On Appro on n Meshless Me hods

The

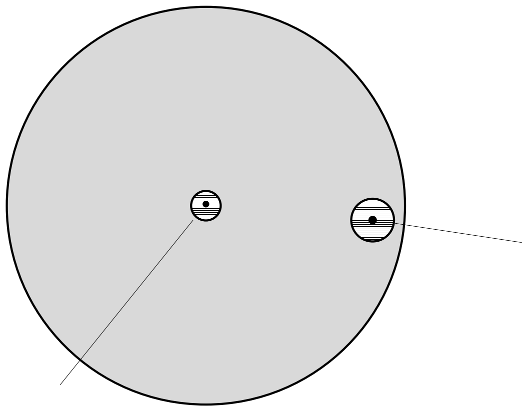
Notation for Particle Methods

tches

1.2 The notion of optimality

Remark 1.4.

2 Polynomial Reproducing Systems



Proof of Theorem 2.6. [illegible]

$$\sum_{j=1}^{N} \quad (8)$$

[illegible]

$$\sum_{i=1}^{N} \quad (9)$$

[illegible] 'd (u583/R48 6. 0 Td +.65974 -1. [illegible]

[illegible] $\mathbb{R}^d$. A [illegible]

Corollary [illegible] *Let* [illegible] $\mathbb{R}^d$ *be a Lipschitz domain. Assume that the* [illegible] e_i *of Theorem 2.6 satisfy additionally an overlap condition, i.e., for some* [illegible] $\in \mathbb{N}$ *we have*

$$\sup_{x \in \mathbb{R}^d} \# \{ \, \in \mathbb{N} \mid \in e_i \}$$

Then there exists a linear map [illegible] $\to$ [illegible] *such that for each* [illegible] $\in \mathbb{N}_0$ *there exists* [illegible] *with*

$$\| \cdot \|_{H^s} \leq \; ^{\min\{p+1;k\}-s} \| \cdot \|_{H^k}$$

Example 2.9. L ... 2 ... $2 \mathbb{N}$

$\frac{p \quad 1=}{-1=}$ ... T ... i

$)^{1-1=}$

$v \; S^{p;} \; k \; k_H$ ... $-p$

... A ... T ...

... i ... b_i)

) ... b_i ...

... 7

... i ...

e_i ...

$G \in \mathbb{R}^{Q\times Q}$ … $G_{kl} = \sum_{i=1}^{N}$ … $a \in \mathbb{R}^{Q}$ …

$$a \cdot Ga = \sum_{i=1}^{N} \Big(\sum_{k=1}^{Q} a_k \dots \Big)$$

… G … $a \in \mathbb{R}^{Q}$ … $a \neq 0$ … $a \cdot Ga = 0$ … $\sum_{k=1}^{Q} a_k$ …

$$\dots = \sum_{l=1}^{Q} \dots \sum_{i;k;l} \dots \sum_{i;k} \dots$$

Exercise 2.14. …

$$\dots = \frac{\dots}{\sum_{j=1}^{N} \dots} = \frac{\dots}{\sum_{j} \dots}$$

∎

A … L …

Exercise 2.15. … A …

$$\sum_{i=1}^{N} \dots \qquad \forall \dots \in \dots \qquad \forall \dots \in P_p$$

∎

Remark 2.16. … L …

∎

... window function ...

... radial ... tensor product ...

Remark 2.17. ...

Regularity of the shape functions ...

Assumption 2.18. ...

Remark 2.19. ...

Theorem 2 20 *Let tisfy cone condition ith ng e nd r dius .*
Let 2), - N − f i j − g $\mathbb{R}^d$ *nd* f i j − g $\mathbb{R}$.
Set

f i g

nd ssume the co ering condition

t

[N i 1 1 d l 1 1 0 d

comp

k j k comp

3. step:

