, Th. 2.8] for $n>3$, as a perturbation of the Stokes problem. As a consequence, the assumptions of the generalised Kuhn-Tucker theorem hold true (see e.g. Zeidler [94, Cor. 48.10 & Th. 48B]). Hence, there exists a Lagrange multiplier

$$\Lambda_p \in \Big(L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n)\Big)^*$$

such that

$$\Big(\mathrm{dE}_p\Big)_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y) \ = \ \Big\langle \Big(\mathrm{dG}\Big)_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y), \Lambda_p \Big\rangle,$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega)$. By standard duality arguments, the Riesz representation theorem and by taking into account the form of the differentials $\mathrm{d}\mathrm{E}_p$ and $\mathrm{d}\mathrm{G}$, we may identify Λ_p with a pair of Lagrange multipliers

$$(\varphi_p, \Psi_p) \in L^p(\Omega_T;\mathbb{R}^n) \times \Big(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n) \Big)$$

such that, the constrained minimiser $\big(u_p, p_p, y_p\big) \in \mathfrak{X}^p(\Omega_T)$ satisfies the equation

$$\begin{aligned}
&(1-\lambda) \int_{\Omega_T} \Big(\mathrm{K}_\eta[u_p, \mathrm{p}_p] \cdot u \;+\; \mathrm{K}_{(A,a)}[u_p, \mathrm{p}_p] : \mathrm{D}u \;+\; \mathrm{K}_r[u_p, \mathrm{p}_p]\,\mathrm{p} \Big) \\
&\qquad \cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p, \mathrm{p}_p]\Big)\, \mathrm{d}\mathcal{L}^{n+1} \;+\; \lambda \int_{\Omega_T} \mathfrak{M}_p(y_p) \cdot y \,\mathrm{d}\mathcal{L}^{n+1} \\
&= \int_{\Omega_T} \Big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p \cdot \mathrm{D})u + \mathrm{D}\mathrm{p} - y \Big) \cdot \varphi_p \,\mathrm{d}\mathcal{L}^{n+1} \;+\; \big\langle \Psi_p, u(\cdot, 0) \big\rangle ,
\end{aligned}$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega_T)$. We note that here we have tacitly rescaled (φ_p, Ψ_p) by multiplying them with the factor $p(\mathcal{L}^{n+1}(\Omega_T))^{-1}$, in order to remove the averages arising from E_p on the left hand side and to be able to obtain non-trivial limits as $p \to \infty$ of the multipliers themselves later on. By using linear independence, the above equation actually decouples to the triplet of relations

$$(1-\lambda) \int_{\Omega_T} \Big(\mathrm{K}_\eta[u_p, \mathrm{p}_p] \cdot u \;+\; \mathrm{K}_{(A,a)}[u_p, \mathrm{p}_p] : \mathrm{D}u \Big) \cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p, \mathrm{p}_p]\Big)\, \mathrm{d}\mathcal{L}^{n+1}$$

$$= \int_{\Omega_T} \Big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p \cdot \mathrm{D})u \Big)_p$$

ΔJ/(E8λ) 11.9552 Tf 6.503 0 Td [(L)]TJ/F44 7.97 Ω_T $\mathfrak{M}_p$ K[u_p, p$_p$] dL^{n+1} η1.7F40ψ11.955ψ1f ψ4.555ψ1.793ψTdψ[([ )

2.5 The equations for L^∞

In order to derive the desired estimate on $(\Psi_p)_{p>n+2}$, we argue as follows. Consider (2.22) for $\mathrm{K}_a \equiv 0$ (the first equation appearing in this proof) and let us fix the initial value on $\Omega \times \{0\}$

$$u(\cdot, 0) \equiv \hat{u} \in W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$$

of the arbitrary test function u, but we will select u on Ω_T such that the term in the bracket in the integral on the right-hand-side becomes a gradient. Then, this term will vanish identically as a consequence of (2.23) when $\mathrm{K}_r \equiv 0$ (the second equation appearing in this proof). Indeed, let $p > n+2$ and let also $(\tilde{u}, \tilde{\mathrm{p}})$ be the (unique) solution to

$$\left\{\begin{array}{rcll} \partial_t \tilde{u} - \nu\Delta\tilde{u} + (\tilde{u}\cdot \mathrm{D})u_p + (u_p\cdot \mathrm{D})\tilde{u} + \mathrm{D}\tilde{\mathrm{p}} &=& 0, & \text{in } \Omega_T,\\ \operatorname{div}\tilde{u} &=& 0, & \text{in } \Omega_T,\\ \tilde{u}(\cdot,0) &=& \hat{u}, & \text{on } \Omega,\\ \tilde{u} &=& 0, & \text{on } \partial\Omega\times(0,T), \end{array}\right.$$

The solvability of the above problem is a consequence of the classical result of Solonnikov [87, Th. 4.2] for $n = 3$ and of Giga-Sohr [54, Th. 2.8] for $n > 3$, as a perturbation of the Stokes problem: by choosing $q > n+2$ in Solonnikov's assumption (4.14), a solution as claimed does exist. Further, since $\hat{u}$ is in $W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$, by [87, Cor. 2, p. 489] we have the uniform estimate

$$\|\tilde{u}\|_{W^{2,1;r}_{\mathrm{L},\sigma}(\Omega_T)} + \|\tilde{\mathrm{p}}\|_{W^{1,0;r}(\Omega_T)} \leq C(r)\,\|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for any $r \in (1,\infty)$. By Lemmas 2.2.1 and 2.2.3 and Remark 2.2.2, if we restrict our attention to $r \in (n+2,\infty)$, we additionally have the bound

$$\|\tilde{u}\|_{L^\infty(\Omega_T)} + \|\mathrm{D}\tilde{u}\|_{L^\infty(\Omega_T)} \leq C(r)\,\|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for some new constant $C(r)$ (which is unbounded as $r \to n+2$). By setting

$$\left\{\begin{array}{l} K_\infty := \sup\Big\{|\mathrm{K}_\eta| + |\mathrm{K}_A| : \ \Omega_T \times \mathbb{B}^n_{R_\infty}(0) \times \mathbb{B}^{n\times n}_{R_\infty}(0)\Big\},\\ R_\infty := \sup\limits_{p>n+2}\Big\{\|u_p\|_{L^\infty(\Omega_T)} + \|\mathrm{D}u_p\|_{L^\infty(\Omega_T)}\Big\}, \end{array}\right.$$

where $\mathbb{B}^n_{R_\infty}(0)$ and $\mathbb{B}^{n\times n}_{R_\infty}(0)$ denote the balls of radius ψ6.378ψ-0.996ψTdψ[(1) TJ/F40ψ11.9552ψTfψ

(2.19) we have $\left\| \mathfrak{M}_p\left(\mathrm{K}[u_p]\right)\right\|_{L^1(\Omega_T)} \leq 1$ (for the normalised L^1 norm):

$$\begin{aligned}\left\langle \Psi_p, \hat{u} \right\rangle &\leq \lambda \left| \int_{\Omega_T} \mathrm{D}\tilde{\mathrm{p}} \cdot \mathfrak{M}_p(y_p)\, \mathrm{d}\mathcal{L}^{n+1} \right| \\ &\quad + (1-\lambda) \left| \int_{\Omega_T} \left(\mathrm{K}_\eta[u_p] \cdot \tilde{u} + \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \right) \cdot \mathfrak{M}_p\left(\mathrm{K}[u_p]\right) \mathrm{d}\mathcal{L}^{n+1} \right| \\ &= (1-\lambda) T \mathcal{L}^n(\Omega) \left| -\!\!\!\!\!\!\int_{\Omega_T} \left(\mathrm{K}_\eta[u_p] \cdot \tilde{u} + \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \right) \cdot \mathfrak{M}_p\left(\mathrm{K}[u_p]\right) \mathrm{d}\mathcal{L}^{n+1} \right|\end{aligned}$$

(in the locally convex sense). Hence, as it can be seen by a customary diagonal argument in the scale of Banach spaces $\big\{W^{2-2/r,r}_{0,\sigma}(\Omega_T;\mathbb{R}^n) : r > n+2\big\}$ comprising the Fréchet space, there exists a continuous linear functional

$$\Psi_\infty \ : \ \bigcap_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n) \longrightarrow \mathbb{R}$$

and a further subsequence as $p \to \infty$ such that along which we have $\Psi_p \rightharpoonup \Psi_\infty$ in the locally convex sense. Additionally, since

$$\Psi_\infty \in \Big(\bigcap_{r>n+2} \Big(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)\Big)^*$$

the convergence $\Psi_p \rightharpoonup \Psi_\infty$ is equivalent to weak* convergence in the Banach space $W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)$ for any fixed $r > n+2$. In conclusion, we see that (2.27)-(2.28) have now been established.

Now we complete the proof of Theorem 2.1.3 by establishing (2.31)-(2.32). Since $\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty]$ in $C\big(\overline{\Omega_T};\mathbb{R}^N\big)$, by applying [64, Prop. 10], we immediately obtain that Σ_∞ concentrates on the set whereon $|\mathrm{K}[u_\infty]|$ is maximised over $\overline{\Omega_T}$:

$$\Sigma_\infty\Big(\Big\{\big|\mathrm{K}[u_\infty]\big| < \max_{\overline{\Omega_T}}\big|\mathrm{K}[u_\infty]\big|\Big\}\Big) = 0.$$

This proves (2.31). For (2.32), we argue as follows. We first note that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)}$$

as $p \to \infty$, along a subsequence. In view of (2.6) and (2.13), this is a consequence of (2.18) and the fact that $\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty]$ uniformly on $\overline{\Omega_T}$, which implies

$$\big\|\mathrm{K}[u_p]\big\|_{\dot{L}^p(\Omega_T)} \longrightarrow \big\|\mathrm{K}[u_\infty]\big\|_{L^\infty(\Omega_T)}.$$

As a consequence of the convergence of $\|y_p\|_{\dot{L}^p(\Omega_T)}$ to $\|y_\infty\|_{L^\infty(\Omega_T)}$, for any $\varepsilon > 0$ we may choose p large so that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \geq \|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}.$$

Let us define now the following subset of Ω_T, which without loss of generality we may assume it has positive $\mathcal{L}^{n+1}$-measure:

$$A_{p,\varepsilon} := \Big\{|y_p| \leq \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big\}.$$

In particular, if $\mathcal{L}^{n+1}(A_{p,\varepsilon}) > 0$, then necessarily $\|y_\infty\|_{L^\infty(\Omega_T)} > 0$. For any Borel set $B \subseteq \Omega_T$ such that $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we estimate by using (2.30), (2.19), (2.7) and the above:

$$
\begin{aligned}
\sigma_p(A_{p,\varepsilon} \cap B) &\leq \left(\frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{L^p(\Omega_T)}}\right) -\!\!\!\!\!\!\int_{A_{p,\varepsilon}\cap B} \Big(|y_p|_{(p)}\Big)^{p-1} \,\mathrm{d}\mathcal{L}^{n+1} \\
&\leq \left(\frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{L^p(\Omega_T)}}\right) -\!\!\!\!\!\!\int_{A_{p,\varepsilon}\cap B} \Big(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big)^{p-1} \,\mathrm{d}\mathcal{L}^{n+1} \\
&\leq \left(\frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{L^p(\Omega_T)}}\right) \Big(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big)^{p-1} \\
&\leq \mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B) \left(\frac{\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}{\|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}}\right)^{p-1}.
\end{aligned}
$$

As a result, for any $\varepsilon > 0$ small, any p large enough and any Borel set $B \subseteq \Omega_T$ with $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we have obtained the density estimate

$$
\frac{\sigma_p(A_{p,\varepsilon} \cap B)}{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)} \leq \left(1 - \frac{\varepsilon}{2\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}\right)^{p-1}.
$$

The above estimate in particular implies that $\sigma_p(A_{p,\varepsilon}) \longrightarrow 0$ as $p \to \infty$ for any $\varepsilon > 0$ fixed, therefore establishing (2.32). The proof of Theorem 2.1.3 is now complete. $\square$

Remark 2.5.1. It is perhaps worth noting (in relation to the preceding arguments in the proof of (2.32)) that the modes of convergence

$$
\|y_p\|_{L^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)} \quad \text{and} \quad y_p \overset{*}{-\!\!\rightharpoonup} y_\infty \text{ in } L^\infty(\Omega_T;\mathbb{R}^n)
$$

as $p \to \infty$, in general by themselves do not suffice to obtain $y_p \longrightarrow y_\infty$ in any strong sense, hence precluding the derivation of a stronger property than (2.32), along the lines of (2.31). A simple counter-example, even in one dimension, is the following: let $p \in 2\mathbb{N}$ and set

$$
y_p := \sum_{j=0}^{(p-2)/2} \left(\chi_{\left(\frac{2j}{p},\frac{2j+1}{p}\right)} - \chi_{\left(\frac{2j+1}{p},\frac{2j+2}{p}\right)}\right) + \chi_{(1,2)},
$$

and also $y_\infty := \chi_{(1,2)}$. Then, we have $|y_p| = 1$ $\mathcal{L}^1$-a.e. on

Chapter 3

On the Isosupremic L^∞ Vectorial Minimisation Problem with PDE Constraints

3.1 Introduction and main results

Let $n, N \in \mathbb{N}$ and let also $\Omega \Subset \mathbb{R}^n$ be a bounded open set with Lipschitz continuous boundary. Consider two functions

$$f, g \ : \ \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}, \tag{3.1}$$

which will be assumed to satisfy certain natural structural assumptions. Additionally, let $\overline{p} > n$ be fixed and consider a given nonlinear operator

$$\mathrm{Q} \ : \ W_0^{1,\overline{p}}\big(\Omega;\mathbb{R}^N\big) \longrightarrow \mathrm{E}, \tag{3.2}$$

where $(\mathrm{E}, \|\cdot\|)$ is an arbitrary Banach space. In this paper we are interested in the following variational problem: given $G \geq 0$ and the supremal functionals

$$\mathrm{F}_\infty, \mathrm{G}_\infty \ : \ W_0^{1,\infty}(\Omega;\mathbb{R}^N) \longrightarrow \mathbb{R},$$

defined by

$$\begin{cases} \mathrm{F}_\infty(u) := \operatorname*{ess\,sup}_\Omega f(\cdot, u, \mathrm{D}u), \\ \mathrm{G}_\infty(u) := \operatorname*{ess\,sup}_\Omega g(\cdot, u, \mathrm{D}u), \end{cases} \tag{3.3}$$

find u_∞ $W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ such that

$$F_\infty(u_\infty) = \inf \quad F_\infty(u$$

best of our knowledge, the only work which directly studies isosupremic problems is [11] by Aronsson-Barron. Among other questions answered therein, it considers some aspects of the one-dimensional case for $n = 1$, but with no additional constraints of any type (which amounts to $Q \equiv 0$ in our setting).

