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# Simple Models of Changing Bathymetry

# Simple Models of Changing Bathymetry with Data Assimilation

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Abstract

**Data assimilation is a means for combining observational data with model predictions to produce a model state that most accurately estimates the current and future states of the true system. The technique is commonly used in atmospheric and oceanic modelling, but in this report we consider its application within a coastal environment. A simplified one-dimensional model**

Hence, we obtain the following expression for $a(z, q) = a(z)$

$$a(z) = \frac{nAF^n}{1-\varepsilon}(h-z)^{-(n+1)}. \tag{2.6}$$

The sediment conservation equation (2.3) can now be solved using the method of characteristics [LeVeque (1992)].

## 2.3 Characteristics

Consider the chain rule for taking the total derivative of z with respect to t,

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{dx}{dt}\frac{\partial z}{\partial x}.$$

Using equation (2.3) to substitute for $\frac{\partial z}{\partial t}$ we obtain

$$\frac{dz}{dt} = \left(\frac{dx}{dt} - a(z)\right)\frac{\partial z}{\partial x}.$$

Note that

$$\frac{dz}{dt} = 0 \text{ along the curves given by } \frac{dx}{dt} = a(z). \tag{2.7}$$

These curves are called the characteristics of the equation, and from (2.7) we can see that $z(x,t)$ takes a constant value on each. We can use this property to construct a solution to (2.3).

Since $z(x,t)$ is constant on any given characteristic so too is the slope of the characteristic $a(z)$, thus the characteristics are straight lines given by

$$x = x_0 + a(z_0)t. \tag{2.8}$$

Here, $z_0 = z(x_0, 0)$ is the value of z on the characteristic, determined by the initial data and $x_0$ is the point of intercept of the characteristic with the x axis (figure 2.1).

To obtain a value for the solution $z(x,t)$ at a point x at time t we trace back along the characteristic (2.8) to the initial data. The solution is given implicitly by

$$z(x,t) = z(x_0, 0) = z(x - a(z(x,t))t, 0) \qquad \text{for } t \geq 0. \tag{2.9}$$

This concept is illustrated in figure 2.1 for an initial bathymetry given by the Gaussian exponential function

$$z(x, 0) = \alpha e^{-\beta(x-\gamma)^2}$$

OI seeks to minimize the analysis error variance by finding an analysis state that is as close as possible to the true state in a root mean square (r.m.s) sense. 3D Var data assimilation can be viewed as a different approach to solving the same problem as OI. It is based on a maximum a posteriori estimate approach and derives the analysis by looking for a state that minimizes a cost

The operator $K \in \mathbb{R}^{m \times p}$ is called the gain matrix [Nichols (2003)] and determines the weight given to the observations. It is given by

$$K = BH^T(HBH^T + R)^{-1}. \tag{3.4}$$

By choosing $K$ correctly, we can ensure that the analysis states will converge to the true states of the system over time [Jazwinski (1970)].

This is the formal solution to the optimization problem. The OI method uses (3.4) to calculate the matrix $K$ explicitly and solve (3.3) directly. The idea of 3D Var is to avoid computation of the gain matrix $K$ and in practice the analysis is obtained iteratively through use of a suitable descent/minimization algorithm [Bouttier and Courtier (2002), Lewis et al. (2006)].

When the observation operator $h$ is linear the 3D Var and OI solutions are equivalent and we can use OI to understand how variational assimilation works [Lewis et al. (2006)]. Since the dimensions of our model are small $K$ is relatively easy to compute. We will therefore adopt the OI method in this work. On an operational scale OI becomes impractical and it is more efficient to apply a 3D Var approach.

## 4 The model

The primary purpose of this report is to investigate and understand some of the basic principles of data assimilation. For simplicity, we wis.248413(s)0.0917563(.248413(s)0.0917563(.24841394(o)0.0492351(n8

2. The analysis phase in which the difference between the predicted observations given by the new background state $z^b(t_{k+1})$ and the vector of measured observations $y(t_{k+1})$ is used in equation (3.3) to produce an updated analysis state $z^a(t_{k+1})$. Observations are taken at the start of each analysis phase; they are used only once, at the correct time, and not again.

Once the analysis phase is complete we advance to the start of the next cycle and the process is repeated.

## 4.4 The error covariance matrices

Before we can implement our assimilation algorithm we need to make estimates of the background and observation error covariance matrices $B$ and $R$. We are assuming that our model is perfect, but in practice the model equations do not describe the system behaviour completely, the background state is not known exactly and the measured observations are imprecise. Our assimilation scheme needs to take account of the errors that arise as a result of these inaccuracies as the precision of the analysis is determined by the precision of the background and observations. The error covariance matrices $B$ and $R$ represent our uncertainty in the background $z^b$ and observations $y$ and their specification has an important effect on the quality of the analysis.