2. step:

$$\| \|_{L^\infty(B(x))} \quad - \quad p \quad \| \|_{L^\infty(B)}$$

$\| \|_{L^\infty(B)}$ $j \in N \setminus i$ $j \in N \setminus i$ $\| \|_{L^\infty(B)}$ $2 \bar{e}$

$$\| \|_{L^\infty(B)} - j \quad j$$

$j \in N \setminus e \setminus \overline{\quad i}$ e

$$- \quad - \quad \frac{1}{x - x_*} \quad 6$$

$\|- \quad \|$ i e $j \in N \setminus$ i $N \setminus e$ $j \in i$

3. step:

$$\frac{-}{\|\, j \quad \|} \quad j \quad 6$$

j $\in \mathbb{R}^d$ $\| \|$

$$\{ \quad j \in \tfrac{1}{\ } \} \quad \bar{e}$$

6

$$\|- \quad j\| \quad \|- \quad \| -$$

$$\| j \quad \|$$

$\in e$ $\in \tfrac{1}{\ }$

$$j - - \quad j \; - \quad - \quad \frac{}{\|- \quad \|} \quad - \quad j \; -$$

$$\| \quad \| \quad \| \quad \| \quad \frac{}{\| j \quad \|} \quad \frac{\| j - \|}{\| j \quad \|}$$

4. step:

2 Bibliographical Remarks

Example 3.4.

		$\mathbb{R}^d$
1;0	0	—
1;1		—
1;		—
;0	0	
;1		
;		

1 Analysis of a class of RBFs

Assumption 3.5.

$$\|k\|^{-1} \qquad \forall \in \mathbb{R}^d$$

Exercise 3.6.

[2] $\hat{\ }( ) = \frac{1}{(2\ )^d}\int_{\mathbb{R}^d} ( )e^{-ix\cdot}\,d$ denotes the Fourier transform $\hat{\ }$ of a function . The inversion formula takes the form $( ) = \int_{\mathbb{R}^d} \hat{\ }( )e^{ix\cdot}\,d$.

A

Z

h j f jk k h j 1g

Proposition *Let tisfy Assumption 3.5. Then*

1. $\mathbb{R}^d$).
2. $\mathbb{R}^d$) *ith equi ent norms.*
3. 2 .
4. *is positive definite.*

Proof. $\mathbb{R}^d$).

Theorem *Let Assumption 3.5 be id. Then for distinct points* N f j g *nd* 2 *the ttered interp tion prob em:*

Find 2 N f k k) j g
such th t j) j)

s unique so ution, hich tisfies

h j 8 2 N

nd

k k k k

Proof. 7 k k) k k k) k 2 N b k)) ix k

Z

h k j C ix k k) k)

k k L

Corollary **(stability of scattered data interpolation)** *Let* $\mathbb{R}^d$ *be Lipschitz dom in (or* $\mathbb{R}^d$*). Let* N f j g *nd suppose Assumption 3.5. Then for* 2)

k k H

Proof. $\mathbb{R}^d$ $\mathbb{R}^d$

$$\| \|_{H^{ }( )} \| \| \| \|_{H^{ }( )} \|_{H^{ }( )} \|_{H^{ }( )}$$

Corollary 10 *Let Assumption 3.5 be satisfied and let* $\mathbb{R}^d$ *be a Lipschitz domain. Define the*

$$\min_{i=1,\dots,N} \| \|$$

Then there exists such that for $\in$ *there holds*

$$\| \|_{H^{s}( )} \le {}^{-s} \| \|_{H^{ }( )}$$

Proof.

1. step: $\| \|_{H^{ }( )}$ $\| \|_{H^{ }( )}$ $\|_{L^{ }( )}$ $\| \|_{H^{s}( )}$

2.step: $\in \mathbb{N}_0$

$$\| \|_{L^\infty(B)} \quad \forall \in \mathcal{P}_p$$

B_j $\bigcup_{j=1}^{M}$ B_j

$\mathbb{R}^d$ $\| \|_{H^{ }( )}$ $\bigcup_{j=1}^{M} B_j$ $\in \mathcal{P}_p$

$$\| \|_{L^{ }(B)} \| \|_{L^{ }(B)} \| \|_{L^{ }(B)} \| \|_{H^{ }(B)} \| \|_{L^\infty(B)}$$

