On Source Analysis by Wave Splitting with Applications in Inverse Scattering of Multiple Obstacles

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Abstract

We study wave splitting procedures for acoustic or electromagnetic scattering problems. The idea of these procedures is to split some scattered field into a sum of fields coming from different spatial regions such that this information can be used either for inversion algorithms or for active noise control.

Splitting algorithms can be based on general boundary layer potential representation or Green's representation formula. We will prove the unique decomposition of scattered wave outside the specified reference domain G and the unique decomposition of far-field pattern with respect to different reference domain G. Further, we employ the splitting technique for field reconstruction for a scatterer with two or more separate components, by combining it with the point source method for wave recovery. Using the decomposition of scattered wave as well as its far-field pattern, the wave splitting procedure proposed in this paper gives an efficient way to the computation of scattered wave near the obstacle, from which the multiple obstacles which cause the far-field pattern can be reconstructed separately. This considerably extends the range of the decomposition methods in the area of inverse scattering. Finally, we will provide numerical examples to prove the feasibility of the splitting method.

Keywords. Inverse scattering, wave splitting, potential theory, near field,

examinations. Nondestructive testing employs inverse problems techniques for quality control. For a given incident wave, the impenetrable obstacle D will generate a scattered wave outside D, which is in general governed by the Helmholtz equation for acoustic waves or Maxwell equations for electromagnetic waves. The scattered the curves [8, 11].

Motivated by these problems, we present an efficient way to reconstruct the scattered wave from the far-field pattern caused by multiple obstacles. The basic idea is to *split the far-field pattern* into several parts which are essentially related to each obstacle. Correspondingly, the scattered wave is also decomposed. Please observe that our splitting *avoids any approximation* as for example employed for the Born approximation or physical optics approximation. Using this idea based on general potential theory or Green representation formula and combining it with the *point source method*, we propose a scheme which provides a reconstruction of the scattered wave at all points outside of some scatterer D with several components. This *splitting method* enables the recovery of the scattered wave outside of multiple

Denote by $\Phi(\cdot,\cdot)$ the free-space fundamental solution to the Helmholtz equation $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. For G given in Definition 2.1, the single- and double-layer potentials are defined by

$$(S\phi)(x) := \int_{\partial G} \Phi(x,y)\phi(y)ds(y), \tag{1}$$

$$(K\phi)(x) := \int_{\partial G} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\phi(y)ds(y) \tag{2}$$

for $x \in \mathbb{R}^m$ and solve the Helmholtz equation in $\mathbb{R}^m \setminus \partial G$. Moreover we introduce

$$(K'\phi)(x) := 2\int_{\partial G} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\phi(y)ds(y), \quad x \in \partial G, \tag{3}$$

$$(T\phi)(x) := 2\frac{\partial}{\partial \nu(x)}\int_{\partial G} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\phi(y)ds(y), \quad x \in \partial G. \tag{4}$$

It is well known the above four integrals called potential functions are well-defined for $x \in \partial G$ with density ϕ in suitable Hölder or Sobolev spaces (see [5]).

2.1 Uniqueness of source splitting

This section serves to establish the uniqueness of a general scattered field splitting for some domain G given by Definition 2.1. Here, we do not need to specify the concrete form of the potentials under consideration.

Theorem 2.2. *Consider domains G_j as given in Definition 2.1. Assume that we are given a decomposition $u^s = u_1^s + u_2^s$ of the scattered field u^s such that*

1. *u_j^s satisfies the radiation condition for $j = 1, 2$;*
2. *u_j^s soles the Helmholtz equation in the exterior of G_j for $j = 1, 2$;*
3. *Both $(u_j^s)^+$ and $\frac{\partial (u_j^s)^+}{\partial \nu}$ exist in ∂G_j, where*

$$(u_j^s)^+|_{\partial G_j} := \lim_{x \in \mathbb{R}^m \setminus \overline{G}_j,\, x \to \partial G_j} u_j^s(x).$$

Then the splitting of u^s is unique, i.e. for ev4.5520TF5111.955Tf15.4170Td[(u)]TF367.97Tf6.6634.338