The observation error covariance matrix $R$ gives a statistical description of the errors in $y$. Observation errors originate from instrumental error, errors in the forward model $h$ and representativeness errors [Bouttier and Courtier (2002)]. Generally, it is reasonable to assume that errors in measurements taken at different locations are uncorrelated, in which case the matrix $R$ is diagonal.

The background error covariance matrix $B = \{b_{ij}\}$ describes the estimation errors of the background state, where element $b_{ij}$ defines the error covariance between components $i$ and $j$ of $z^b$. It is the last operator to act in (3.4) and is therefore fundamental in determining the nature of the analysis increment. The correlations in $B$ govern the smoothing and spreading of information from the observations, determining how an observation at one point influences the analysis at nearby points [Bouttier ubijc a898(s)0.093.210368397.04923752351(r)-0.21054531.210368(o)0.0492351(r)-31(6(i)-0.2338 where $y_j$ is the observation of the true bathymetry $z_j^t$, given by (4.2), at the grid point $x_j$. The set of grid points $x_j$ at which observations are to be taken is determined at the start of the assimilation process and remains fixed throughout. As our algorithm is sequential, a new set of observations is used during each cycle. Since the observations are taken from the truth, we weight in their favour, setting the observation and background variances to be $\sigma_o^2 = 0.1$ and $\sigma_b^2 = 1$ respectively.

We assume that the observation errors are uncorrelated and take the observation error covariance matrix $\mathrm{R}$ to be diagonal with variance $\sigma_o^2$. We consider three different ways of computing the background error covariance matrix $\mathrm{B}$:

To begin, we assume that the matrix $\mathrm{B}$ is diagonal with variance $\sigma_b^2$, i.e.

$$\mathrm{B} = \sigma_b^2 \mathrm{I}, \qquad \mathrm{I} \in \mathbb{R}^{m \times m}. \tag{4.3}$$

However, this is a poor approximation as it ignores correlations between grid points and means that observations have no effect on their neighbouring points.

Next, we add entries above and below the main diagonal by setting

$$b_{i-1,i} = b_{i,i-1} = \frac{\sigma_b^2}{2}, \qquad i = 2, \dots, m.$$

This gives a tri-diagonal matrix $\mathrm{B}$ of the form

$$\mathrm{B} = \begin{pmatrix} \sigma_b^2 & \frac{\sigma_b^2}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{\sigma_b^2}{2} & \sigma_b^2 & \frac{\sigma_b^2}{2} & 0 & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \frac{\sigma_b^2}{2} & \sigma_b^2 & \frac{\sigma_b^2}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\sigma_b^2}{2} & \sigma_b^2 \end{pmatrix}$$

The assimilation was run for each of the different $\mathrm{B}$ matrices, experimenting with various combinations of observations, and validating the results against the analytic solution. The effect of L in (4.5) was also investigated, using the analysis errors $\varepsilon_a = z_a - z_t$ to try to determine its optimal value for different observation strategies. Results are presented in the following section.

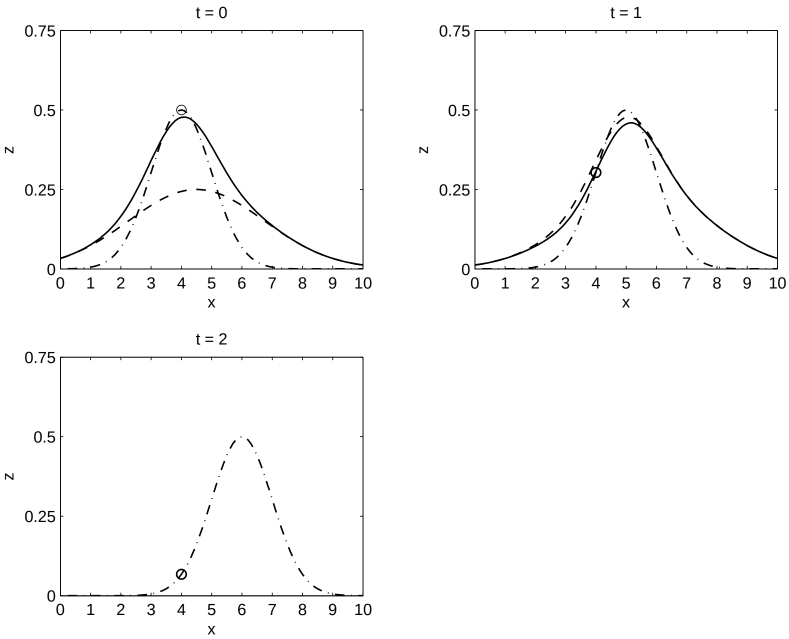
# 5 Results

## 5.1 The Matrix B

information and degradation of the analysis. In order to be able to accurately reconstruct the model state we must ensure that sufficient weight is given to the background state.

