

M A T H E M A T I C S D E P A R T M E N T

Black Oil Reservoir Simulation in 1-D via the Higher Order
Godunov Method

E.M. Dicks

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U N I V E R S I T Y O F R E A D I N G

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Department of Mathematics
P.O. Box 220
University of Reading
Whiteknights
Reading
RG6 2AX
United Kingdom

Abstract

In this report we review the Black-Oil model for petroleum reservoir simulation. The model includes compressibility and general mass transfer effects. The flow equation are formulated sequentially and split into a parabolic pressure equation and a hyperbolic system. We also review the Higher Order Godunov method which is used to discretise the hyperbolic part of the system. A possible technique of reducing the volume error discrepancy inherent in this sequential formulation is presented and numerical results given.

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The Black-Oil model is the industry standard phase behaviour model most often used in reservoir simulation and includes compressibility and mass transfer effects between the phases. The fluid flow exhibits both hyperbolic and parabolic behaviour, both of which can be treated by a fully implicit discretisation. However to effectively treat both types of behaviour present in the flow equations different types of numerical procedure are required for each and so a splitting of the flow equations into a system of hyperbolic conservation laws and a parabolic pressure

2 The Black-Oil Equations

2.1 Overview of model

The Black-Oil model [2] is the Industry standard phase behaviour model most often used in petroleum reservoir simulation. It includes compressibility and general mass transfer effects between the phases that are needed to model primary (pressure depletion) and secondary (water injection) recovery.

The flow equations are derived from four main principles:

- i) Phase equilibrium - determines how the components combine to form phases,
- ii) equation of state - states that the fluid fills the pore volume,
- iii) Darcy's law for the volumetric flow rates,
- iv) Mass conservation equations for each component.

The flow equations are formulated sequentially, therefore they cannot be satisfied exactly at each step of the computation, and so some incompatibility is introduced. We satisfy phase equilibrium, Darcy's law and the component conservation laws exactly but the equation of state is linearised so that it is only satisfied approximately. This splitting of the flow equation is termed a 'volume-discrepancy splitting'. We now consider the equations in detail.

2.2 Thermodynamic equilibrium

2.2.1 Components and Phases

The reservoir is considered to be composed of three components oil, gas, and water. To reach phase equilibrium these components combine to form at most three phases, liquid, vapour and aqua. At each point in the reservoir the components associate into phases in order to attain thermodynamic equilibrium. The components flow in phases but their total mass is conserved, thus it is necessary to find how the mass of each component is apportioned into the phases. This phase equilibrium problem can be expressed as follows: given the pressure p and the vector $\mathbf{n} = [n_o, n_g, n_w]^T$ of mass component densities, find the matrix

$$N = \begin{bmatrix} n_{ol} & n_{ov} & 0 \\ n_{gl} & n_{gv} & n_{ga} \\ 0 & 0 & n_{wa} \end{bmatrix}$$

of component densities in each of the phases, subject to the mass balance condition

$$\mathbf{n} = N\mathbf{e},$$

where $\mathbf{e}$ is a vector of one's. Thus it can be seen from the definition of N that there is a restriction on the way the components are allowed to mix in the phases. i.e. oil may be allowed in the liquid and vapour phases, gas in all three phases and water in the aqueous phase only. Oil and water do not mix and steam is not treated.

With each phase we associate a principal component, oil with the liquid phase, gas with the vapour phase and water with the aqueous phase. The amount of each component in each phase is related to the amount of the principal component in that phase by the ratio matrix $\mathbf{r}$. When all three phases are formed $\mathbf{r}$ is given by

$$\mathbf{r} = \begin{pmatrix} 1 & r_v & 0 \\ r_l & 1 & r_a \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{N}^{-1}$$

where $\mathbf{N}$ is the diagonal part of $\mathbf{r}$. The ratios r_l , r_v , and r_a must be non-negative functions of pressure and are supplied as part of the model.

Now we can determine how many phases are present. Firstly we define $\mathbf{p} = \mathbf{r}^{-1}$, then form the vector $\mathbf{p} \cdot \mathbf{c}$. If all three entries of $\mathbf{p} \cdot \mathbf{c}$ are positive then all three phases are formed and the fluid is termed saturated. Hence it can be shown that $\mathbf{N}$ is given by

$$\mathbf{N} =$$

and the solution to phase equilibrium by

$$\mathbf{N}$$

$$l$$

$$v$$

$$l \quad v \quad a$$

$$b$$

$$l \quad b \quad o \quad g \quad a \quad b \quad w$$

$$\begin{array}{c} \text{---} \\ v \end{array} \quad \begin{array}{c} \text{---} \quad o \\ g \quad a \quad w \end{array}$$