Proof. We subtract the two representations and use the definition

(5) $$[illegible]\; v_j(x) := u^s(x) - [illegible]_j(x), \quad x \in \mathbb{R}^m \setminus \overline{G_j}, \quad j = 1, 2$$

to obtain [illegible] $\mathbb{R}^m \setminus \overline{G_1 \cup G_2}$ that

(6) [illegible]

with $\gamma = 1/(4\pi)$ in $\mathbb{R}^3$ and $\gamma = e^{i\pi/4}/\sqrt{8\pi\kappa}$ in $\mathbb{R}^2$, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^m$. Here, the density φ lives on $\partial G = \partial G_1 \cup \partial G_2$. We denote

$$\varphi_j(y) := \varphi(y) \text{ for } y \in \partial G_j \tag{11}$$

and denote the corresponding single-layer potential operators by S_j, i.e.

$$(S_j\varphi_j)(x) := \int_{\partial G_j} \Phi(x,y)\varphi_j(y)ds(y), \quad x \in \mathbb{R}^m \tag{12}$$

and we have

$$S\varphi = S_1\varphi_1 + S_2\varphi_2. \tag{13}$$

Algorithm 2.3. *The splitting of the far field of a scatterer $D = D_1 \cup D_2$ is obtained from the following three steps.*

1. *Solve the far-field equation*

$$S^\infty \varphi = u^\infty \tag{14}$$

   to generate density function φ defined in ∂G, where S^∞ is given via (10).

2. *Define two functions*

$$u_j^s(x) := (S_j\varphi_j)(x), \quad x \in \mathbb{R}^m \setminus \overline{G_j}, \ j = 1, 2, \tag{15}$$

   which can be considered as a scattered wave outside G_j, in the sense that it solves the Helmholtz equation in $\mathbb{R}^m \setminus \overline{G}_j$ and meets the radiation condition.

3. *Compute the far field patterns of u_j^s defined by*

$$u_j^\infty := S_j^\infty \varphi_j, \quad j = 1, 2. \tag{16}$$

In this way, the far field pattern u^∞ is decomposed as

$$u^\infty = u_1^\infty + u_2^\infty. \tag{17}$$

Correspondingly, the scattered wave u^s related to u^∞ has the splitting

$$u^s(x) = u_1^s(x) + u_2^s(x), \quad x \in \mathbb{R}^m \setminus \overline{G} \tag{18}$$

from the linear superposition principle and Rellich lemma, where u_j^s is computed via (15). Moreover, u_j^s outside G_j is the scattered wave related to u_j^∞ with $j = 1, 2$ again from Rellich lemma, noticing $u_j^s(x)$ defined by (15) is the radiation solution.

For the feasibility of the above scattered wave splitting based on the far-field pattern decomposition we need to investigate the following questions.

1. Is (14) uniquely solvable? If so, then the decomposition (17) and the functions u_1^s, u_2^s in (15) are uniquely defined.

2. For given G, is (18) a decomposition of u^s in the sense of Theorem 2.2? If this is the case then the single-layer approach is a *constructive method* for this unique decomposition of the scattered field.

Theorem 2.4. *Assume that G is chosen in the way of Definition 2.1 such that $-\kappa^2$ is not the interior Dirichlet eigenvalue of Δ in G_j for $j = 1, 2$. Then there exists a unique solution ϕ $L^2(\partial G)$ to (14).*

Remark. The proof contains two parts. Firstly, we prove that the far field operator S^∞

from the following Theorem 2.10 again that u_j^s can be extended analytically to $\mathbb{R}^m \setminus \overline{G}_j$. Therefore condition 3 is also met. □

This result gives a positive answer to the second question. Since the choice of G meeting the previous conditions is not unique, we must consider the uniqueness of far-field decomposition (17) and the scattered wave decomposition for diﬀerent choice of G. This uniqueness can be stated as