More broadly, very few previous works involve vectorial problems with general constraints in L^∞. Certain vectorial and higher order problems involving eigenvalues in L^∞ have been considered in [65, 69]. Examples of problems with PDE and other constraints are considered in [30, 63, 64, 66]. In the paper [15] of Barron-Jensen, a scalar L^∞ constrained problem was considered, but the constraint was integral. With the exception of the paper [11], it appears that vectorial variational problems in L^∞ involving isosupremic constraints have not been studied before, especially including additional nonlinear constraints which cover numerous different cases, as in this work. For assorted interesting works within the wider area of Calculus of Variations in L^∞ we refer to [10, 14, 16, 17, 19, 20, 26, 27, 68, 76, 80, 81, 84].

Let us note that, in this work, we refrain from discussing the question of defining and studying localised versions of L^∞

$$\begin{cases} \text{Exists a continuous } C : \overline{\Omega} \times \mathbb{R}^N \longrightarrow [0, \infty) \text{ and an } \alpha > 2: \\ \quad \begin{cases} 0 \le f(x, \eta, P) \le C(x, \eta)\left(|P|^\alpha + 1\right), \\ 0 \le g(x, \eta, P) \le C(x, \eta)\left(|P|^\alpha + 1\right), \end{cases} \\ \text{for a.e. } x \text{ and all } (\eta, P). \end{cases} \tag{3.6}$$

$$\text{For a.e. } x \text{ and all } \eta,\ f(x, \eta, \cdot) \text{ and } g(x, \eta, \cdot) \text{ are quasiconvex on } \mathbb{R}^{N\times n}. \tag{3.7}$$

$$\begin{cases} \text{Either } f \text{ or } g \text{ is coercive, namely exist } c, C > 0 \text{ such that either} \\ \qquad f(x, \eta, P) \ \ge\ c|P|^\alpha - C, \\ \text{or} \\ \qquad g(x, \eta, P) \ \ge\ c|P|^\alpha - C, \\ \text{for a.e. } x \text{ and all } (\eta, P). \end{cases} \tag{3.8}$$

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Our first main result concerns the existence of F_p-minimisers in $\mathfrak{X}^p(\Omega)$ and the existence of F_∞-minimisers in $\mathfrak{X}^\infty(\Omega)$, obtained as subsequential limits as $p \to \infty$.

Theorem 3.1.1 (F_∞-minimisers, F_p-minimisers & convergence as $p \to \infty$). *Suppose that the mappings f, g and Q satisfy the assumptions (3.5) through (3.9). If the next compatibility condition is satisfied*

$$\inf\Big\{ \mathrm{G}_\infty : \mathrm{Q}^{-1}\big(\{0\}\big) \cap W_0^{1,\infty}(\Omega;\mathbb{R}^N) \Big\} < G, \tag{3.12}$$

then, for any $p \in [\bar{p}, \infty]$, the functional F_p has a constrained minimiser u_p in the admissible class $\mathfrak{X}^p(\Omega)$, namely

$$\mathrm{F}_p(u_p) = \inf\Big\{ \mathrm{F}_p(v) : v \in \mathfrak{X}^p(\Omega) \Big\}. \tag{3.13}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that, the sequence of respective F_{p_j}-minimisers satisfies $u_p \longrightarrow u_\infty$ p1 unifo/F110 1wTJ/F4ak9nTd [[(11.9552 f 5.95 -11.557 (8J/F

$$\left\{\begin{array}{l}\text{The partial derivatives } g_{\eta\eta}, g_{\eta P}, g_{P\eta}, g_{PP} \text{ of } g \text{ are continuous on} \\ \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n}, \text{ and for } C, \alpha \text{ as in (3.6), we have the bounds} \\ \quad \left\{\begin{array}{l} |g_{\eta\eta}(x,\eta,P)| \leq C(x,\eta)\left(|P|^{\alpha-2}+1\right), \\ |g_{\eta P}(x,\eta,P)| \leq C(x,\eta)\left(|P|^{\alpha-2}+1\right), \\ |g_{PP}(x,\eta,P)| \leq C(x,\eta)\left(|P|^{\alpha-2}+1\right), \\ |g_{P\eta}(x,\eta,P)| \leq C(x,\eta)\left(|P|^{\alpha-2}+1\right), \end{array}\right. \\ \text{for all } (x,\eta,P). \end{array}\right. \tag{3.15}$$

It follows that

$$\left\{\begin{array}{l}\text{The partial derivatives } f_\eta, f_P, g_\eta, g_P \text{ of } f \text{ and } g \text{ are continuous on} \\ \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n}, \text{ and for } C, \alpha \text{ as in (3.6), we have the bounds} \\ \quad \left\{\begin{array}{l} |f_\eta|(x,\eta,P) + |f_P|(x,\eta,P) \leq C(x,\eta)\left(|P|^{\alpha-1}+1\right), \\ |g_\eta|(x,\eta,P) + |g_P|(x,\eta,P) \leq C(x,\eta)\left(|P|^{\alpha-1}+1\right), \end{array}\right. \\ \text{for all } (x,\eta,P). \end{array}\right. \tag{3.16}$$

Further, we will assume that:

$$\left\{\begin{array}{l} Q \text{ is continuously differentiable, and its Fréchet derivative} \\ \qquad (dQ)_{\overline{u}} \ : \ W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow E \\ \text{has closed range in } E, \text{ for any } \overline{u} \in Q^{-1}(\{0\}) \subseteq W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N). \end{array}\right. \tag{3.17}$$

where $(E^*, \|\cdot\|_*)$ *is the dual space of* E*, such that not all vanish simultaneously:*

$$|\lambda_p| + |\mu_p| + \|\psi_p\|_* \neq 0. \tag{3.19}$$

Then, the minimiser $u_p \in \mathfrak{X}^p(\Omega)$ *satisfies the equation*

$$\lambda_p \left(-\!\!\!\!\!\!\int_\Omega f[u_p]^{p-1} f_\eta \right)$$

need to impose some natural additional hypotheses. These hypotheses, although they restrict considerably the classes of f, g, Q that were utilised in order to prove existence of minimisers, they do nonetheless include the interesting case of F_∞ being the L^∞ norm of the gradient. Firstly, let us introduce some convenient notation and rewrite (3.20) in a way which will be more appropriate for the statement and the subsequent proof. By introducing for each $p \in (\bar{p}, \infty)$ the non-negative Radon measures $\sigma_p, \tau_p \in \mathcal{M}(\overline{\Omega})$ given by

$$\sigma_p := \left(\frac{1}{\mathcal{L}^n(\Omega)} \frac{f[u_p]}{\mathrm{F}_p(u_p)} \right)^{p-1} \mathcal{L}^n \chi_\Omega$$

Now we state the additional assumptions which we need to impose:

$$\mathrm{E} \text{ is a separable Banach space.} \tag{3.27}$$

$$\left\{\begin{array}{l} \text{The restriction of the differential } (u,v) \to (\mathrm{dQ})_u(v)\text{, considered as} \\ \qquad \mathrm{dQ} \ : \ \mathrm{Q}^{-1}(\{0\}) \times W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow \mathrm{E}, \\ \text{satisfies the following conditions:} \\ \text{If } u_m \rightharpoonup u \text{ in } \mathrm{Q}^{-1}(\{0\}) \text{ as } m \to \infty, \text{ and } \varphi \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N), \text{ then} \\ \qquad \left\{\begin{array}{l} (\mathrm{dQ})_{u_m}(u_m) \longrightarrow (\mathrm{dQ})_u(u), \\ (\mathrm{dQ})_{u_m}(\varphi) \longrightarrow (\mathrm{dQ})_u(\varphi), \end{array}\right. \\ \text{as } m \to \infty. \end{array}\right. \tag{3.28}$$

The above assumption requires that dQ be weakly-strongly continuous on the diagonal of $\mathrm{Q}^{-1}(\{0\}) \times \mathrm{Q}^{-1}(\{0\})$ and on subsets of the form $\mathrm{Q}^{-1}(\{0\}) \times \{\varphi\}$, when $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \times W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ is endowed with its weak topology and E with its norm topology. We assume further that:

$$\left\{\begin{array}{l} \text{(i) } g \text{ does not depend on } P\text{, namely } g(x,\eta,P) = g(x,\eta), \\ \text{(ii) } f \text{ is quadratic in } P \text{ and independent of } \eta\text{, namely} \\ \qquad f(x,\eta,P) = \mathrm{A}(x):P \otimes P, \\ \text{for some continuous positive symmetric fourth order tensor} \\ \mathrm{A}:\overline{\Omega} \longrightarrow \mathbb{R}^{N\times n} \otimes \mathbb{R}^{N\times n}\text{, which satisfies} \\ \qquad \mathrm{A}(x):P\otimes P > 0\,, \quad \mathrm{A}(x):P \otimes Q = \mathrm{A}(x):Q \otimes P, \\ \text{for all } x \in \overline{\Omega} \text{ and all } P,Q \in \mathbb{R}^{N\times n}\setminus\{0\}. \end{array}\right. \tag{3.29}$$

The above requirements are compatible with the previous assumptions on f. In fact, by [65, Lemma 4, p. 8] and our earlier assumptions, the positivity and symmetry requirements for A are superfluous and can be deduced by merely assuming that f is quadratic in P (up to a replacement of A by its symmetrisation), but we have added them to (3.29) for simplicity. We may finally state our last principal result.

Theorem 3.1.3 (The equations in L^∞). *Suppose we are in the setting of Theorem 3.1.2 and that the same assumptions are satisfied. Additionally we assume that* (3.27) *through* (3.29) *hold true. Then, there exist*

$$\Lambda_\infty \in [0,1], \quad \mathrm{M}_\infty \in [0,1], \quad \Psi_\infty \in \overline{\mathbb{B}}_1^{\mathbf{E}^*}(0), \tag{3.30}$$

which are Lagrange multipliers associated with the constrained minimisation problem (3.4). *There also exist Radon measures*

$$\sigma_\infty \in \mathcal{M}(\overline{\Omega}), \quad \tau_\infty \in \mathcal{M}(\overline{\Omega}), \tag{3.31}$$

and a Borel measurable mapping $\mathrm{D}u_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^{N\times n}$ *which is a version of* $\mathrm{D}u_\infty \in L^\infty(\Omega;\mathbb{R}^{N\times n})$, *such that the minimiser* $u_\infty \in \mathfrak{X}^\infty(\Omega)$ *satisfies the equation*

$$\Lambda_\infty \int_{\overline{\Omega}} f_P(\cdot, \mathrm{D}u_\infty) : \mathrm{D}\varphi \, \mathrm{d}\sigma_\infty + \mathrm{M}_\infty \int_{\overline{\Omega}} g_\eta(\cdot, u_\infty) \cdot \varphi \, \mathrm{d}\tau_\infty = \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\varphi) \Big\rangle, \tag{3.32}$$

for all test maps $\varphi \in C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)$, *coupled by the condition*

$$\mathrm{M}_\infty \Big(\mathrm{G}_\infty(u_\infty) - G \Big) = 0. \tag{3.33}$$

Additionally, the map $\mathrm{D}u_\infty$ *can be represented (modulo Lebesgue null sets) as follows:*

$$\left\{ \begin{array}{l} \text{For any sequence } (v_j)_1^\infty \subseteq C^1_0\big(\overline{\Omega};\mathbb{R}^N\big) \text{ satisfying} \\ \qquad \left\{ \begin{array}{l} \lim\limits_{j\to\infty} \| v_j - u_\infty \|_{(W^{1,1}_0 \cap L^\infty)(\Omega)} = 0, \\ \limsup\limits_{j\to\infty} \mathrm{F}_\infty(v_j) \leq \mathrm{F}_\infty(u_\infty), \end{array} \right. \\ \text{exist a subsequence } (j_k)_1^\infty \text{ such that} \\ \mathrm{D}u_\infty(x) = \left\{ \begin{array}{ll} \lim\limits_{k\to\infty} \mathrm{D}v_{j_k}(x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{array} \right. \end{array} \right. \tag{3.34}$$

(Such an explicit sequence $(v_j)_1^\infty$ *is constructed in the proof.) Finally, the Lagrange multipliers* $\Lambda_\infty, \mathrm{M}_\infty, \Psi_\infty$ *and the measures* $\sigma_\infty, \tau_\infty$ *can be approximated as follows:*

$$\left\{ \begin{array}{lll} \Psi_p \overset{*}{\rightharpoonup} \Psi_\infty, & \text{in } \overline{\mathbb{B}}_1^{\mathbf{E}^*}(0), \\ \Lambda_p \longrightarrow \Lambda_\infty, & \text{in } [0,1], \\ \mathrm{M}_p \longrightarrow \mathrm{M}_\infty, & \text{in } [0,1], \end{array} \right. \tag{3.35}$$

and

$$\left\{ \begin{array}{ll} \sigma_p \overset{*}{\rightharpoonup} \sigma_\infty, & \text{in } \mathcal{M}(\overline{\Omega}), \\ \tau_p \overset{*}{\rightharpoonup} \tau_\infty, & \text{in } \mathcal{M}(\overline{\Omega}), \end{array} \right. \tag{3.36}$$

along a subsequence $p_j \to \infty$.