$$\text{---} \quad \text{---} \quad N \quad T \quad N$$

For example in the case of the vapour phase being undersaturated, $\mathbf{f}_v$ is given by

$$\mathbf{f}_v = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Throughout, quantities denoted with an overbar refer to the reduced matrix or vector in the undersaturated case. The solution to phase equilibrium is then given by

$$\mathbf{f} = \mathbf{f}_N^{-1} \mathbf{T}$$

Water and oil have small compressibilities, but the relatively high compressibility of gas, and the swelling effects caused by gas dissolving in liquid lead to important volume changes at reservoir pressures. The Black-Oil model incorporates these volume changes by relating the volumes of each of the phases to the amounts of the principal component in that phase. To quantify this relationship we first define $\mathbf{V}_u$ to be the diagonal matrix of phase volumes per pore volume. The entries of $\mathbf{V}_N$ and $\mathbf{V}_u$ are related by the formation volume factors,

$$V_{l,i} = \frac{V_{l,i}}{V_{v,i}} = \frac{V_{l,i}}{V_{a,i}}$$

which are functions of pressure and the undersaturation variable u_i in the case of undersaturated flow. In the case of undersaturated liquid u_i is the bubble

Darcy’s law specifies how the phases flow through the reservoir and gives the vector of flow volumes per unit area per unit time, i.e. phase velocities, which in matrix-vector form is

$$= \left(\begin{array}{c} - \\ - \end{array} \right) \tag{2.1}$$

i.e. Darcy’s law states that the fluid flow is due to a pressure gradient and a gravitational potential. is the diagonal matrix of phase mobilities which are phase relative permeabilities divided by phase viscosities ($\frac{k_p}{\mu_p}$). Relative

$$\overline{T}$$

$$T$$

$$\frac{\partial d}{\partial x}$$

$$T \quad T \quad T \quad \text{---} \quad T \quad \text{---}$$

$$\frac{T}{T} \quad \frac{T}{T} \quad \text{---}$$

$$T$$

$$T_{t+\Delta t} \quad T_t \quad T \text{---} \quad T \text{---}$$

$$T$$

$$T \quad \text{---} \quad T \text{---} \quad T \text{---}$$

and the quantity $\mathbf{e}^T \mathbf{u} - 1$ is termed the volume-error-discrepancy.

Multiplication by the porosity ϕ and use of the mass conservation laws to replace the time derivative of $\mathbf{n}$ results in

$$c \frac{\partial p}{\partial t} + \mathbf{w}^T \frac{\partial}{\partial x} (\mathbf{f} v_T + \mathbf{g} \tau) = q, \quad (2.5)$$

with the total velocity v_T being given by

$$v_T = [-\frac{\partial p}{\partial x} + \gamma] \tau, \quad (2.6)$$

where the coefficients c , $\mathbf{w}$, $\mathbf{f}$, $\mathbf{g}$, γ , q and τ depend on the pressure p and component density $\mathbf{n}$. Since we use c and τ explicitly we define them here

$$c = \phi \mathbf{e}^T \frac{\partial \mathbf{u}}{\partial p} - \mathbf{e}^T \mathbf{u} \frac{\partial \phi}{\partial p},$$

$$\tau = \mathbf{e}^T L \mathbf{e} \kappa.$$

For more details and definitions of the other coefficients see [5].

To examine the character of (2.5) we need to examine the coefficient c of $\frac{\partial p}{\partial t}$ and the coefficient of $\frac{\partial^2 p}{\partial x^2}$ which is obtained after substitution of (2.6) into (2.5) and is given by

$$-\mathbf{w}^T \mathbf{f} \tau = -\mathbf{e}^T L \mathbf{e} \kappa,$$

since $\mathbf{w}$ and $\mathbf{f}$ satisfy $\mathbf{w}^T \mathbf{f} = 1$. The transmissibility τ is positive, hence the coefficient of $\frac{\partial^2 p}{\partial x^2}$ is negative, and therefore for (2.5) to be parabolic we require c to be negative. As pressure increases the rock occupies a smaller volume hence porosity ϕ is a non-decreasing function of pressure, $\frac{\partial \phi}{\partial p} \geq 0$. Therefore for c to be negative we require $\mathbf{e}^T \frac{\partial \mathbf{u}}{\partial p} < 0$ which is the condition of negative total fluid compressibility. This is guaranteed by placing restrictions on the formation volume factors B and ratios R .

Given the sequential formulation described, it is therefore accepted that mass is conserved at the expense of developing a volume error discrepancy .