Theorem 2.6. *Denote by u^∞ the far-field pattern caused by obstacle D and u^s the scattered wave outside $\overline{D}$ related to u^∞. Assume that $\tilde{G} := \tilde{G}_1 \cup \tilde{G}_2$ diﬀerent from G is the other configuration satisfied the same requirement on G given previously. If we decompose the far-field pattern u^∞ as*

$$u^\infty = \tilde{u}_1^\infty + \tilde{u}_2^\infty \tag{19}$$

using the same algorithm given above for $\tilde{G}$ and construct $\tilde{u}_j^s(x)$ outside $\tilde{G}_j$ by the density function $\tilde{\varphi}_j$ related to $\partial\tilde{G}_j$, then we have for $i = 1, 2$ that

$$\tilde{u}_j^\infty(\hat{x}) = u_j^\infty(\hat{x}) \tag{20}$$

and

$$\tilde{u}_j^s(x) = \tilde{u}_j^s(x), \quad x \in \mathbb{R}^m \setminus \overline{G_j \cup \tilde{G}_j}, \tag{21}$$

provided that $(G, \tilde{G})$ meets the following separation condition

$$\overline{G_1 \cup \tilde{G}_1} \cap \overline{G_2 \cup \tilde{G}_2} = \emptyset. \tag{22}$$

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Proof. We prove this theorem splitting the proof into the following two cases. First, we treat the case where G_j contains G_j in its interior for both indices $j = 1, 2$. Secondly, we reduce the general case to this special case.

Case 1: $G_j \cup \tilde{G}_j = G_j$ with $j = 1, 2$. For given $G, \tilde{G}$, it follows from $\tilde{u}_1^\infty + \tilde{u}_2^\infty = u^\infty = u_1^\infty + u_2^\infty$ that

$$(\tilde{u}_1^\infty - u_1^\infty) + (\tilde{u}_2^\infty - u_2^\infty) = 0. \tag{23}$$

On the other hand, noticing the correspondence be1711.91520Td[(:)]TJ/F1711.95417sp9nstruct

Now applying the same argument as that in the proof of Theorem 2.2 with G there replaced by $\tilde{G}$, we get that

$$\tilde{u}_j^s = u_j^s, \ x \quad \mathbb{R}^m \setminus \overline{\tilde{G}_j}$$

for $j = 1, 2$, which proves (21), noticing in this case $G_j \,\Box\, \tilde{G}_j = \tilde{G}_j$. Now using the relation between $(\tilde{u}_j^\infty, u_j^\infty)$ and $(\tilde{u}_j^s, u_j^s)$ in terms of the density $(\tilde{\phi}_j, \phi_j)$ again, we know $(\tilde{u}_j^\infty, u_j^\infty)$ are the far-field pattern of scattered wave $(\tilde{u}_j^s, u_j^s)$. Therefore (21) leads to (20) immediately.

Notice, in this case, our proof does not need the condition (22), which is guaranteed automatically by the definition of $\tilde{G}$.

Case 2: $G_j \,\Box\, \tilde{G}_j = G_j$ for at least one $j = 1, 2$. In this case, the separation

On the other hand, the radiating solution u^s to the Helmholtz equation in the exterior of G_j has the representation ([5], Theorem 2.4)

$$\int_{\partial G_j} \left\{ u^s(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} - \frac{\partial u^s(y)}{\partial\nu(y)}\Phi(x,y) \right\} ds(y) = \begin{cases} 0 & x \quad G_j \\ u^s(x) & x \quad G_j. \end{cases} \tag{27}$$

We will use these formulas to derive a general splitting procedure which does not need to avoid interior eigenvalues of the domain G_j.

Using the potential operators, Green's formula for the radiating solution u^s of the Helmholtz equation can be written in the form

$$u^s = K u^s - S\frac{\partial u^s}{\partial\nu} \quad \text{in } \mathbb{R}^m \setminus \overline{G}. \tag{28}$$

Then, the normal derivative $\frac{\partial u^s}{\partial\nu}$ on $\Lambda \quad \partial G$ for $u^s(x)$ outside G meets

$$\frac{\partial u^s}{\partial\nu} = T u^s - K' \frac{\partial u^s}{\partial\nu} \quad \text{in } \partial G \tag{29}$$

due to the jump relation of potential functions. This equation is not adequate to calculate the normal derivative from its boundary values, since for interior eigenvalues for the negative Lapcacian it lacks uniqueness and thus existence for general boundary values. For this reason we use the following operator representation of the the Dirichlet-to-Neumann map $B : u^s|_{\partial G} \to \frac{\partial u^s}{\partial\nu}|_{\partial G}$ or Steklov-Poincare operator, respectively. Following [5], page 48, with some parameter $\eta > 0$, the operator is given by

$$B := (i\eta I - i\eta K' + T)(I + K - i\eta S)^{-1} : C^{1,\beta}(\partial G) \to C^{0,\beta}(\partial G). \tag{30}$$