The weak interpretation of (3.32) is

$$-\Lambda_\infty \operatorname{div}(f_P(\cdot, \mathrm{D}u_\infty)\sigma_\infty) + \mathrm{M}_\infty\, g_\eta(\cdot, u_\infty)\tau_\infty = \langle \Psi_\infty, (\mathrm{dQ})_{u_\infty} \rangle,$$

in $\big(C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)\big)^*$, up to the identifications

$$\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty} \rangle \equiv \langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\cdot) \rangle, \quad g_\eta \equiv g_\eta \cdot (\cdot), \quad f_P \equiv (\cdot) \cdot f_P.$$

Note that in Theorem 3.1.3, the equations obtained depend on certain measures not a priori known explicitly. Therefore, their significance is understood to be largely theoretical, rather than computational. For the proof of this result, we will utilise some machinery developed in the recent paper [65] for some related work on generalised ∞-eigenvalue problems. The main points of this approach are recalled in the course of the proof, for the convenience of the reader.

We conclude this lengthy introduction with some comments concerning the composition of this paper. In Sections 3.2 and

Consequently, in view of (3.11), both constraints are satisfied by u_0, hence $u_0 \in \mathfrak{X}^p(\Omega) \neq \emptyset$. Next, note that f^p is a (Morrey) quasiconvex function. To see this, let $h : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ be an arbitrary quasiconvex function, in our case we will take $h(P) = f(x,\eta,P)$ for fixed (x,η). Then, by assumption (3.7), for any $\varphi \in W_0^{1,\infty}(U;\mathbb{R}^N)$ with $U \Subset \mathbb{R}^n$ open and $P \in \mathbb{R}^{N\times n}$ fixed,

$$h(P) \leq -\!\!\!\!\!\!\int_U h(P + \mathrm{D}\varphi)\,\mathrm{d}\mathcal{L}^n.$$

Hence, by Jensen's inequality and the convexity of $t \to t^p$, we conclude that

$$h(P)^p \leq \left(-\!\!\!\!\!\!\int_U h(P + \mathrm{D}\varphi)\,\mathrm{d}\mathcal{L}^n \right)^p \leq -\!\!\!\!\!\!\int_U h(P + \mathrm{D}\varphi)^p\,\mathrm{d}\mathcal{L}^n.$$

We now proceed to bound f^p. By (3.6), we estimate

$$0 \leq f(x,\eta,P)^p \leq \mathrm{C}(x,\eta)^p \left(1 + |P|^\alpha \right)^p \leq 2^{p-1}\mathrm{C}(x,\eta)^p \left(1 + |P|^{\alpha p} \right).$$

By standard results (see [36]), F_p is weakly lower semicontinuous on $W_0^{1,\alpha p}(\Omega;\mathbb{R}^N)$. Let $(u^{(i)})_1^\infty \subseteq \mathfrak{X}^p(\Omega)$ denote a minimising sequence. By virtue of (3.6) we have $f \geq 0$, hence clearly $\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) \geq 0$. We now show that the infimum is finite. To this aim, by (3.6) we estimate

$$\begin{aligned}
\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) &\leq \mathrm{F}_p(u_0) \\
&= \left(-\!\!\!\!\!\!\int_\Omega f(\cdot,u_0,\mathrm{D}u_0)^p\,\mathrm{d}\mathcal{L}^n \right)^{1/p} \\
&\leq \left(-\!\!\!\!\!\!\int_\Omega \Big(\mathrm{C}(\cdot,u_0)(1 + |\mathrm{D}u_0|^\alpha) \Big)^p\,\mathrm{d}\mathcal{L}^n \right)^{1/p},
\end{aligned}$$

which yields

$$\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) \leq \Big\| \mathrm{C}(\cdot,u_0) \ldots$$

$\mathbb{R}^N)10.41811.9552\psi Tf$

By using the Poincaré and Hölder inequalities, we infer that

$$C + \left(-\!\!\!\!\!\!\int_\Omega |h(\cdot, u^{(i)}, Du^{(i)})|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \geq c \, \| u^{(i)} \|^{\alpha}_{W^{1,\alpha p}(\Omega)}$$

for some new constant $c > 0$ which is independent of $i \in \mathbb{N}$. If $h = f$, then by the previously derived estimates we have the uniform bound

$$\| u^{(i)} \|^{\alpha}_{W^{1,\alpha p}(\Omega)} \leq \frac{1}{c} \left(C + \mathrm{F}_p(u^{(i)}) \right) \leq \frac{C}{c} + \frac{1}{c} \| \mathrm{C}(\cdot, u_0) \|_{L^\infty(\Omega)} \left(1 + \| Du_0 \|^{\alpha}_{L^\infty(\Omega)} \right),$$

and if $h = g$, then by the isosupremic constraint we have the uniform bound

$$\| u^{(i)} \|^{\alpha}_{W^{1,\alpha p}(\Omega)} \leq \frac{1}{c} \left(C + \mathrm{G}_p(u^{(i)}) \right) \leq \frac{C + G}{c}.$$

In either case, we have that $(u^{(i)})_1^\infty$ is weakly precompact in $W_0^{1,\alpha p}(\Omega; \mathbb{R}^N)$. By passing to a subsequence if necessary, standard strong and weak compactness arguments imply that there exists a map $u_p \in W_0^{1,\alpha p}(\Omega; \mathbb{R}^N)$ and a subsequence denoted again by $(u^{(i)})_1^\infty$ such that

$$\left\{ \begin{array}{ll} u^{(i)} \longrightarrow u_p, & \text{in } L^{\alpha p}(\Omega; \mathbb{R}^N), \\ Du^{(i)} \rightharpoonup Du_p, & \text{in } L^{\alpha p}(\Omega; \mathbb{R}^{N\times n}), \end{array} \right.$$

as $i \to \infty$. Further, since $p > n$, by the Morrey estimate we have that $(u^{(i)})_1^\infty$ is also bounded in $C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^N)$ for $\gamma < 1 - n/(\alpha p)$. By the compact embedding of Hölder spaces, we conclude that

$$u^{(i)} \longrightarrow u_p$$

for the constants $c, C > 0$ of (3.8) (which are independent of p and q). If $h = f$, by applying our earlier estimates we deduce the uniform bound

$$\|Du_p\|^{\alpha}_{L^q(\Omega)} \le \frac{1}{c}\Big(C + \mathrm{F}_q(u_p)\Big) \le \frac{C}{c} + \frac{1}{c}\,\|\mathrm{C}(\cdot, u_0)\|_{L^\infty(\Omega)}\Big(1 + \|Du_0\|^{\alpha}_{L^\infty(\Omega)}\Big).$$

If $h = g$, then again as in our earlier estimates we have the uniform bound

$$\|Du_p\|^{\alpha}_{L^q(\Omega)} \le \frac{1}{c}\Big(C + \mathrm{G}_q(u_p)\Big) \le \frac{C + G}{c}.$$

In either case, we see that under (3.8), our estimates above imply that

$$\|Du_p\|_{L^q(\Omega)} \le K,$$

for some constant $K > 0$ independent of p, q. Further, by the Poincaré inequality, we deduce that

$$\|u_p\|_{W^{1,q}(\Omega)} \le K\Big(1 + C(q)\Big),$$

where $C(q)$ is the constant of the Poincaré inequality in $W^{1,q}_0(\Omega;\mathbb{R}^N)$. Hence, the sequence of minimisers $(u_p)_{p\ge\bar{p}}$ is bounded in $W^{1,q}_0(\Omega;\mathbb{R}^N)$ for any fixed $q \in (1,\infty)$, and therefore it is weakly precompact in this collection of spaces. By a standard diagonal argument, there exists a sequence $(p_j)_1^\infty$ and a mapping

$$u_\infty \in \bigcap_{\bar{p}<q<\infty} W^{1,q}_0(\Omega;\mathbb{R}^n),$$

such that $u_p \rightharpoonup u_\infty$ in $W^{1,q}_0(\Omega;\mathbb{R}^n)$ as $p_j \to \infty$, for any fixed $q \in (\bar{p},\infty)$. By standard compactness arguments in Sobolev and Hölder spaces, we infer that

$$\begin{cases} u_p \longrightarrow u_\infty, & \text{in } C\big(\overline{\Omega};\mathbb{R}^N\big), \\ Du_p \rightharpoonup Du_\infty, & \text{in } L^q(\Omega;\mathbb{R}^{N\times n}), \end{cases}$$

as $p_j \to \infty$, for any $q \in (\bar{p},\infty)$. We will now show that $u_\infty \in \mathfrak{X}^\infty(\Omega)$. In view of (3.11), we need to show that $u_\infty \in W^{1,\infty}_0(\Omega;\mathbb{R}^N)$ and that $\mathrm{G}_\infty(u_\infty) \le G$ and also $\mathrm{Q}(u_\infty) = 0$. By the weak lower semi-continuity of the L^q norm for $q \ge \bar{p}$ fixed, we have

$$\|Du_\infty\|_{L^q(\Omega)} \le \liminf_{p_j\to\infty} \|Du_p\|_{L^q(\Omega)} \le K.$$

By letting $q \to \infty$, this yields that $Du_\infty \in L^\infty(\Omega;\mathbb{R}^N)$. By the Poincaré inequality in $W^{1,\infty}_0(\Omega;\mathbb{R}^N)$, we infer that $u_\infty \in W^{1,\infty}_0(\Omega;\mathbb{R}^N)$. Next, since $\mathrm{G}_p(u_p) \le G$ for all $p \in (\bar{p},\infty)$, via the Hölder inequality and weak lower semi-continuity, we have

$$\mathrm{G}_\infty(u_\infty) = \lim_{q\to\infty} \mathrm{G}_q(u_\infty) \le \liminf_{q\to\infty}\Big(\liminf_{p_j\to\infty} \mathrm{G}_q(u_p)\Big) \le \liminf_{p_j\to\infty} \mathrm{G}_p(u_p) \le G,$$

yielding that indeed $G_\infty(u_\infty) \leq G$. We now show that $Q(u_\infty) = 0$. We have already shown in Proposition 3.2.1 that $Q^{-1}(\{0\})$ is a weakly closed subset of $W_0^{1,q}(\Omega;\mathbb{R}^N)$ for any $q \in (\bar{p},\infty)$. Since $Q(u_p) = 0$ for all $p \geq \bar{p}$ and $u_p \rightharpoonup u_\infty$ in $W_0^{1,q}(\Omega;\mathbb{R}^N)$ as $p_j \to \infty$, we deduce that $Q(u_\infty) = 0$, as desired.