2.5 Conservation of mass

We require that the mass of each component is conserved. The matrix N_u^{-1} represents the density of each component in each phase, hence the conservation of mass equations are

$$\frac{\partial(\mathbf{n}\phi)}{\partial t} + \frac{\partial(N_u^{-1} \mathbf{v})}{\partial x} = 0. \quad (2.7)$$

The flux function $\mathbf{h} = N_u^{-1} \mathbf{v}$ can also be expressed as $\mathbf{h} = R B^{-1} \mathbf{v}$, which is the form that will be used in the characteristic analysis. The flux is a function of $\mathbf{n}$, p and v_T , hence in quasi-linear form we have

$$\phi \frac{\partial \mathbf{n}}{\partial t} + \frac{\partial \mathbf{h}}{\partial \mathbf{n}} \frac{\partial \mathbf{n}}{\partial x} = -\frac{\partial \mathbf{h}}{\partial p} \frac{\partial p}{\partial x} - \frac{\partial \mathbf{h}}{\partial v_T} \frac{\partial v_T}{\partial x}. \quad (2.8)$$

The system is hyperbolic if and only if

$$H = \frac{\partial \mathbf{h}}{\partial \mathbf{n}}$$

has real eigenvalues for all values of $\frac{\partial \mathbf{h}}{\partial \mathbf{n}}$. The terms on the right hand side of (2.8) are source terms which must be included in the numerical scheme to obtain

$$T$$

$$-1\qquad T$$

$$c$$

$$c$$

$$T$$

$$\rule{1cm}{0.4pt}$$

$$c$$

$$c$$

$$c$$

$$c$$

$$c\quad c$$

$$\frac{1}{\mathbf{e}^T\mathbf{u}}\quad c$$

$$\rule{1cm}{0.4pt}-1\qquad \overline{v}$$

$$\overline{v}\qquad \overline{T}\qquad \overline{v}$$

$$\rule{1cm}{0.4pt}$$

$$\overline{v}$$

$$\overline{v}$$

$$\overline{v}$$

$$\begin{matrix} * \\ a \end{matrix}$$

$$\overline{v}$$

$$\frac{\partial v_a}{\partial s_a}$$

$$\frac{\partial v_a}{\partial s_{\star}}$$

$$*$$

$$\frac{1}{\mathbf{e}^T\mathbf{u}}\quad \overline{v}$$

$$\frac{l\quad o\quad \frac{v_l}{u_l}\qquad a\quad w\quad \frac{v_a}{u_a}}{l\quad o\qquad a\quad w}$$

$$\frac{v}{v}$$

To solve the system of flow equations arising from the Black-Oil model we need to use a numerical approximation. To provide accurate resolution of the sharp fronts typically found in oil reservoir simulations, requires a high order method. Hence we use Higher Order Godunov discretisations of the conservation laws. The pressure equation is solved via an implicit discretisation.

The Higher Order Godunov method [1] achieves second order accuracy in both space and time. The scheme's framework includes modifications for both eigenvector deficiencies and local linear degeneracies and hence gives better performance on systems of conservation laws which possess these properties, than do other second order schemes.

We now give a review of the second-order Godunov scheme, which can be considered as a 5-step procedure.

1. Beginning with the piecewise constant approximation $\mathbf{U}_j^n$, compute 'centred', 'left' and 'right' slopes, whilst maintaining conservation.
2. 'Limit' the slopes using monotonicity criteria, again whilst maintaining conservation. This provides a piecewise discontinuous linear approximation to the solution at time t .
3. Trace along characteristics, using a Taylor series extrapolation, to derive left and right states at grid block interfaces at time $t + \frac{1}{2}\Delta t$. It is possible that the traced states are unphysical, if this is the case the physical cell centred value is used instead.
4. Solve the Riemann problem with these left and right states.
5. Use a conservative difference approximation to (2.7) to produce a piecewise constant approximation to $\mathbf{U}_j^{n+1}$.

We now describe the main steps of the method in more detail.

We begin with a piecewise constant approximation $\mathbf{U}_j^n$ to the solution in each grid block. We wish to calculate an approximation to the slope $\frac{\partial \mathbf{U}}{\partial x}$ so that we may express the data as a linear profile, which will be piecewise discontinuous. For example in the scalar case a piecewise constant approximation would look something like that in Figure 1. We require a discontinuous piecewise linear approximation which would look something like that in Figure 2.

We define a monotonised centred-difference approximation,

$$\frac{\partial \mathbf{U}}{\partial x} \Big|_{x=x_j} \approx \frac{\Delta \mathbf{U}_j}{\Delta x}, \quad (3.1)$$

where in order to calculate $\Delta \mathbf{U}_j$ we compute undivided centered, right and left differences and expand these differences in the right eigenvectors $\mathbf{r}_k$ of the system