1. *Solve the integral equation*

   $$u^{\infty} = (K^{\infty} - S^{\infty} B)\phi \tag{32}$$

   to obtain the boundary values $\phi = u^s$ *on* ∂G *via the solution* $\phi \quad C^{1,\beta}(\partial G)$, *noticing (27).*

2. *Use (30) to evaluate the Dirichlet-to-Neumann map*

   $$\psi := B\phi \tag{33}$$

   to calculate the normal derivative $\psi = \partial u^s / \partial \nu$ *on* ∂G *of the field* u^s *outside* G *in terms of (29).*

3. *Compute*

   $$u_1^{\infty} \tag{34}$$

By the same argument as that in the proof of Theorem 2.2, we get that

$$v_j^s(x) \equiv 0, \quad x \quad \mathbb{R}^m \setminus \overline{G}_j, \quad j = 1, 2.$$

On the other hand, $v_1^s(x)$ is the radiation solution outside G_1, it follows from (27) that $v_1^s(x) \equiv 0$ in G_1. Since both double-layer potential $K_1\phi_1(x)$ and single-layer potential $S_1B\phi_1(x)$ solve the Helmholtz equation in $\mathbb{R}^m \setminus \partial G_1$, using the jump relation of K_1 and the continuity of S_1 with on ∂G_1 in the continuous density setting, we finally get from $v_1^s(x) \equiv 0$ in $\mathbb{R}^m \setminus \partial G_1$ that $\phi_1(y)|_{\partial G_1} \equiv 0$. Similarly, we get $\phi_2(y)|_{\partial G_2} \equiv 0$. □

Here we decompose the far-field pattern by Green formula, where $u^s|_{\partial G}$ is considered as the density. Comparing the decomposition of far-field pattern by general potential theory method in the previous section, the advantage of wave splitting based on Green formula is that we get u^s as well as its normal derivative directly. This kind of technique has been used in the reconstruction of Neumann data from scattered wave in terms of the density functions $(u_i^s|_{\partial G_i}, \frac{\partial u_i^s}{\partial \nu}|_{\partial G_i})$ corresponding to the far-field decomposition (34). So we omit this result.

By the above theorem we can calculate u_j^s outside of the domains G_j. In $G_j \setminus \overline{D}_j$ we will show below that the scattered wave u_j^s can be calculated from u_j^∞ via *point source method*. We combine these two methods to calculate the total wave

$$u = u^i + u^s = u^i + u_1^s + u_2^s$$

around each obstacle D_j, $j = 1, 2$. Then we can use the zero points set of u to construct the boundary ∂D_j.

2.4 Determination of splitting domains via the range test

So far we have used the assumption that we know two domains G_1 and G_2 which contain the two components D_1 and D_2 of a scatterer D with the important condition $G_1 \cap G_2 = \emptyset$. Here we will discuss how these domains can be determined from the knowledge of the far field pattern u^∞ from one scattered time-harmonic wave. We will employ the *range test* as suggested by Kusiak, Potthast and Sylvester [12].

The range test exploits solvability arguments for the equation (14). Consider the equation in dependence of the unknown domain $G = G_1 \cup G_2$. Then we have the following result proven in [12].

Theorem 2.10. *If the scattered field u^s defined by its far field pattern u^∞ can be analytically extended into the set $\mathbb{R}^m \setminus G$, then the far field equation*

$$S^\infty \phi = u^\infty \tag{37}$$

does have a solution in $L^2(\partial G)$. In this case, u^s expressed in terms of the density ϕ can be extended to $\mathbb{R}^m \setminus \overline{G}$. If the field cannot be analytically extended into $\mathbb{R}^m \setminus \overline{G}$, then the equation (37) does not have a solution.

The solvability of the equation (37) can be numerically tested by calculating the regularized Tikhonov solution

$$\phi_\alpha := (\alpha I + S^{\infty'} S^\infty)^{-1} S^{\infty'} u^\infty \tag{38}$$

and observing the behaviour of the norm $||\phi_\alpha||_{L^2(S)}$ for $\alpha \to 0$. The key ingredient is Theorem 3.7 of [12] adapted to our notation.