1 Approximation Theory

Theorem 1 *Let* $\mathbb{R}^d$ *be Lipschitz dom in nd et* f j − g *be co ection of* $^{1;}$) *functions. Set* i − pp i) , i − i nd ssume

k ikL∞)1   k r ikL∞)1 G/i −

N X i on , p r f 2 ℕj 2 ig
i=1 x

Assume th t e ch i, − , *is Lipschitz dom in s e .*
For e ch 2 f g *et* i 1 i) *be given nd set*

( ) N X − i i − N X i ij i 2 i)
i=1 i=1

Then
Assu

1.43994 Td (2)Tj /R28 9.96264 Tf 9.4752

G

V

$\sum_{i=1}^{N}$

$\sum_{i=1}^{N}$ $\sum_{i=1}^{N}$

$\sum_{j=1}^{M}$ $\sum_{j=1}^{M}$

$\sum_{i=1}^{N}$ $\sum_{i=1}^{N}$

$\sum_{i=1}^{N}$

$\sum_{i=1}^{N}$ $\sum_{i=1}^{N}$

$\sum_{i=1}^{N}$

Remark 4.2.

Example 4.3. $\mathbb{R}^d$

Remark 4.4.

i i j ; 6 i

■

B_i f $i;j$ j g

B f i $i;j$ j g

[redacted]

0 1 N

L i P_p f j j g

f i j j

P_N

$i=0$ i i

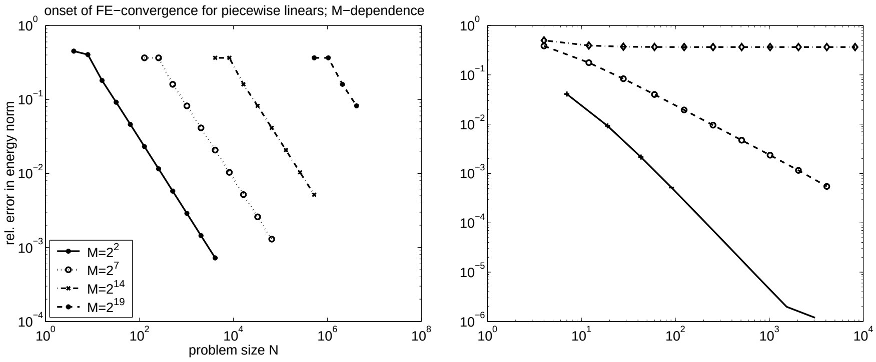
■

Exercise 4.8.

5 Examples of operator adapted approximation spaces

1 A one dimensional example

V 0 Tdp(Theorem) /R148 0

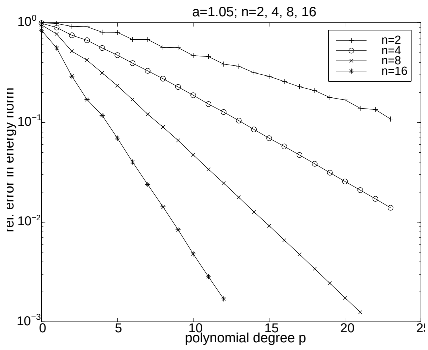

onset of FE−convergence for piecewise linears; M−dependence
rel. error in energy norm
problem size N
M=2^2
M=2^7
M=2^14
M=2^19

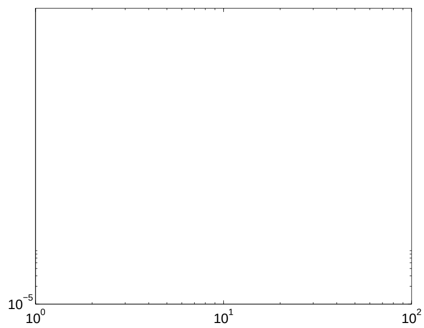
$$| \quad | \quad \int_{x_0}^{x} \quad \| \quad \|_{L^\infty}$$

$$| \quad |$$

independent

2 Laplace's Equation


a=1.05; n=2, 4, 8, 16
n=2
n=4
n=8
n=16
rel. error in energy norm
polynomial degree p