It remains to show that u_∞ is indeed a minimiser of F_∞ in $\mathfrak{X}^\infty(\Omega)$, and additionally that the energies converge. Fix an arbitrary $u \in \mathfrak{X}^\infty(\Omega)$. By minimality and by noting that $\mathfrak{X}^\infty(\Omega) \subset \mathfrak{X}^p(\Omega)$ for any $p \in [\bar{p},\infty]$, we have the estimate

$$
\begin{aligned}
F_\infty(u_\infty) &= \lim_{q\to\infty} F_q(u_\infty) \\
&\leq \liminf_{q\to\infty} \left(\liminf_{p_j\to\infty} F_q(u_p) \right) \\
&\leq \liminf_{p_j\to\infty} F_p(u_p) \\
&\leq \limsup_{p_j\to\infty} F_p(u_p) \\
&\leq \limsup_{p_j\to\infty} F_p(u) \\
&= F_\infty(u),
\end{aligned}
$$

forp

This reformulation is a labour-saving device, drastically shortening the proof of this result. In view of assumption (3.16), first we will show that the following functionals are Fréchet differentiable

$$\frac{1}{p}(\mathrm{F}_p)^p \ : \ W_0^{1,\alpha p}\big(\Omega;\mathbb{R}^N\big) \longrightarrow \mathbb{R},$$
$$\frac{1}{p}(\mathrm{G}_p)^p - \frac{G^p}{p} \ : \ W_0^{1,\alpha p}\big(\Omega;\mathbb{R}^N\big) \longrightarrow \mathbb{R}.$$

A direct computation gives the next formal expressions for their Gateaux derivatives

$$\mathrm{d}\Big(\frac{1}{p}(\mathrm{F}_p)^p\Big)_u(v) = -\!\!\!\!\!\!\int_\Omega f[u]^{p-1}\Big(f_\eta[u]\cdot v + f_P[u]:\mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n,$$
$$\mathrm{d}\Big(\frac{1}{p}(\mathrm{G}_p)^p - \frac{G^p}{p}\Big)_u(v) = -\!\!\!\!\!\!\int_\Omega g[u]^{p-1}\Big(g_\eta[u]\cdot v + g_P[u]:\mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n,$$

for all $u, v \in W_0^{1,\alpha p}(\Omega;\mathbb{R}^N)$. We will now show the above formal expressions indeed define Fréchet derivatives, by employing relatively standard estimates through the Hölder inequality. We argue only for $\frac{1}{p}(\mathrm{F}_p)^p$, as the estimates for $\frac{1}{p}\big((\mathrm{G}_p)^p - G^p\big)$ are identical. Since $\alpha > 1$ and $p \geq \overline{p} > n$, by Morrey's estimate we have

$$\begin{aligned}
&\left| -\!\!\!\!\!\!\int_\Omega f[u]^{p-1}\Big(f_\eta[u]\cdot v + f_P[u]:\mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n\right| \\
&\quad\leq -\!\!\!\!\!\!\int_\Omega |f[u]|^{p-1}\Big(|f_\eta[u]||v| + |f_P[u]||\mathrm{D}v|\Big)\,\mathrm{d}\mathcal{L}^n \\
&\quad\leq -\!\!\!\!\!\!\int_\Omega C(\cdot,u)^p\Big(1+|\mathrm{D}u|^\alpha\Big)^{p-1}\Big(1+|\mathrm{D}u|^{\alpha-1}\Big)(|v|+|\mathrm{D}v|)\,\mathrm{d}\mathcal{L}^n \\
&\quad\leq 2^p\Big\||v|C(\cdot,u)^p\Big\|_{L^\infty(\Omega)} -\!\!\!\!\!\!\int_\Omega \Big(1+|\mathrm{D}u|^{\alpha-1}+|\mathrm{D}u|^{\alpha p-\alpha}+|\mathrm{D}u|^{\alpha p-1}\Big)\,\mathrm{d}\mathcal{L}^n
\end{aligned}$$

$F \in C^2(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n})$, with $F[u] \equiv F(\cdot, u, Du)$, where

$$E(u) = ⨍_\Omega F[u]\, dL^n,$$

and

$$(dE)_u(v) = ⨍_\Omega F_\eta[u] \cdot v + F_P[u] : Dv \, dL^n,$$

for all $u, v \in W_0^{1,\alpha p}(\Omega; \mathbb{R}^N)$. As F is arbitrary we can choose $F = f^p$ (to investigate our functionals of interest), such that f satisfies (3.14). We have

$$\begin{aligned} F_P &= p f^{p-1} f_P, \\ F_{PP} &= p f^{p-2} \left(f f_{PP} + (p-1) f_P \otimes f_P \right). \end{aligned}$$

By (3.14), it follows that

$$\begin{aligned} |f_P(x,\eta,P)| &\leq C(x,\eta)\left(|P|^{\alpha-1} + 1 \right), \\ |f(x,\eta,P)| &\leq C(x,\eta)\left(|P|^{\alpha} + 1 \right), \end{aligned}$$

for some new continuous functions at each step. Hence,

$$|F(x,\eta,P)| \leq C(x,\eta)\left(|P|^{\alpha p} + 1 \right).$$

Additionally,

$$\begin{aligned} |F_P(x,\eta,P)| &\leq C(x,\eta)\left(|P|^{\alpha(p-1)} + 1 \right)\left(|P|^{\alpha-1} + 1 \right) \\ &\leq C(x,\eta)\left(|P|^{\alpha p-1} + 1 \right). \end{aligned}$$

Furthermore,

$$|F_{PP}(x,\eta,P)| \leq C\big(|P|^{\alpha-}$$

$$
\begin{aligned}
&\leq \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta\eta}[u + (\lambda + \mu)v] \cdot \lambda v + \mathrm{F}_{\eta P}[u + (\lambda + \mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \| v \|_{L^{\alpha p}(\Omega)} \\
&\leq \left(\fint_\Omega \int_0^1 \int_0^1 \Big(\left| \mathrm{F}_{\eta\eta}[u + (\lambda + \mu)v] \cdot \lambda v \right| + \left| \mathrm{F}_{\eta P}[u + (\lambda + \mu)v] : \lambda \mathrm{D}v \right| \Big)^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \| v \|_{L^{\alpha p}(\Omega)} \\
&\leq C_0 \left(\fint_\Omega \int_0^1 \int_0^1 \Big(\left| \mathrm{F}_{\eta\eta}[u + (\lambda + \mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p - 1}} \right. \\
&\qquad \left. + \left| \mathrm{F}_{\eta P}[u + (\lambda + \mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p - 1}} \Big) \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \| v \|_{L^{\alpha p}(\Omega)} \\
&= C_0 \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta\eta}[u + (\lambda + \mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right. \\
&\qquad \left. + \fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta P}[u + (\lambda + \mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \| v \|_{L^{\alpha p}(\Omega)} \\
&\leq C_1 \left[\left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta\eta}[u + (\lambda + \mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \right. \\
&\qquad \left. + \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta P}[u + (\lambda + \mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p - 1}} \mathrm{d}\mu \, \mathrm{d}\lambda \, \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p - 1}{\alpha p}} \right] \| v \|_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

Combining both of these we bounds, we obtain,

$$\fint_\Omega \left| \int_0^1 \mathrm{F}_\eta[u+\lambda v] - \mathrm{F}_\eta[u]\,\mathrm{d}\lambda \cdot v \right| \mathrm{d}\mathcal{L}^n + \fint_\Omega \left| \int_0^1 \mathrm{F}_P[u+\lambda v] - \mathrm{F}_P[u]\,\mathrm{d}\lambda : \mathrm{D}v \right| \mathrm{d}\mathcal{L}^n$$

$$\leq C_1 \left(\left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right.$$

$$\left. + \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right) \|v\|_{L^{\alpha p}(\Omega)}$$

$$+ C_1 \left(\left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{P\eta}[u+(\lambda+\mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right.$$

$$\left. + \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{PP}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v \right|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right) \|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}.$$

We proceed to bound the first term,

$$C_1 \left(\fint_\Omega \int_0^1 \int_0^1 \left| \mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v \right|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)}$$

$$\leq C_1 \left(\fint_\Omega \int_0^1 \int_0^1 \left(C(x,\eta) \left(\left(|\mathrm{D}u| + |(\lambda+\mu)\mathrm{D}v| \right)^{\alpha p-2} + 1 \right) \right)^{\frac{\alpha p}{\alpha p-1}} |v|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}}$$

$$\|v\|_{L^{\alpha p}(\Omega)}$$

$$\leq \|c_1(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\fint_\Omega \left(|\mathrm{D}u|^{\alpha p-2} + |\mathrm{D}v|^{\alpha p-2} + 1 \right)^{\frac{\alpha p}{\alpha p-1}} |v|^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)}$$

$$\leq \|c_1(\cdot,\eta)\| \left(\fint_\Omega |\mathrm{D}$$

$$-_{\Omega} |V|^{t \ \alpha p}$$

$$
\begin{aligned}
&+ \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\|Du\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|Dv\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \right) \|Dv\|_{L^{\alpha p}(\Omega)} \|v\|_{L^{\alpha p}(\Omega)} \\
&+ \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\|Du\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|Dv\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \right) \|v\|_{L^{\alpha p}(\Omega)} \|Dv\|_{L^{\alpha p}(\Omega)} \\
&+ \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\|Du\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|Dv\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \right) \|Dv\|^2_{L^{\alpha p}(\Omega)} \\
= {}& \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\|Du\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|Dv\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \right) \\
&\quad \left(\|v\|^2_{L^{\alpha p}(\Omega)} + 2\|Dv\|_{L^{\alpha p}(\Omega)} \|v\|_{L^{\alpha p}(\Omega)} + \|Dv\|^2_{L^{\alpha p}(\Omega)} \right) \\
= {}& \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\|Du\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|Dv\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \right) \|v\|^2_{W^{1,\alpha p}(\Omega)} \\
\leq {}& o\left(\|v\|_{W^{1,\alpha p}(\Omega)} \right).
\end{aligned}
$$

This estimate establishes that the functional $\frac{1}{p}(\mathrm{F}_p)^p$ (and therefore $\frac{1}{p}\left((\mathrm{G}_p)^p - G^p\right)$) is indeed Fréchet differentiable.

We now show that the equations that the constrained minimiser satisfies take the form as given in (3.20) and (3.21). Given the Fréchet derivatives and our assumption (3.17) on the range of dQ, we can invoke the generalised Kuhnlp3.203.17)

on the ra -14.446 Td [(on)-348(the)-348(range)-348(of)]TJ/F40 11.9y8(raF405he)-E the space of Radon measures. Indeed, if $\mathrm{F}_p(u_p) > 0$, then since $f \geq 0$ we have

$$
\begin{aligned}
\sigma_p\,(\overline{\Omega}) &= \sigma_p(\overline{\Omega}) \\
&= \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega \frac{f(\cdot, \mathrm{D}u_p)^{p-1}}{\mathrm{F}_p(u_p)^{p-1}}\, \mathrm{d}\mathcal{L}^n \\
&= \frac{1}{\mathrm{F}_p(u_p)^{p-1}} ⨍_\Omega f(\cdot, \mathrm{D}u_p)^{p-1}\, \mathrm{d}\mathcal{L}^n \\
&\leq \frac{1}{\mathrm{F}_p(u_p)^{p-1}} \left(⨍_\Omega f(\cdot, \mathrm{D}u_p)^{p}\, \mathrm{d}\mathcal{L}^n \right)^{\frac{p-1}{p}} \\
&= 1,
\end{aligned}
$$

whilst if $\mathrm{F}_p(u_p) = 0$, then trivially $\sigma_p\,(\overline{\Omega}) = 0$. In both cases, $\sigma_p\,(\overline{\Omega}) \leq 1$ for all $p \in (\overline{p}, \infty)$. The estimate for $\tau_p\,(\overline{\Omega})$ is completely analogous, yielding $\tau_p\,(\overline{\Omega}) \leq 1$ for all $p \in (\overline{p}, \infty)$.

Step 2. By using assumption (3.29) and definition (3.22), we have the following differential identity: for any fixed $v \in C^1_0\big(\overline{\Omega}; \mathbb{R}^N\big)$ and any $p \in (\overline{p}, \infty)$ we have

$$
\begin{aligned}
\int_\Omega f(\cdot, \mathrm{D}v - \mathrm{D}u_p)\, \mathrm{d}\sigma_p &= \int_\Omega f(\cdot, \mathrm{D}v)\, \mathrm{d}\sigma_p - \int_\Omega f(\cdot, \mathrm{D}u_p)\, \mathrm{d}\sigma_p \\
&\quad + \int_\Omega f_P(\cdot, \mathrm{D}u_p) : (\mathrm{D}u_p - \mathrm{D}v)\, \mathrm{d}\sigma_p.
\end{aligned}
$$

Indeed, by using that $f_P(x,P) = \mathrm{A}(x) : \big((\cdot)\ P + P\ (\cdot)\big)$, we may compute

$$
\int_\Omega f(\cdot, \mathrm{D}v - \mathrm{D}u_p)\, \mathrm{d}\sigma_p = \int_\Omega \mathrm{A}(
$$

$+$

for all test maps $\varphi \in W_0^{1,\alpha p}(\Omega;\mathbb{R}^N)$, whilst we also have that

$$\Lambda_p \in [0,1], \quad \mathrm{M}_p \in [0,1] \quad \text{and} \quad \Psi_p \in \overline{\mathbb{B}}_1^{\mathbf{E}^*}(0).$$

Further, by assumption (3.27), the weak* topology of the dual space $\mathbf{E}^*$ is sequentially (pre-) compact on bounded sets. Thus, the previous memberships imply that, upon passing to a further subsequence as $j \to \infty$, symbolised again by $(p_j)_1^\infty$, there exist

$$\Lambda_\infty \in [0,1], \quad \mathrm{M}_\infty \in [0,1] \quad \text{and} \quad \Psi_\infty \in \overline{\mathbb{B}}_1^{\mathbf{E}^*}(0),$$

such that the modes of convergence (3.35) hold true as $p_j \to \infty$.