Theorem 2.11. *If the scattered field u^s defined by its far field pattern u^∞ can be analytically extended into the set $\mathbb{R}^m \setminus G$, then*

$$||\phi_\alpha||_{L^2(S)} < \infty, \quad \alpha \to 0. \tag{39}$$

On the contrary, if the field cannot be analytically extended into

4. *Calculate an approximation $u_{(\alpha)}$ to the total field u in $G_2 \setminus \overline{D}_2$ by adding u_1^s and the incident field u^i*

$$u_{(\alpha)} = u^i + S_1\phi_1 + u^s_{2,\alpha} \tag{41}$$

in case of splitting by potential theory, or

$$u_{(\alpha)} = u^i + K_1\phi_1 - S_1\psi_1 + u^s_{2,\alpha} \tag{42}$$

with $\psi_1 = B\phi_1$, if the wave splitting is based on the Green formula.

5. *Search for the zero curve of u_α to calculate an approximation to the boundary ∂D_2, provided that the component D_2 has the sound-soft type boundary.*

6. *∂D_1 can be reconstructed analogously.*

Obviously the above shape reconstruction scheme can be applied to multiple obstacle with other kinds of boundary conditions on each component of D.

In the remaining part of this section we will give more details and a convergence analysis of step 3, that is, reconstruction of u_j^s, $j = 1, 2$, in $G_j \setminus \overline{D}_j$ from its far-field pattern by point source method. Notice, the expression (15) (or (35)) gives the scattered wave $u_j^s(x)$ only outside $\overline{G}_j$. Here we give the basic idea of the point source method based on potential theory as suggested by Liu, see [8, 11]. This approach to the point source method extends it to the reconstruction of general radiating fields, whereas the use of reciprocity relations as employed in [10] limits it to fields arising from scattering of plane waves.

Since in our boundary reconstruction problem, ∂D_2 is unknown, we try to approximate ∂D_2 by the zero-curve of total wave near ∂D_2. So in practice, we compute u_2^s outside some chosen domain $H_2 \supset D_2$. The initial specification of H_2 depends on some *a-priori* information of D_2. With some rough zero-curve of total wave outside H_2, we can shrink H_2 continuously to get a better reconstruction of ∂D_2.

Algorithm 3.2. *The point source method for the recovery of u_2^s in the known domain $G_2 \setminus \overline{H}_2$ for given H_2 uses the following steps.*

1. *Approximate the point source $\Phi(\cdot, x)$ for any fixed $x \in G_2 \setminus \overline{H_2}$ by a superposition of plane waves*

$$\Phi(y, x) = \int_S e^{i\kappa y\cdot d} g_x(d)\, ds(d), \quad y \in \partial H_2. \tag{43}$$

2. *Express the scattered wave $u_2^s(x)$ outside H_2 as well as its far-field pattern in terms of the density function by*

$$u_2^s(x) = \int_{\partial H_2} \Phi(y, x)\rho(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \overline{H}_2, \tag{44}$$

$$u_2^\infty(\hat{x}) = \gamma \int_{\partial H_2} e^{-i\kappa\hat{x}\cdot y}\rho(y)ds(y), \quad \hat{x} \in \mathbb{S}. \tag{45}$$

3. *By inserting (43) into (44) and exchanging the order of integral, it follows that*

$$u_2^s(x) = \frac{1}{\gamma}\int_{\mathbb{S}} u_2^\infty(-d)g_x(d)ds(d), \quad x \in G_2 \setminus \overline{H}_2 \tag{46}$$

in terms of (45), which reconstructs $u_2^s(x)$ from its far-field pattern.