## 6 Conclusions and further work

The aim of this work was to use a simple one-dimensional model of changing bathymetry to illustrate the basic theory of data assimilation and examine some of the issues associated with its practical implementation. We began by introducing the sediment conse



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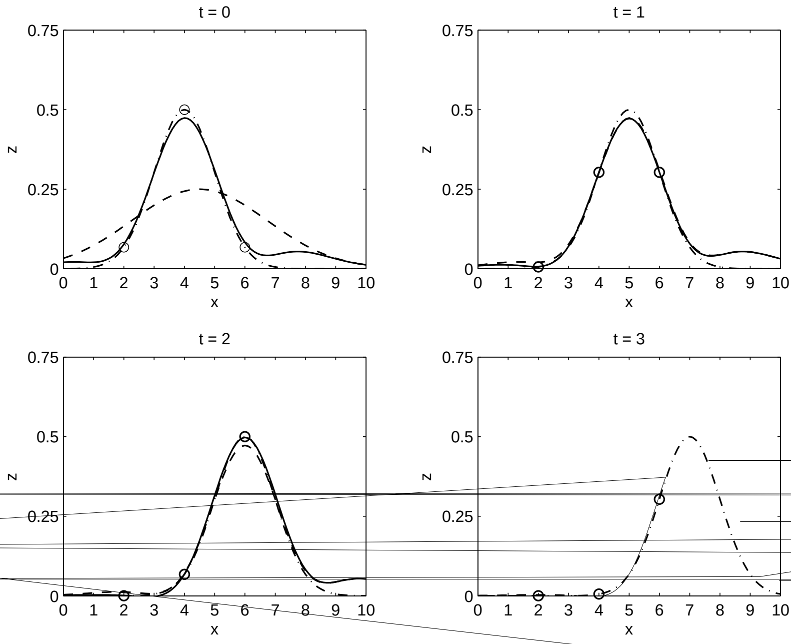
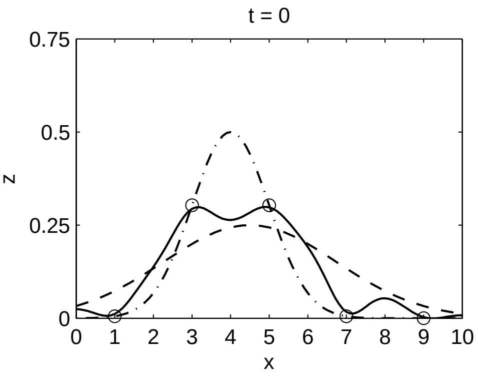
t = 1
t = 2
z
x
0
0.25
0.5
0.75
0 1 2 3 4 5 6 7 8 9 10