Step 4. By Steps 2 and 3, for $\varphi := u_p - v$, where $v \in C_0^1(\overline{\Omega};\mathbb{R}^N)$ is an arbitrary fixed map, for any fixed $p \in (\bar{p}, \infty)$ we have the identity

$$\Lambda_p \int_{\overline{\Omega}} f(\cdot, \mathrm{D}u_p - \mathrm{D}v)\,\mathrm{d}\sigma_p = \Big\langle \Psi_p, (\mathrm{d}\mathrm{Q})_{u_p}(u_p$$

$u_p - \mathrm{D}v$

$_p\,; u_p$

f

2

M_p

u $\mathrm{D}u_p$ p417701 Td [(3)]TJ 91.9265 Tf 6.75 4.315 Td [(Z)]TJ 28 0 -3 7 1 Td [(f)]TJ

whilst for $F_p(u_p) = 0$ the equality follows trivially. To establish (3.37), it suffices to note that by assumption (3.29) and by the variational representation of the minimum eigenvalue of the symmetric linear operator $A(x) : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^{N\times n}$, we have that $\alpha_0 > 0$ and the inequality

$$\alpha_0|Q|^2 \leq A(x) : Q \quad Q$$

for all $x \quad \overline{\Omega}$ and all $Q \quad \mathbb{R}^{N\times n}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^{N\times n}$.

Step 6. By Steps 1, 3 and 5, and by using that $F_p(u_p) \longrightarrow F_\infty(u_\infty)$ as $p_j \to \infty$ (as shown in Theorem 3.1.1), we may invoke Hutchinson's theory of measure-function pairs, in particular [57, Sec. 4, Def. 4.1.1, 4.1.2, 4.2.1 and Th. 4.4.2], to infer that there exists a map

$$V_\infty \quad L^2\left(\overline{\Omega}, \sigma_\infty; \mathbb{R}^{N\times n}\right)$$

such that, along perhaps a further subsequence $(p_j)_1^\infty$

map

$_p)$

Step 7. The equations established in Step 6 will complete the proof of the theorem, upon establishing that

$$V_\infty \ = \ \mathrm{D}u_\infty \quad \sigma_\infty\text{-a.e. on } \overline{\Omega},$$

where $\mathrm{D}u_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^{N\times n}$ is some Borel measurable mapping which is a version of $\mathrm{D}u_\infty$ $L^\infty(\Omega;\mathbb{R}^{N\times n})$, namely such that

$$\mathrm{D}u_\infty \ = \ \mathrm{D}u_\infty \quad \mathcal{L}^n\text{-a.e. on } \overline{\Omega},$$

(recall that $\partial\Omega$ is a nullset for the Lebesgue measure $\mathcal{L}^n$). The remaining steps are devoted to establishing this claim, together with the approximability properties claimed in (3.34) for some sequence of mappings $(v_j)_1^\infty$ $C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)$, which will be constructed explicitly.

Step 8. If $\Lambda_\infty = 0$, then Step 6 completes the proof of Theorem 3.1.3 as the first term involving V_∞ vanishes. Hence, we may henceforth assume that $\Lambda_\infty > 0$. Therefore, by passing perhaps to a further subsequence if necessary, we may assume that

$$\Lambda_{p_j} \ \geq \ \frac{\Lambda_\infty}{2} \ > \ 0,$$

Step 11. To complete the proof of Theorem 3.1.3, it remains to show that at least one sequence of mapping $(v_j)_1^\infty \subseteq C_0^1(\overline{\Omega}; \mathbb{R}^N)$ exists, which satisfies the modes of convergence required by (3.34). To this end we utilise (for the first time) the assumption that the bounded domain Ω

- For any $\Theta \in C\left(\overline{\Omega} \times \mathbb{R}^{N\times n}\right)$, satisfying for any $x \in \overline{\Omega}$ that $\Theta(x, \cdot)$ is convex on $\mathbb{R}^{N\times n}$ with $\Theta(x, \cdot) \geq \Theta(x, 0) = 0$, and also that the partial derivative Θ_P exists and is continuous on $\overline{\Omega} \times \mathbb{R}^{N\times n}$, we will show that there exists a modulus of continuity $\omega \in C\left([0, \infty); [0, \infty)\right)$ with $\omega(0) = 0$ whic9626 Tf 9.234 11.557 Td [(“)]552 Tf 11.955 0 Td [())]TJ/F49 9.9624 0 Td [(–)0 we have

$$\Theta\left(x, K^{\varepsilon}(Du)(x)\right) = \Theta\left(x, \int_{\mathbb{R}^n}\right.$$

and assumption (3.9) *is always satisfied.*

(ii) If for any $x \in \Omega$ *we have*

$$\big(\Pi(x,\cdot) = 0\big) \implies \big(\Pi_\eta(x,\cdot) = 0\big),$$

namely when all points in the zero set are critical points, then Q *satisfies* (3.17)*.*

(iii) Assumptions (3.27) *and* (3.28) *are always satisfied.*

The choice of E is deliberately made "as large as possible", as then the Lagrange multipliers of Theorems 3.1.2 and 3.1.3 are valued in the smaller space $\mathrm{E}^* = L^\infty(\Omega;\mathbb{R}^M)$.

Proof of Proposition 3.4.1. (i) Follows directly from the definitions, by the continuity of Π and by Morrey's estimate, because $\bar{p} > n$.

(ii) Indeed, since

$$(\mathrm{dQ})_u(\varphi) \;=\; \Pi_\eta(\cdot,u)\cdot\varphi,$$

if $u \in \mathrm{Q}^{-1}(\{0\})$, then $\Pi(\cdot,u) = 0$ a.e. on Ω and therefore $\Pi_\eta(\cdot,u) = 0$ a.e. on Ω, which implies that $(\mathrm{dQ})_u = 0$, hence its image is the closed trivial subspace $\{0\} \subseteq L^1(\Omega;\mathbb{R}^M)$.

(iii) Note first that $L^1(\Omega;\mathbb{R}^M)$ is separable. Also, if we have $u_m \rightharpoonup u$ and $\varphi_m \rightharpoonup \varphi$ in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ as $m \to \infty$, then by Morrey's theorem and the compactness of the imbedding of Hölder spaces we have $u_m \longrightarrow u$ and also $\varphi_m \longrightarrow \varphi$ in $C\big(\overline{\Omega};\mathbb{R}^N\big)$ as $m \to \infty$. Hence, we have as $m \to \infty$ that

$$(\mathrm{dQ})_{u_m}(\varphi_m) = \Pi_\eta(\cdot,u_m)\cdot\varphi_m \;\longrightarrow\; \Pi_\eta(\cdot,u)\cdot\varphi = (\mathrm{dQ})_u(\varphi),$$

in $C\big(\overline{\Omega};\mathbb{R}^M\big)$, which a fortiori implies strong convergence in $L^1(\Omega;\mathbb{R}^M)$. □

We note that the proof of (iii) above is immediate if one assumes the additional hypothesis of (ii), since then $(\mathrm{dQ})_{u_m} = 0$ for any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$.

Proposition 3.4.2 (Case 2). *Let* $\Pi \in C^1(\overline{\Omega}\times\mathbb{R}^N)$ *and let us define* $\pi : \mathbb{R} \longrightarrow \mathbb{R}$

for $\mathsf{E} := L^1(\Omega)$*, we have the following:*

(i) The zero set of Q *equals*

$$\mathsf{Q}^{-1}(\{0\}) = \left\{ v \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \ : \ \Pi(x, v(x)) \leq 0, \ a.e.\ x \in \Omega \right\},$$

and assumption (3.9) *is always satisfied.*

(ii) If for any $x \in \Omega$ *it holds that*

$$\left\{\Pi(x,\cdot) = 0\right\} \subseteq \left\{\Pi_\eta(x,\cdot) = 0\right\},$$

then Q *satisfies assumption* (3.17)*.*

(iii) Assumptions (3.27) *and* (3.28) *are always satisfied.*

Proof of Proposition 3.4.2. (i) Follows as in the proof of Proposition 3.4.1(i), upon noting that $\{\pi \leq 0\} = (-\infty, 0]$.

(ii) Since

$$(\mathrm{d}\mathsf{Q})_u(\varphi) = \pi'\big(\Pi(\cdot,u)\big)\,\Pi_\eta(\cdot,u)\cdot\varphi,$$

if $u \in \mathsf{Q}^{-1}(\{0\})$, then $\Pi(\cdot,u) \leq 0$ a.e. on Ω and therefore $\pi'(\Pi(\cdot,u)) = 0$ a.e. on Ω because $\{\pi' = 0\} = (-\infty,0]$, which implies that $(\mathrm{d}\mathsf{Q})_u = 0$, hence its image is the closed trivial subspace $\{0\} \subseteq L^1(\Omega;\mathbb{R}^M)$ and (3.17) is satisfied.

(iii) Similar to the proof of Proposition 3.4.1(iii), using the diﬀerent expression for the diﬀerential $\mathrm{d}\mathsf{Q}$ as above. □

Proposition 3.4.3 (Case 3). *Let* $K \subseteq \mathbb{R}^N$ *be a closed set with* $K \neq \emptyset$*. Then, there exists* $\Pi \in C^\infty(\mathbb{R}^N)$ *satisfying* $K = \{\Pi = 0\} \subseteq \{\Pi_\eta = 0\}$*. Further, if one defines*

$$\mathsf{Q} \ : \ W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow L^1(\Omega), \qquad \mathsf{Q}(u) := \Pi(u),$$

and $\mathsf{E} := L^1(\Omega)$*, then we have*

$$\mathsf{Q}^{-1}(\{0\}) = \left\{ v \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \ : \ v(x) \in K, \ a.e.\ x \in \Omega \right\},$$

and Q *satisfies* (3.9)*,* (3.17)*,* (3.27) *and* (3.28)*.*

Proof of Proposition 3.4.3. It is well-known that for every such set K, there exists a function $\Pi \in C^\infty(\mathbb{R}^N)$ with the claimed properties. A proof of this fact can be found e.g. in [82, Sec. 1.1.13, p. 25] (the claimed inclusion is not explicitly stated, but follows from the method of proof by the smooth Urysohn lemma). The rest follows from Proposition 3.4.1. □

3.4.2 Integral and isoperimetric constraints

The nonlinear operator of (3.2) can also cover the following important case of constraint:

$$\int_\Omega h(\cdot, u, \mathrm{D}u)\,\mathrm{d}\mathcal{L}^n \leq H,$$

when $h : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ and $H \in \mathbb{R}$ are given.

Proposition 3.4.4. *Let $h : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ satisfy the assumptions (3.5)-(3.7) and (3.16) that f, g are assumed to satisfy, with $\alpha \leq \bar{p}$. Let also $H \in \mathbb{R}$ be given and let $\pi : \mathbb{R} \longrightarrow \mathbb{R}$ be as in (4.2). Then, by defining the operator*

$$\mathrm{Q} \ :\ W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathbb{R}, \qquad \mathrm{Q}(u) := \pi\left(\int_\Omega h(\cdot, u, \mathrm{D}u)\,\mathrm{d}\mathcal{L}^n - H\right),$$

and setting $\mathbb{E} := \mathbb{R}$, we have the following:

(i) The zero set of Q equals

$$\mathrm{Q}^{-1}(\{0\}) = \left\{ v \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) : \int_\Omega h(\cdot, v, \mathrm{D}v)\,\mathrm{d}\mathcal{L}^n \leq H \right\}$$

and assumption (3.9) is satisfied.

(ii) Q *satisfies (3.17), (3.27) and (3.28).*

Proof of Proposition 3.4.4. (i) If $\mathrm{Q}(u_m) = 0$ and $u_m \rightharpoonup u$ in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$

and assumption (3.16) for h implies that dQ is (jointly) continuous on $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \times W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$. Further, if $u \in \mathrm{Q}^{-1}(\{0\})$, then by part (i) we have

$$\int_\Omega h(\cdot,u,\mathrm{D}u)\,\mathrm{d}\mathcal{L}^n - H \le 0,$$

and therefore the first factor of $(\mathrm{dQ})_u(\varphi)$ vanishes because $\{\pi = 0\} = (-\infty, 0]$. Thus, $(\mathrm{dQ})_u = 0$ when $u \in \mathrm{Q}^{-1}(\{0\})$, and hence its image is the closed trivial subspace $\{0\} \subseteq \mathbb{R}$, yielding that (3.17) is satisfied.

(iii) For any sequences $u_m \rightharpoonup u$ in $\mathrm{Q}^{-1}(\{0\}) \subseteq W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ and $\varphi_m \rightharpoonup \varphi$ in $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ as $m \to \infty$, by part (ii) we have

$$(\mathrm{dQ})_{u_m}(\varphi_m) = 0 \longrightarrow 0 = (\mathrm{dQ})_u(\varphi)$$

as $m \to \infty$, hence (3.27) and (3.28) are satisfied. □

3.4.3 Quasilinear second order differential constraints

The operator Q of (3.2) can also cover the case of various types of nontrivial PDE constraints. As an example, we discuss the case of quasilinear divergence second order systems of PDE of the form

$$\mathrm{div}\big(A(\cdot,u,\mathrm{D}u)\big) = B(\cdot,u,\mathrm{D}u) \quad \text{in } \Omega, \tag{3.41}$$

where the coefficients maps $A : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^{N\times n}$ and $B : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N$ are given. Given the plethora of possibilities on the assumptions for such systems, the discussion in this subsection is less formal and is only aimed as a general indication of the admissible choices for Q.