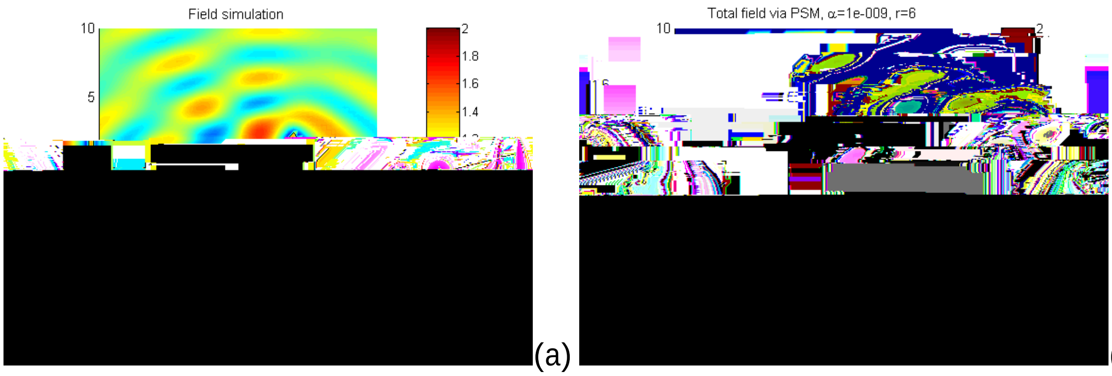


Figure 1: (a) Simulation of the scattered field (b)Point source method using some circular approximation domain without splitting and without modifications which might take into account the non-convexity of the scatterer. The non-convex part of the fields and domains cannot be reconstructed since it is outside of the illuminated area of the method

4 Numerical examples

In this last section we demonstrate the feasibility of the splitting procedure by an application to the inverse acoustic scattering problem from two obstacles with Dirichlet boundary condition. For simplicity we restrict our attention to the two-dimensional case.

We have carried out a simulation of the wave scattering problem via a Brackhage-Werner potential approach

$$u^s(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \phi(y)\, ds(y) - i \int_{\partial D} \Phi(x,y)\phi(y)\, ds(y), \quad x \in \mathbb{R}^m \setminus \partial D, \tag{50}$$

leading to boundary integral equations of the second kind

$$(I + K - iS)\phi = -2u^i \quad \text{on } \partial D, \tag{51}$$

compare [5] or [10] for a detailed presentation. Employing Nystöm's method for the numerical solution of the integral equation and quadrature based on the trapezoidal rule the density potential can be evaluated on subsets of $\mathbb{R}^m$. Figures 1(a) and 2(a) show a plot of the modulus of the total field $u = u^i + u^s$ in a rectangle $Q = [-10, 10]x[-10, 10]$. The wave number has been chosen to $\kappa = 1$.