t = 0
t = 1
t = 2
t = 3
z
x



t = 0
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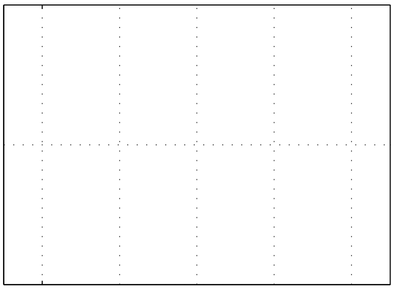
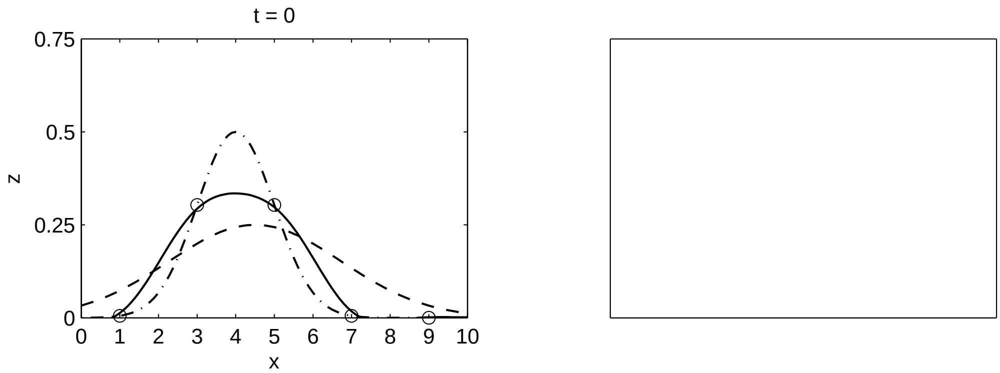
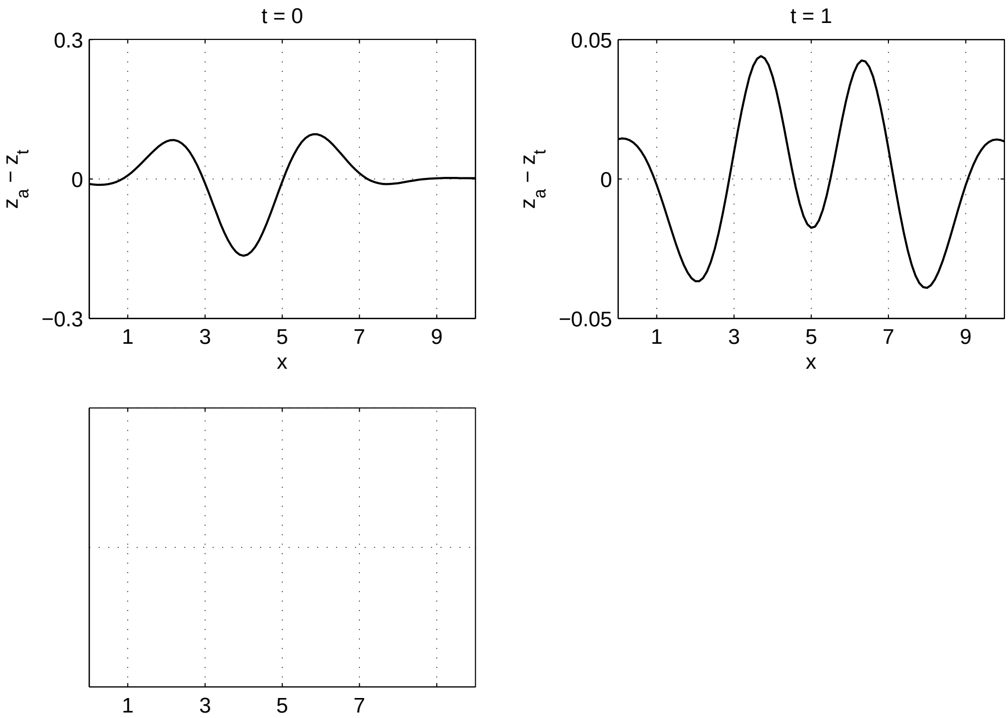
0.5
0.25
0
z
0
1
2
3
4
5
6
7
8
9
10
x




t = 0
0.75
0.5
0.25
0
z
0
1
2
3
4
5
6
7
8
9
10
x



t = 0
0.3
0
−0.3
z_a − z_t
1
3
5
7
9
x
t = 1
0.05
0
−0.05
z_a − z_t
1
3
5
7
9
x
1
3
5
7

## Acknowledgements

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## Glossary of Symbols

| | |
|---|---|
| $A$ | parameter for calculating the sediment transport rate |
| $a(z, q)$ | advection velocity/ bed celerity |
| $F$ | water flux |
| $h$ | water height |
| $n$ | parameter for calculating the sediment transport rate |
| $t$ | time |
| $u(x, t)$ | depth averaged current |
| $x$ | horizontal coordinate |
| $z$ | bathymetry |
| $z_0$ | initial bathymetry |
| $q$ | sediment transport rate |
| $\varepsilon$ | sediment porosity |
| $\alpha$, $\beta$, $\gamma$ | parameters of the gaussian function |
| | |
| $\mathbf{B}$ | background error covariance matrix (dimension $m \times m$) |
| $\mathbf{h}$ | observation operator (from dimension m to p) |
| $\mathbf{H}$ | linearised observation operator (dimension $p \times m$) |
| $J(\mathbf{z})$ | cost function |
| $\mathbf{K}$ | gain matrix (dimension $m \times p$) |
| $m$ | dimension of the state vector |
| $p$ | dimension of the observation vector |
| $\mathbf{R}$ | observation error covariance matrix (dimension $p \times p$) |
| $r_{ij}$ | distance $(x_i - x_j)$ between grid points $x_i$ and $x_j$ |
| $\mathbf{y}$ | vector of observations (dimension p) |
| $\mathbf{z}^t$ | true model state (dimension m) |

| | |
|---|---|
| $\mathbf{z^b}$ | **background state (dimension m)** |
| $\mathbf{z^a}$ | **analysis (dimension m)** |
| $\boldsymbol{\sigma}_{\mathbf{b}}^2$ | **background error variance** |
| $\boldsymbol{\sigma}_{\mathbf{o}}^2$ | **observation error variance** |
| $\boldsymbol{\varepsilon}_{\mathbf{a}}$ | |

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