Suppose that A, B are C^1 and satisfy appropriate growth bounds, and also that $P \mapsto A(\cdot,\cdot,P)$ a monotone map, and that the set of weak solutions to the system (3.41) is strongly precompact in $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$. A sufficient conditions for strong precompactness in $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ for the set of weak solutions is for example a global $C^{1,\gamma}$ or a $W^{2,1+\gamma}$ a priori uniform bound on the set of solutions, for some $\gamma \in (0,1)$. Appropriate assumptions on the coefficients A, B that allow the derivation of such a priori bounds can be found e.g. in [55] for $N = 1$ and in [52] for $N \ge 2$. Then, by defining the operator

$$\mathrm{Q} : W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow W^{-1,\bar{p}}(\Omega;\mathbb{R}^N)$$

as

$$\big\langle \mathrm{Q}(u), \psi \big\rangle := \int_\Omega \Big(A(\cdot,u,\mathrm{D}u) : \mathrm{D}\psi + B(\cdot,u,\mathrm{D}u)\cdot\psi \Big)\,\mathrm{d}\mathcal{L}^n,$$

and setting also $\mathbb{E} := W^{-1,\bar{p}}(\Omega;\mathbb{R}^N)$, assumptions (3.9), (3.17), (3.27) and (3.28) are satisfied, with

$$\mathrm{Q}^{-1}(\{0\}) = \Big\{ u \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \ : \ \mathrm{div}\big(A(\cdot,u,\mathrm{D}u)\big) = B(\cdot,u,\mathrm{D}u) \text{ weakly in } \Omega \Big\}.$$

Note first that the expression of $\mathrm{Q}^{-1}(\{0\})$ is immediate by the definition of the differential operator Q. Next, note that by assumption, for any sequence of weak solutions $(u_m)_1^\infty \subseteq W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ to (3.41), there exists $u \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ such that $u_m \longrightarrow u$ strongly along a subsequence $m_j \to \infty$. By applying this to any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ (namely sequence of solutions) for which $u_m \rightharpoonup u$ as $m \to \infty$, by passing to the limit in the weak formulation for fixed $\psi \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$, which reads

$$\int_\Omega \Big(A(\cdot,u_m,\mathrm{D}u_m) : \mathrm{D}\psi + B(\cdot,u_m,\mathrm{D}u_m)\cdot\psi \Big)\, \mathrm{d}\mathcal{L}^n = 0,$$

we get that $u \in \mathrm{Q}^{-1}(\{0\})$, as the convergence is in fact strong. Hence, (3.9) is satisfied. Further, under appropriate bounds, the operator Q is Fréchet differentiable and

$$\begin{aligned}\Big\langle (\mathrm{dQ})_u(\varphi), \psi \Big\rangle &= \int_\Omega \Big(A_\eta(\cdot,u,\mathrm{D}u)\cdot\varphi + A_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\varphi \Big) : \mathrm{D}\psi \, \mathrm{d}\mathcal{L}^n \\ &\quad + \int_\Omega \Big(B_\eta(\cdot,u,\mathrm{D}u)\cdot\varphi + B_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\varphi \Big)\cdot\psi \, \mathrm{d}\mathcal{L}^n.\end{aligned}$$

To see that the image of $(\mathrm{dQ})_u : W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow \mathbb{E}$ is closed for any fixed $u \in \mathrm{Q}^{-1}(\{0\})$, let $(T_m)_1^\infty \subseteq \mathrm{Rg}\big((\mathrm{dQ})_u\big) \subseteq \mathbb{E}$ be a sequence in the range with $T_m \longrightarrow T$ strongly in $\mathbb{E}$ as $m \to \infty$. Since $T_m \in \mathrm{Rg}\big((\mathrm{dQ})_u\big)$, exists $\varphi_m \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ solving the following linear second order system

$$\begin{aligned}-\mathrm{div}\Big(A_\eta(\cdot,u,\mathrm{D}u)\cdot\varphi_m + A_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\varphi_m \Big)& \\ + B_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\varphi_m + B_\eta(\cdot,u,\mathrm{D}u)\cdot\varphi_m &= T_m.\end{aligned}$$

By the monotonicity of the above system (due to our earlier assumption), under appropriate conditions one has a uniform bound in $W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$, yielding the weak compactness of the sequence of solutions $(\varphi_m)_1^\infty$, which establishes the closedness of $\mathrm{Rg}\big((\mathrm{dQ})_u\big) \subseteq \mathbb{E}$ and (3.17) ensues.

Finally, for any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ satisfying $u_m \rightharpoonup u$ as $m \to \infty$ and any $\varphi \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N)$, there exists $m_j \to \infty$ such that $u_m \longrightarrow u$ as $m_j \to \infty$. These facts imply that $(\mathrm{dQ})_{u_m}(u_m) \longrightarrow (\mathrm{dQ})_u(u)$ and also $(\mathrm{dQ})_{u_m}(\varphi) \longrightarrow (\mathrm{dQ})_u(\varphi)$, both strongly in $\mathbb{E}$ as $m \to \infty$. Hence, (3.27) and (3.28) are satisfied.

3.4.4 Null Lagrangians and determinant constraints

We close this paper with the observation that Theorem 3.1.1 holds true even when Q expresses a fully nonlinear pointwise Jacobian determinant constraint, or even a more general pointwise PDE constraint driven by a null Lagrangian. As an explicit example, let $n = N$ and consider the differential operator

$$\mathrm{Q} \;:\; W_0^{1,\bar{p}}(\Omega;\mathbb{R}^n) \longrightarrow W^{-1,(\bar{p}/n)}(\Omega),$$

by setting

$$\mathrm{Q}(u) := \det(\mathrm{D}u) - h,$$

for a fixed $h \in L^{\bar{p}/n}(\Omega)$, satisfying the necessary compatibility condition

$$\int_\Omega h \,\mathrm{d}\mathcal{L}^n = 0.$$

We also take

$$\mathrm{E} := W^{-1,(\bar{p}/n)}(\Omega) = \left(W_0^{1,\bar{p}/n}(\Omega)\right)^{*}.$$

Then, we have

$$\mathrm{Q}^{-1}(\{0\}) = \left\{ u \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^n) \;:\; \det(\mathrm{D}u) = h \text{ a.e. in } \Omega \right\}.$$

It follows that (3.9) is satisfied by the well-known property of weak continuity for Jacobian determinants (see e.g. [36, Th. 8.20, p. 395]). However, the situation is more complicated regarding the satisfaction of the remaining assumptions. If additionally $n = 2$, then (3.27) and (3.28) are also satisfied. Indeed, since

$$(\mathrm{dQ})_u(\varphi) = \mathrm{cof}(\mathrm{D}u) : \mathrm{D}\varphi,$$

and since for $u = \varphi$ we have the identity

$$(\mathrm{dQ})_u(u) = \mathrm{cof}(\mathrm{D}u) : \mathrm{D}u = n\det(\mathrm{D}u),$$

for any $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ with $u_m \rightharpoonup u$ as $m \to \infty$, we have

$$(\mathrm{dQ})_{u_m}(u_m) = n\det(\mathrm{D}u_m) \rightharpoonup n\det(\mathrm{D}u) = (\mathrm{dQ})_u(u)$$

in $L^{\bar{p}/2}(\Omega)$ as $m \to \infty$, whilst for any $\varphi \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^2)$ we have

$$(\mathrm{dQ})_{u_m}(\varphi) \rightharpoonup (\mathrm{dQ})_u(\varphi)$$

in $L^{\bar{p}/2}(\Omega)$ as $m \to \infty$, by the linearity of the cofactor operator when $n = 2$. Then, the compactness of the imbedding

$$L^{\bar{p}/2}(\Omega) \Subset W^{-1,(\bar{p}/2)}(\Omega)$$

implies that the above modes of convergence are in fact strong in $\mathsf{E} = W^{-1,(\bar{p}/2)}(\Omega)$. However, it is not clear when assumption (3.17) is satisfied, or when (3.28) is satisfied in

Chapter 4

Generalised Second Order Vectorial ∞-Eigenvalue Problem

4.1 Introduction and main results

Let $n, N \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \Subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\Omega$. In this paper we are interested in studying nonlinear second order L^∞ eigenvalue problems. Specifically, we investigate the problem of finding a minimising map $u_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^N$, that solves

$$
\|f(\mathrm{D}^2 u_\infty)\|_{L^\infty(\Omega)} = \inf\Big\{ \|f(\mathrm{D}^2 v)\|_{L^\infty(\Omega)} : \\ v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N), \quad \|g(v, \mathrm{D}v)\|_{L^\infty(\Omega)} = 1 \Big\}. \tag{4.1}
$$

Additionally, we pursue the necessary conditions that these constrained minimisers must satisfy, in the form of PDEs. In the above, $f : \mathbb{R}^{N\times n^2}_s \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ are given functions that will be required to satisfy some natural assumptions, to be discussed later in this section. We merely note now that $\mathbb{R}^{N\times n^2}$

The space $W^{2,\infty}_{C}(\Omega;\mathbb{R}^N)$ encompasses the case of so-called clamped boundary conditions, which can be seen as first order Dirichlet or as coupled Dirichlet-Neumann conditions, requiring $|u| = |Du| = 0$ on $\partial\Omega$. On the other hand, $W^{2,\infty}_{H}(\Omega;\mathbb{R}^N)$ encompasses the so-called hinged boundary conditions, which are zeroth order Dirichlet conditions, requiring $|u| = 0$ on $\partial\Omega$. This is standard terminology for such problems, see e.g. [69].

Problem (4.1) lies within the Calculus of Variations in L^∞, a modern area, initiated by Gunnar Aronsson in the 1960s. Since then this field has undergone a substantial transformation. There are some general complications one must be wary of when tackling L^∞ variational problems. For example, the L^∞ norm is generally not Gateaux differentiable, therefore the analogue of the Euler-Lagrange equations cannot be derived directly by considering variations. Any supremal functional also has issues with locality in terms of minimisation on subdomains. Further, the space itself lacks some fundamental functional analytic properties, such as reflexivity and separability. Higher order problems and problems involving constraints present additional difficulties and have been studied even more sparsely, see e.g. [11, 15, 30, 31, 63, 64, 66, 65, 68, 72]. In fact, this paper is an extension

To state our main result, we now introduce the required hypotheses for the functions f and g:

(*a*) $f \quad C^1(\mathrm{R}^N$

Finally, we observe that (4.3)(c), implies that $f > 0$ on $\mathbb{R}_s^{N\times n^2} \setminus \{0\}$, $f(0) = 0$ and f is radially increasing, meaning that *and* (u_p,Λ_p) *satisfies*

$$\begin{cases} \displaystyle -\!\!\!\!\!\!\int_\Omega f(\mathrm{D}^2u_p)^{p-1}\partial f(\mathrm{D}^2u_p):\mathrm{D}^2\varphi\,\mathrm{d}\mathcal{L}^n \\ \displaystyle = (\Lambda_p)^p -\!\!\!\!\!\!\int_\Omega g(u_p,\mathrm{D}u_p)^{p-1}\Big(\partial_\eta g(u_p,\mathrm{D}u_p)\cdot\varphi+\partial_P g(u_p,\mathrm{D}u_p):\mathrm{D}\varphi\Big)\,\mathrm{d}\mathcal{L}^n \end{cases} \tag{4.9}$$

for all test maps $\varphi \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$*. Finally, the measures* M_p, ν_p *are given by*

$$\begin{cases} \displaystyle \mathrm{M}_p = \frac{1}{\mathcal{L}^n(\Omega)}\left(\frac{f(\mathrm{D}^2u_p)}{\Lambda_p}\right)^{p-1}\partial f(\mathrm{D}^2u_p)\,\mathcal{L}^n \llcorner_{\Omega}, \\ \displaystyle \nu_p = \frac{1}{\mathcal{L}^n(\Omega)}\,g(u_p,\mathrm{D}u_p)^{p-1}\,\mathcal{L}^n \llcorner_{\Omega}. \end{cases} \tag{4.10}$$

We note that one could pursue optimality in Theorem 4.1.1 (A) by using L^∞ versions of quasiconvexity, as developed by Barron-Jensen-Wang [17] but adapted to this higher order case, in regards to the existence of L^∞ minimisers. However, for parts (B) and (C) of Theorem 4.1.1 regarding the necessary PDE conditions, we do need Morrey quasiconvexity, as we rely essentially on the existence of solutions to the corresponding Euler-Lagrange equations and the theory of Lagrange multipliers in the finite p case. Further, the measures $\mathrm{M}_\infty, \nu_\infty$ *depend* on the minimiser u_∞ in a non-linear fashion, hence one more could perhaps symbolise them more concisely as $\mathrm{M}_\infty(u_\infty), \nu_\infty(u_\infty)$. Consequently, the significance of these equations is currently understood to be mostly of conceptual value, rather than of computational nature. However, it is possible to obtain further information about the underlying structure of these parametric measure coefficients. This requires techniques such as measure function pairs and mollifications up to the boundary as in [31, 57, 65], but to keep the presentation as simple as possible, we refrain from pursuing this considerably more technical endeavour, which also requires stronger assumptions.