ne es

ik ! n (x;y) j n — O

n — n n p p

Gener ized rmonic po ynom s

f n n n j n — g

n

ppr pr n

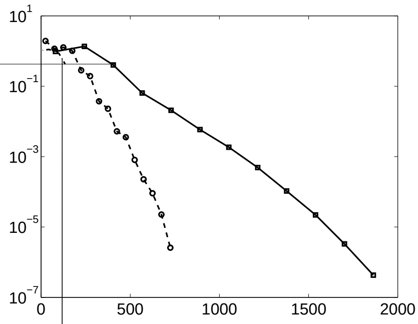
Theorem 1 *Let* $\mathbb{R}$ *be simp y connected dom in, be comp ct subset. Let so e (5.6). Then there exist , such th t for* $2\ \mathbb{N}$, :

k kH −bp k kH −bp= log p

Proof. pr pr

Theorem 1 *Let* $\mathbb{R}$ *be st r-sh ped ith respect to . Let tisfy n exterior cone condition ith ng e . Let 2 k , , so e (5.6). Then there exists such th t*

k kH


10^1
10^−1
10^−3
10^−5
10^−7
0
500
1000
1500
2000

Theorem 1 *Let* $\mathbb{R}$ *be st r-sh ped ith respect to . Let t-isfy n exterior cone condition ith ng e . Let* $2\,\mathbb{N}$, 2 *nd s-sume th t the disp cement fie d* 2 *tisfies the homogeneous sticity eq tions (5.13). Then the function* $\mathbf{u}$ *i n be pproxim ted from* V_p^{elast} *such th t*

Proof.

Rem rk 5.18.

Further examples

ye2078bdf

for some numbers $\in \mathbb{R}$ *and* ...

Proof. [illegible]

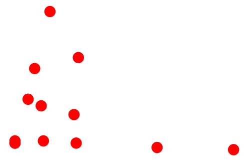
Exercise 6.2. [illegible]

$$\sum_{j=1} \sum_{i < k} \ldots$$

$$\min\{p;k\}$$

Exercise 6.3. L[illegible] T [illegible] $\mathbb{R}$. L[illegible]

[illegible]



Remark 6.5.

$\mathbf{2}\ \mathbf{P}_p$

7 Enforcement of essential boundary conditions

Conforming methods:

Non-conforming methods:

1 Conforming methods

A simple approach

Exercise 7.1.

Combination with the classical FEM

Ex mp e 7.5.

2 Non conforming methods: Lagrange Multiplier methods and collocation techniques

Exercise 7.8. [illegible] ∎

[illegible]

Non conforming methods: penalty method

[illegible]

[illegible]

[illegible]

[illegible]

Theorem (penalty method). *Let* [illegible] $\subset \mathbb{R}^d$ *be a Lipschitz domain. Let* [illegible]*. Assume* [illegible] $\in$ [illegible]^{k}[illegible] *is the solution of (7.1). Let* [illegible] $\in$ [illegible]$^{k-1}$[illegible] *solve*

[illegible] *on* [illegible] *on* [illegible]

Assume that the approximation space [illegible] *satisfies:*

[illegible]

[illegible]

Then there holds for [illegible] *independent of* [illegible] *and* [illegible]

[illegible]

Setting — *ith the optim* *ue* — $\frac{k-1}{}$ *gives*

Remark 7.10. 2 $k-1$

Proof of Theorem 7.9.

Rem rk 7.11.

Remark 7.16.

Lemma 1 *Set*

Exercise 7.19.

$$\|v - v_N\| \quad \|_{L} \quad \| \quad \|_{H} \quad \| \quad \| \quad \|_{H^k} \quad \| \quad \| \quad \|_{H^k}$$

∎

Remark 7.20.

∎

A Results from Analysis

Theorem A 1 (universal extension operator). *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain. Then there exists a linear operator* $E: L^1(\Omega) \to L^1(\mathbb{R}^d)$ *with the following properties:*

(i) $Eu|_{\Omega} = u$ *for all* $u \in L^1(\Omega)$.
(ii) *For each* $k \in \mathbb{N}_0$, $p \in [1, \infty]$*, there exists* C *such that* $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \le C \|u\|_{W^{k,p}(\Omega)}$ *for all* $u \in W^{k,p}(\Omega)$.