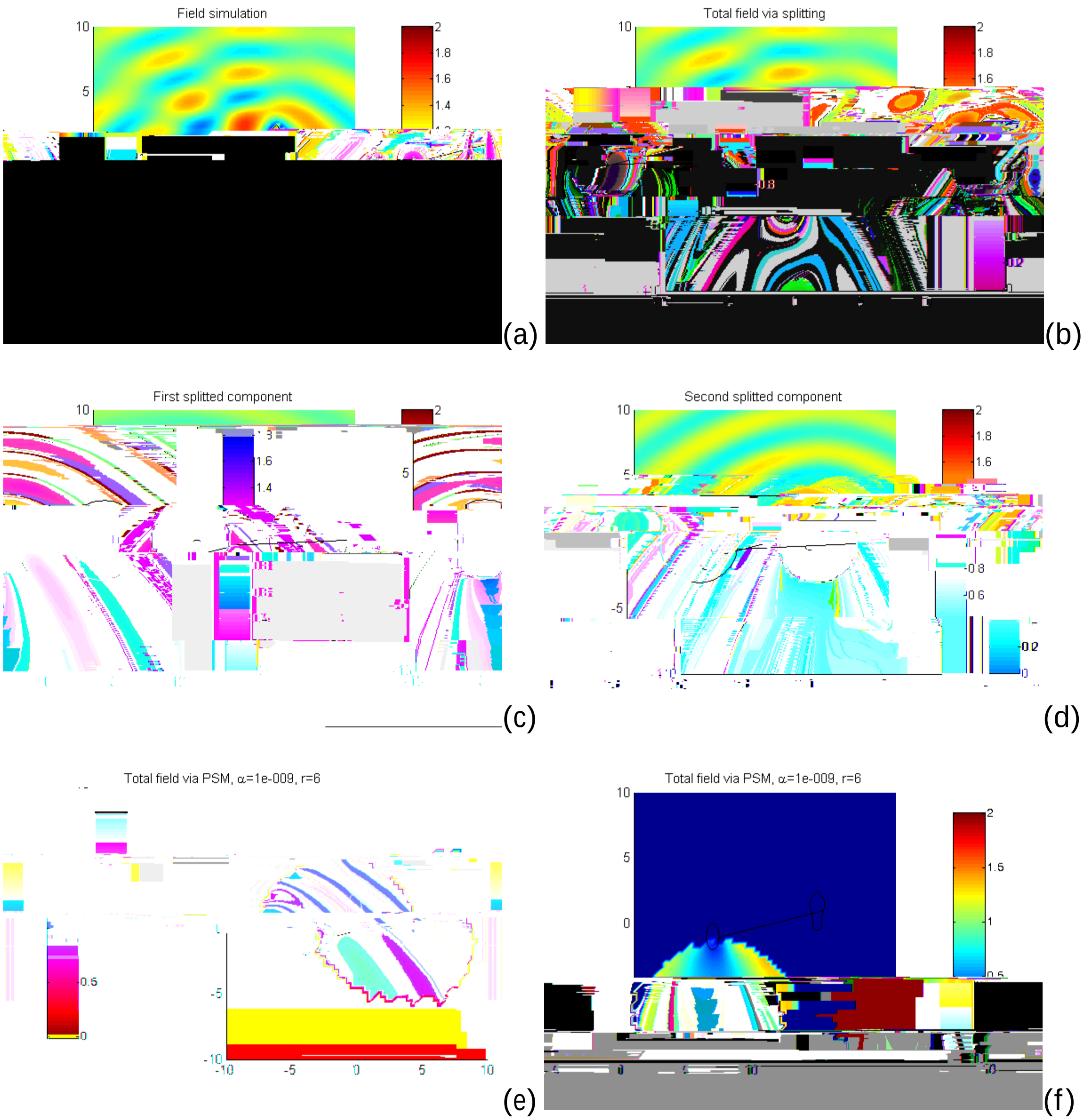
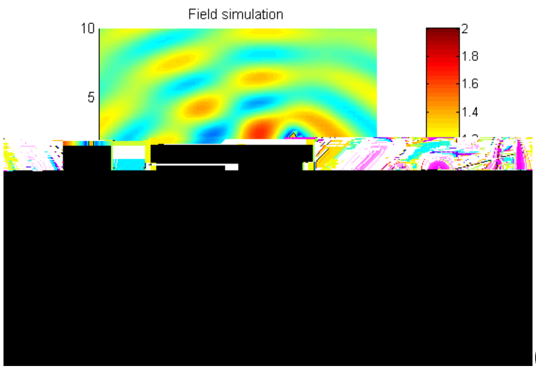
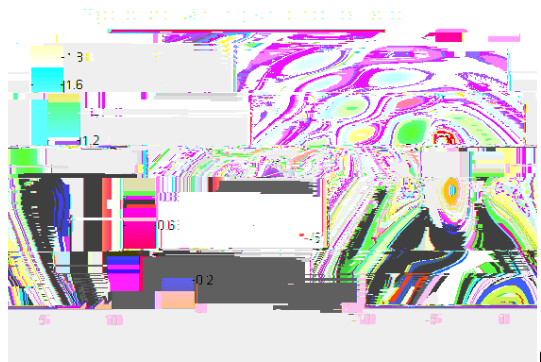


Figure 2: The images show the simulated field for scattering by two obstacles (a), the full reconstructed scattered field u^s via the splitting procedure with a single-layer approach following Algorithm 2.3 in figure (b), the field $u_1^s + u^i$ calculated via the splitting procedure in (c) and the field $u_2^s + u^i$ in (d). Reconstruction of $u_1^s + u^i$ from u_1^∞ on two illuminated areas around D_1 via the point source method is shown in (e) and (f). In particular, in (e) we obtain a reconstruction in an area where the point source method in its simple implementation cannot reconstruct the field.



(a)


(a)

ing operations into a full reconstruction of the total field in $\mathbb{R}^m \setminus D = \mathbb{R}^m \setminus (D_1 \cup D_2)$. This is shown in Figure 3. From the reconstructed total field we are able to find the shape of the domains D_1 and D_2 searching for points where $|u(x)|$ is zero or close to zero. Since this is along the lines of [10] we omit further details and point to the literature.

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