4.2 Proofs

In this section we establish Theorem 4.1.1. Its proof is not labeled explicitly, but will be completed by proving a combination of smaller subsidiary results, including a selection of lemmas and propositions.

Before introducing the approximating problem (the L^p case for finite p), we need to establish a convergence result, which shows that the admissible classes of the p-problems are non-empty. It is required because the function g appearing in the constraint is not assumed to be homogeneous, therefore a standard scaling argument does not suffice.

Lemma 4.2.1. *For any $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \setminus \{0\}$, there exists $(t_p)_{p\in(n/\alpha,\infty]}$ with $t_p \to t_\infty$ as $p \to \infty$, such that*

$$\left\| g\left(t_p v, t_p \mathrm{D}v\right)\right\|_{L^p(\Omega)} = 1,$$

for all $p \in (n/\alpha, \infty]$. Further, if $\|g(v, \mathrm{D}v)\|_{L^\infty(\Omega)} = 1$, then $t_\infty = 1$.

Proof of Lemma 4.2.1. Fix $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \setminus \{0\}$ and define

$$\rho_\infty(t) := \max_{x\in\overline{\Omega}} g\left(tv(x), t\mathrm{D}v(x)\right), \qquad t \geq 0.$$

It follows that $\rho_\infty(0) = 0$ and ρ_∞ is continuous on $[0,\infty)$. We will now show that ρ_∞ is strictly increasing. We first show it is non-decreasing. For any $s > 0$ and $(\eta, P) \in \mathbb{R}^N \times \mathbb{R}^{N\times n} \setminus \{(0,0)\}$, our assumption (4.4)(c) implies

$$\begin{aligned}
0 \;&<\; \frac{C_7 g(s\eta, sP)}{s}\\
&\leq\; \partial_\eta g(s\eta, sP)\cdot\eta \;+\; \partial_P g(s\eta, sP) : P\\
&=\; \partial_{(\eta,P)} g(s\eta, sP) : (\eta, P)\\
&=\; \frac{\mathrm{d}}{\mathrm{d}s}\left(g(s\eta, sP)\right),
\end{aligned}$$

thus $s \to g(s\eta, sP)$ is increasing on $(0,\infty)$. Hence, for any $x \in \overline{\Omega}$ and $t > s \geq 0$ we have $g(sv(x), s\mathrm{D}v(x)) \geq g(tv(x), t\mathrm{D}v(x))$, which yields,

$$\rho_\infty(s) = \max_{x\in\overline{\Omega}} g\left(sv(x), s\mathrm{D}v(x)\right) \leq \max_{x\in\overline{\Omega}} g\left(tv(x), t\mathrm{D}v(x)\right) = \rho_\infty(t).$$

We proceed to demonstrate that $t \to \rho_\infty(t)$ is actually strictly monotonic over $(0,\infty)$. Fix $t_0 > 0$. By Danskin's theorem [37], the derivative from the right $\rho'(t_0^+)$ exists, and is given by the formula

$$\rho'_\infty(t_0^+) = \max_{x\in\Omega_{t_0}} \left\{ \partial_{(\eta,P)} g(t_0 v(x), t_0\mathrm{D}v(x)) : \left(v(x), \mathrm{D}v(x)\right)\right\},$$

where

$$\Omega_{t_0} := \left\{ \overline{x} \in \overline{\Omega} \;:\; \rho_\infty(t_0) = g\left(t_0 v(\overline{x}), t_0\mathrm{D}v(\overline{x})\right)\right\}.$$

Hence, by (4.4)(c) we estimate

$$\begin{aligned}\rho_\infty(t_0^+) &= \frac{1}{t_0}\max_{x\in\Omega_{t_0}}\Big(\partial_{(\eta,P)}g(t_0v(x),t_0\mathrm{D}v(x)):\big(t_0v(x),t_0\mathrm{D}v(x)\big)\Big)\\ &\geq \frac{C_7}{t_0}\max_{x\in\Omega_{t_0}} g\big(t_0v(x),t_0\mathrm{D}v(x)\big)\\ &= \frac{C_7}{t_0}\rho_\infty(t_0)\\ &> 0.\end{aligned}$$

This implies that ρ_∞ is strictly increasing on $(0,\infty)$. Next, recall that g is coercive by assumption (4.4)(b), namely $g(sv$

Since $g(tv, t\mathrm{D}v)^p \to \infty$ pointwise on $\{(v, \mathrm{D}v) \neq (0, 0)\}$ as $t \to \infty$, by the monotone convergence theorem, we infer that

$$\int_{\{(v,\mathrm{D}v)\neq(0,0)\}} g(tv, t\mathrm{D}v)^p \,\mathrm{d}\mathcal{L}^n \longrightarrow \infty,$$

as $t \to \infty$. As a consequence, $\rho_p(t) \to \infty$ as $t \to \infty$. Since $\rho_p(0) = 0$, by the intermediate value theorem there exists $t_p > 0$ such that $\rho_p(t_p) = 1$, namely

$$\big\| g(t_p v, t_p \mathrm{D}v)\big\|_{L^p(\Omega)} = 1.$$

For the sake of contradiction, suppose that $t_p \nrightarrow t_\infty$, as $p \to \infty$. In this case, there exists a subsequence $(t_{p_j})_1^\infty \subseteq (n/\alpha, \infty)$ and $t_0 \in [0, t_\infty) \cup (t_\infty, \infty]$ such that $t_{p_j} \to t_0$ as $j \to \infty$. Further, $(t_{p_j})_1^\infty$ can assumed to be either monotonically increasing or decreasing. We first prove that t_0 is finite. If $t_0 = \infty$, then the sequence $(t_{p_j})_1^\infty$ can be selected to be monotonically increasing. Therefore, by arguing as before, $g(t_{p_j} v, t_{p_j} \mathrm{D}v) \nearrow \infty$ as $j \to \infty$, pointwise on $\{(v, \mathrm{D}v) \neq (0, 0)\}$, and the monotone convergence theorem provides the contradiction

$$1 = \lim_{j\to\infty} -\!\!\!\!\!\!\int_\Omega g(t_{p_j} v, t_{p_j} \mathrm{D}v)^{p_j} \,\mathrm{d}\mathcal{L}^n = -\!\!\!\!\!\!\int_\Omega \lim_{j\to\infty} g(t_{p_j} v, t_{p_j} \mathrm{D}v)^{p_j} \,\mathrm{d}\mathcal{L}^n = \infty.$$

Consequently, we have that $t_0 \in [0, t_\infty) \cup (t_\infty, \infty)$. Since $(t_{p_j} v, t_{p_j} \mathrm{D}v) \to (t_0 v, t_0 \mathrm{D}v)$ uniformly on $\overline{\Omega}$ as $j \to \infty$, we calculate

$$\begin{aligned} 1 &= \big\| g(t_{p_j} v, t_{p_j} \mathrm{D}v)\big\|_{L^{p_j}(\Omega)} \\ &= \big\| g(t_0 v, t_0 \mathrm{D}v)\big\|_{L^{p_j}(\Omega)} + \mathrm{o}(1)_{j\to\infty} \\ &= \big\| g(t_0 v, t_0 \mathrm{D}v)\big\|_{L^\infty(\Omega)} + \mathrm{o}(1)_{j\to\infty} \\ &= \rho_\infty(t_0) + \mathrm{o}(1)_{j\to\infty}. \end{aligned}$$

By passing to the limit as $j \to \infty$, we obtain a contradiction if $t_\infty \neq t_0$, because ρ_∞ is a strictly increasing function and $\rho_\infty(t_\infty) = 1$. In conclusion, $t_p \to t_\infty$ as $p \to \infty$. □

Utilising the above result we can now show existence for the approximating minimisation problem for $p < \infty$.

Lemma 4.2.2. *For any $p > n/\alpha$, the minimisation problem (4.9) has a solution $u_p \in W^{2,\alpha p}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$.*

Proof of Lemma 4.2.2. Let us fix $p \in (n/\alpha, \infty)$ and $v_0 \in W^{2,\infty}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$ where $v_0 \not\equiv 0$. By application of Lemma 4.2.1, there exists $t_p > 0$ such that $\|g(t_p v_0, t_p \mathrm{D} v_0)\|_{L^p(\Omega)} = 1$ implying that $t_p v_0$ is indeed an element of the admissible class of (4.9). Hence, we deduce that the admissible class is non empty. Further, by assumption (4.3)(b), f is (Morrey) quasiconvex. We now confirm that f^p is also (Morrey) quasiconvex function, as a consequence of Jensen's inequality: for any fixed $X \in \mathbb{R}^{N \times n^2}_s$ and any $\varphi \in W^{2,\infty}_0(\Omega; \mathbb{R}^N)$, we have

$$f^p(X) \leq \left(\fint_\Omega f(X + \mathrm{D}^2\varphi)\, \mathrm{d}\mathcal{L}^n \right)^p \leq \fint_\Omega f(X + \mathrm{D}^2\varphi)^p\, \mathrm{d}\mathcal{L}^n.$$

By assumption by assumption (4.3)(d), we have for some new $C_5(p), C_6(p) > 0$ that

$$f(X)^p \leq C_5(p)|X|^{\alpha p} + C_6(p),$$

for any $X \in \mathbb{R}^{N \times n^2}_s$. Moreover, by [95, Theorem 3.6] we have that the functional $v \to \|f(\mathrm{D}^2 v)\|_{L^p(\Omega)}$ is weakly lower semi-continuous on $W^{2,\alpha p}(\Omega; \mathbb{R}^N)$ and therefore the same is true over the closed subspace $W^{2,\alpha p}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$. Let $(u_i)_1^\infty$ be a minimising sequence for (4.9). As $f \geq 0$, it is clear that $\inf_{i \in \mathbb{N}} \|f(\mathrm{D}^2 u_i)\|_{L^p(\Omega)} \geq 0$. Since the admissible class is non-empty, the infimum is finite. Additionally, by (4.3)(d), we have the bound

$$\begin{aligned}
\inf_{i \in \mathbb{N}} \|f(\mathrm{D}^2 u_i)\|_{L^p(\Omega)} &\leq \left\| f\left(\mathrm{D}^2(t_p v_0)\right)\right\|_{L^p(\Omega)} \\
&\leq \left\| C_5 \left|t_p \mathrm{D}^2 v_0\right|^\alpha + C_6 \right\|_{L^\infty(\Omega)} \\
&\leq C_5 (t_p)^\alpha \|\mathrm{D}^2 v_0\|^\alpha_{L^\infty(\Omega)} + C_6 \\
&< \infty.
\end{aligned}$$

Now we show that the functional is coercive in $W^{2,\alpha p}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$, arguing separately for either case of boundary conditions. By assumption (4.3)(d) and the Poincaré inequality, for any $u \in W^{2,\alpha p}_{\mathrm{C}}(\Omega; \mathbb{R}^N)$ (satisfying $|u| = |\mathrm{D}u| = 0$ on $\partial\Omega$), we have

$$\left(\fint_\Omega \left(f(\mathrm{D}^2 u) + C_3\right)^p \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \geq C_4 \left(\fint_\Omega |\mathrm{D}^2 u|^{\alpha p}\, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \geq C_4 \|u\|^\alpha_{W^{1,\alpha p}(\Omega)},$$

for a new constant $C_4 = C_4(p) > 0$. Hence, for any $u \in W^{2,\alpha p}_{\mathrm{C}}(\Omega; \mathbb{R}^N)$,

$$\|f(\mathrm{D}^2 u)\|_{L^p(\Omega)} \geq C_4 \left(\|u\|_{W^{2,\alpha p}(\Omega)} \right)^\alpha - C_3. \tag{4.11}$$

The above estimate is also true when $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega; \mathbb{R}^N)$, but since in this case we have only $|u| = 0$ on $\partial\Omega$, it requires an additional justification. By the Poincaré-Wirtinger inequality involving averages, for any $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$ we have

$$\left\| \mathrm{D}u - \fint_\Omega \mathrm{D}u \, \mathrm{d}\mathcal{L}^n \right\|_{L^{\alpha p}(\Omega)} \leq C \, \|\mathrm{D}^2 u\|_{L^{\alpha p}(\Omega)},$$

where $C = C(\alpha, p, \Omega) > 0$ is a constant. Since $|u| = 0$ on $\partial\Omega$, by the Gauss-Green theorem we have

$$\int_\Omega \mathrm{D}u \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} u \otimes \hat{n} \, \mathrm{d}\mathcal{H}^{n-1} = 0,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. In conclusion,

$$\|\mathrm{D}u\|_{L^{\alpha p}(\Omega)} \leq C \, \|\mathrm{D}^2 u\|_{L^{\alpha p}(\Omega)},$$

for any $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$. The above estimate together with the standard Poincaré inequality applied to u itself allow to infer that (4.11) holds for any $u \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ in both cases of boundary conditions. Returning to our minimising sequence, by standard compactness results, exists $u_p \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$ such that $u_i \rightharpoonup u_p$ in $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, as $i \to \infty$ along a subsequence of indices. Additionally, by the Morrey estimate we have that $u_i \longrightarrow u_p$ in $C^1(\overline{\Omega};\mathbb{R}^N)$ as $i \to \infty$, along perhaps a further subsequence. Since $u \mapsto \|g(u, \mathrm{D}u)\|_{L^p(\Omega)}$ is weakly continuous on $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, the admissible class is weakly closed in $W^{2,\alpha p}(\Omega;\mathbb{R}^N)$ and hence we may pass to the limit in the constraint. By weak lower semicontinuity of the functional, it follows that a minimiser u_p which satisfies (4.9) does indeed exist. □

Now we describe the necessary conditions (Euler-Lagrange equations) that approximating minimiser u_p must satisfy. These equations will involve a Lagrange multiplier, emerging from the constraint $\|g(\cdot, \mathrm{D}(\cdot))\|_{L^p(\Omega)} = 1$.