Proof. ∎

Theorem A 2 (multiplicative trace theorem). *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain,* $s \in$ *. Then there exists a constant* C *such that for all* $u \in H^s(\Omega)$ *the trace* $\gamma u = u|_{\partial\Omega}$ *satisfies*

$$\|\gamma u\|_{L(\partial\Omega)} \le C \|u\|_{L(\Omega)}^{1-1/(2s)} \|u\|_{H^s(\Omega)}^{1/(2s)}$$

Proof.

$$\|\gamma u\|_{L(\partial\Omega)} \le C \|u\|_{B(\Omega)} \qquad (A.)$$

∎

Here, the notation) *represents the function* 7!) — f g. *The constant* p;q;k *depends only on* 2 $\mathbb{N}_0$, 2 1), *and* . *The bound (B.2) also holds for* — 1 *if and are restricted to integer values* , 2 $\mathbb{N}_0$.

If 2 1) *and or if* — *and* , *then additionally*

$$k \quad p\, k_{L^\infty(B)} \quad p;q;k \quad \min\{p \; 1;k\}-d=q\, k\, k_{W^{k;q}(B)}$$

where p;q;k *depends only on* , , *and* .

Proof. [illegible]

1. step: L [illegible]

C Approximation with adapted function systems

C.1 The theory of Bergman and Vekua

Lemma C.1 *Let* $\mathbb{C}$ *be simp y connected Lipschitz dom in. Fix* $_0 \in$ *. Let* H *—* f j *ho omorphic on nd* $_0) \; g$*. Then there exists in r p ith the fo o ing properties:*

1. *) so es (C.1) for every* $\in H$*.*
2. *For every so ution of (C.1) there exists unique* $\in H$ *such th t* $) =$*.*
3. $k \;) k_{H^k}$ $\; k \; k_{H^k}$ *for* $\in H$ *nd .*
4. *for* $\in$ ^{k} *), , so es (C.1), then the corresponding* $-$ $^{-1}$ $) \in H$ *is i e ise in* ^{k} *) nd* $k \; k_{H^k}$ $k \; k_{H^k}$*.*

n the st t o estim tes, the const nt depends on , nd the differenti oper tor.

Proof.

Rem rk C.2.

2. step:

Lemma C *Let ℝ be st r-sh ped ith respect to 0). Let the disp cement fie d* **u** − i **2** k) *for some* **2** ℕ. *Let* $_0$ **2** . *Let , be the ho omorphic functions ppe ring in the represent tion formu (5.14), hich re unique y determined by stipu ting* 0) − . *Then*

k kH k 1 k kH k− 1 kukH k 1

here depends on y on the L mé const nts, upper bounds on , nd o er bounds on .

Proof. ... **2**) k kL 1 kukH 1 ... **2**) k kL 1 kukH 1 ... 0) − ... L ... k kL 1 k kL 1 ... pp r ... 5 ... pr ... **u**

Then [illegible] m *is defined on* $\frac{1}{1-"}$ [illegible] *nd*

$$\| \ {}_m \|_L \ [illegible] \quad \|u\|_{H^m} \ [illegible]$$

$$\| \ {}_m \|_H \ [illegible] \quad {}^{-1}\|u\|_{H^m} \ [illegible]$$

$$\| \ ) \ {}_m \|_L \ [illegible] \quad \| \ ) \ {}_m \|_H \ [illegible] \quad {}^{m} \ {}^{s}\|u\|_{H^{m,s}} \ [illegible]$$

Proof. [illegible]

ap 2 Hp

k 1 apkW j;∞) −pk 1kL (Int(L)) −)

k 1 apkW j;∞) −pk 1kL (Int(L)) −)

h − f) jj j − g ℂ n 1) ! ℂ n

1) − 1 1)

h $\frac{1}{1-"=}$

2 f g

ku ap ap ap) kH j)

m s−

Lemma C *Let* $\mathbb{C}$ *be st r-sh ped ith respect to nd ssume th t*) *. Then for* **2** ^{1}) *ho omorphic on e e*

$$\mathbf{k} \quad) \mathbf{k}_L \quad \mathbf{p}_- \quad - \quad \frac{"}{} \quad \#_{1=} \quad \mathbf{k} \quad \mathbf{k}_L \quad)$$

Proof. [illegible]

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