Lemma 4.2.3. *For any $u \in W$*

In particular, it follows that in both cases u_p is a weak solution in $W^{2,\alpha p}(\Omega;\mathbb{R}^N)$ to

$$\begin{cases} \mathrm{D}^2 : \Big(f(\mathrm{D}^2 u_p)^{p-1} \partial f(\mathrm{D}^2 u_p) \Big) \\ \quad = \lambda_p \Big(g(u_p, \mathrm{D}u_p)^{p-1} \partial_\eta g(u_p, \mathrm{D}u_p) - \mathrm{div}\Big(g(u_p, \mathrm{D}u_p)^{p-1} \partial_P g(u_p, \mathrm{D}u_p) \Big)\Big), \end{cases} \tag{4.12}$$

where we have used the notation $\mathrm{D}^2 : F = \sum_{i,j=1}^{n} \mathrm{D}^2_{ij} F_{ij}$, when $F \in C^2(\Omega;\mathbb{R}^{n\times n})$, which is equivalent to the double divergence (applied once column-wise and once row-wise). Note that in the case of hinged boundary data, we have an additional natural boundary condition arising (since $\mathrm{D}u$ is free on $\partial\Omega$

As $f, g \geq 0$ we can manipulate the respective assumptions (4.3)(c) and (4.4)(c) to produce the following bounds:

$$
\begin{aligned}
C_1 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n &\leq \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1} \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 u_p \,\mathrm{d}\mathcal{L}^n \\
&\leq C_2 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n, \\
C_7 \fint_\Omega g(u_p, \mathrm{D}u_p)^p \,\mathrm{d}\mathcal{L}^n &\leq \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big[\partial_\eta g(u_p, \mathrm{D}u_p) \cdot u_p + \\
&\quad + \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}u_p \Big] \,\mathrm{d}\mathcal{L}^n \\
&\leq C_8 \fint_\Omega g(u_p, \mathrm{D}u_p)^p \,\mathrm{d}\mathcal{L}^n.
\end{aligned}
$$

The above two estimates, combined with the Euler-Lagrange equations, imply that $\lambda_p > 0$. Hence, we may therefore define $\Lambda_p := (\lambda_p)^{\frac{1}{p}} > 0$. We will now obtain the upper and lower bounds. We determine the lower bound as follows:

$$
\begin{aligned}
C_1 (L_p)^p &= C_1 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n \\
&\leq \fint_\Omega f^{p-1}(\mathrm{D}^2 u_p) \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 u_p \,\mathrm{d}\mathcal{L}^n \\
&= \lambda_p \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big[\partial_\eta g(u_p, \mathrm{D}u_p) \cdot \varphi + \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}u_p \Big] \,\mathrm{d}\mathcal{L}^n \\
&\leq \lambda_p C_8.
\end{aligned}
$$

Hence,

$$
\left(\frac{C_1}{C_8} \right)^{\frac{1}{p}} L_p \leq (\lambda_p)^{\frac{1}{p}} = \Lambda_p.
$$

The upper bound isppue7(o)-2rs:

Proof of Proposition 4.2.5. Fix $p > n/\alpha, q \leq p$ and a map v_0 $W^{2,\infty}_{B}(\Omega; R^N) \setminus \{0\}$.
Then, by Lemma 4.2.1 there exists (t_p)

for any such v. By the weak lower semi-continuity of the functional on $W^{2,\alpha q}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, we may let $p_j \to \infty$ to obtain

$$
\begin{aligned}
\big\| f(\mathrm{D}^2 u_\infty)\big\|_{L^q(\Omega)} &\leq \liminf_{p_j\to\infty} L_p \\
&\leq \limsup_{p_j\to\infty} L_p \\
&\leq \limsup_{p_j\to\infty} \big\| f(t_{p_j}\mathrm{D}^2 v)\big\|_{L^p(\Omega)} \\
&= \big\| f(\mathrm{D}^2 v)\big\|_{L^\infty(\Omega)}.
\end{aligned}
$$

Now we may let $q \to \infty$ in the above bound, hence producing

$$
\big\| f(\mathrm{D}^2 u_\infty)\big\|_{L^\infty(\Omega)} \leq \liminf_{p_j\to\infty} L_p \leq \limsup_{p_j\to\infty} L_p \leq \big\| f(\mathrm{D}^2 v)\big\|_{L^\infty(\Omega)}.
$$

for all mappings $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ satisfying the constraint $\| g(v,\mathrm{D}v)\|_{L^\infty(\Omega)} = 1$. If we additionally show that in fact u_∞ satisfies the constraint in (4.1), then the above estimate shows both that it is the desired minimisers (by choosing $v := u_\infty$), and also that the sequence $(L_{p_j})_1^\infty$ converges to the infimum. Now we show that this is indeed the case. In view of assumption (4.3)(d), the previous estimate implies also that $\mathrm{D}^2 u_\infty \in L^\infty\big(\Omega;\mathbb{R}^{N\times n^2}_s\big)$, which toge[illegible]

0. We now show that $\Lambda_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. By our earlier energy estimate, we have $L_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. By Lemma 4.2.4, we have

$$0 < \lim_{p_j \to \infty} \left(\frac{C_1}{C_8}\right)^{\frac{1}{p}} L_p \leq \lim_{p_j \to \infty} \Lambda_p \leq \lim_{p_j \to \infty} \left(\frac{C_2}{C_7}\right)^{\frac{1}{p}} L_p,$$

and therefore $\Lambda_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. The result ensues. □

Lemma 4.2.6. *For any $p > (n/\alpha) + 2$, there exist measures $\nu_\infty \in \mathcal{M}(\overline{\Omega})$ and $\mathrm{M}_\infty \in \mathcal{M}(\overline{\Omega}; \mathbb{R}_s^{N \times n^2})$ such that, along perhaps a further sequence $(p_j)_1^\infty$ of exponents, we have*

$$\begin{cases} \nu_p \rightharpoonup \nu_\infty, \\ \\ \end{cases}$$

Hence,

$$
\begin{aligned}
\mathrm{M}_p(\overline{\Omega}) &\leq \frac{C_5}{\Lambda_p^{p-1}} \left(⨍_{\Omega} f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n \right)^{\frac{p-1+\beta}{p}} + \frac{C_6}{\Lambda_p^{p-1}} \left(⨍_{\Omega} f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n \right)^{\frac{p-1}{p}} \\
&= C_5 \frac{(L_p)^{p-1+\beta}}{\Lambda_p^{p-1}} + C_6 \frac{(L_p)^{p-1}}{\Lambda_p^{p-1}} \\
&= \left(\frac{L_p}{\Lambda_p} \right)^{p-1} \left(C_5 L_p^{\beta} + C_6 \right) \\
&\leq \left(\frac{C_8}{C_1} \right)^{1-\frac{1}{p}} C_5(\Lambda_\infty + 1)
\end{aligned}
$$

Chapter 5

Conclusion and Future Work

In this chapter we discuss the main conclusions drawn from Chapters 2, 3, 4 and how these relate to the aim of our thesis, mentioned in Chapter 1. Several directions for further work are outlined, also how we can surpass some of the limitations within our current work.

5.1 Conclusions

In conclusion, this thesis is a collection of papers, presented as chapters, that are comprised of original research. This work consists of current progress in the field of vectorial Calculus of Variations in L^∞. These contemporary results are varied in nature and include the contemplation of new problems and the generalisation of previously existing theory. For example, Chapter 2 is a novel consideration, whilst Chapters 3 and 4 are extensions of previous publications.

The main results throughout this thesis are concerned with establishing conditions, that constrained supremal functionals must satisfy. Specifically, the results are Theorems 2.1.3, 3.1.3 and 4.1.1. These results are built on the methodology of L^p approximations, where we have explored sophisticated contrasting limiting processes. Given the anatomy of the vectorial environment, we could not employ the intrinsic characterisation that exists for scalar problems. The technique of L^p approximations was the only means we had available to us, to tackle the problem of finding such conditions in the L^∞ setting.

In each chapter we have pursued the same goal, whilst varying the nature of the investigation. We have noted how minor adaptations to a constraint can have far reaching

Once we achieved our main intentions in Chapter 2, we started to investigate more comprehensive problems, beyond the specificity of the Navier-Stokes equations. This led one could investigate the construction of numerical methods. Note that our results for finite p should not be disregarded and could potentially support the discretisation process of the problems presented in this thesis.

A possible extension to Chapter 2 would be to consider the same motivational notion of variational data assimilation, but constrain the process by a different equation. The structure of the equation will certainly modify the techniques required in the limiting process. Depending on the choice of equation, this research could produce outcomes that are of theoretical and applications based interests. For instance, variational data assimilation can also be used to model traffic flow. We could constrain the minimisation process by relevant conservation laws.

A further augmentation of Chapter 2 would be to look at the same problem but relax some of our assumptions. For instance, strong solutions can be quite restrictive, hence we could limit our attention to only weak solutions of the Navier-Stokes equations. This is a completely different investigation. We would need to reestablish coercivity for the functional, to deduce relevant bounds, to substantiate any form of compactness. This would involve deriving a new bound for the solutions of Navier-Stokes equations, under less regularity than (

(*a*) f $C^1(\mathrm{R}_s^{N\times n^3})$.

(*b*) f

Appendix A

Additional Bound for the Operator $\mathfrak{M}_p$

Here we provide a proof for the $L^{p'}$ bound mentioned in Chapter 2 (page 15). Recall that for any $M \in \mathbb{N}$ and $p \in (1, \infty)$, we define the operator

$$\mathfrak{M}_p \ : \ L^p(\Omega_T; \mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T; \mathbb{R}^M),$$

where $p' := p/(p-1)$, by setting

$$\mathfrak{M}_p(V) := \frac{|V|_{(p)}^{p-2} V}{\big\| V \big\|_{L^p(\Omega_T)}^{p-1}}.$$

Here $|\cdot|_{(p)}$ is the regularisation of the Euclidean norm of $\mathbb{R}^M$.

Lemma A.0.1. *. We have*

$$\Big\| \mathfrak{M}_p(V) \Big\|_{L^{p'}(\Omega_T)} \leq 1,$$

and therefore $\mathfrak{M}_p$ is valued in the unit ball of $L^{p'}(\Omega_T; \mathbb{R}^M)$.

Proof of Lemma

Appendix B

The Modified Hölder Inequality

Here we establish the proof for the modified Hölder inequality used in Chapter 2 (page 18, 23).

Lemma B.0.1. *For any $1 \le q \le p < \infty$ and $h \in L^p(X)$, we have the inequality*

$$\|h\|_{L^q(X)} \le \|h\|_{L^p(X)} + \sqrt{q^{-2} - p^{-2}}.$$

Proof of Lemma B.0.1 Set

$$|f|_{(p)} := \left(|f|^2 + p^{-2}\right)^{\frac{1}{2}}, \text{ where } \|f\|_{L^p(X)} = \left(\fint_X |f|_{(p)}^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned}
\|f\|_{L^q(X)} = \left(\fint_X |f|_{(q)}^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} &= \left(\fint_X \left(|f|^2 + q^{-2} + p^{-2} - p^{-2}\right)^{\frac{q}{2}} \mathrm{d}\mu\right)^{\frac{1}{q}} \\
&= \left(\fint_X \left(|f|^2 + p^{-2} + \left(\sqrt{q^{-2} - p^{-2}}\right)^2\right)^{\frac{q}{2}} \mathrm{d}\mu\right)^{\frac{2}{q}\cdot\frac{1}{2}} \\
&\le \sqrt{\left(\fint_X \left(|f|^2 + p^{-2}\right)^{\frac{q}{2}} \mathrm{d}\mu\right)^{\frac{2}{q}} + q^{-2} - p^{-2}} \\
&\le \sqrt{\left(\fint_X \left(|f|^2 + p^{-2}\right)^{\frac{p}{2}} \mathrm{d}\mu\right)^{\frac{2}{p}}} + \sqrt{q^{-2} - p^{-2}} \\
&= \|f\|_{L^p(X)} + \sqrt{q^{-2} - p^{-2}}.
\end{aligned}$$

□

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