A local error analysis of the boundary concentrated FEM

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Abstract

e o nd y concen ed n e e e en e od n o e **hp** e on o
e M p c y ed o e n e c e en o e p c o nd y
e p o e oo coe c en nd o nd y cond on o e y
o non oo eo e e - n p pe e con de e c e o e d c e z on
o c e p o e e c o on $\mathbf{u} \in \mathbf{H}^{1+}$

where N is the problem size. We refer to [9] for a detailed description. In the present paper, we focus on the local error and we will investigate the behavior of the local error on compact subsets of the domain . We prove the existence of a 0 such that these errors behave as $O(N$), up to logarithmic terms. For simplicity of exposition we analyze here as a model problem a Poisson problem in two dimensions. We expect that the local error analysis of Theorem 2.1 can be adapted to a more general class of strongly elliptic operators with analytic coefficients. The restriction to two dimensions is likewise done for simplicity of exposition—the techniques used in this paper are likely to have extentions to higher dimensions. The paper is organized as follows: We start with a brief repetition of the foundations of boundary concentrated FEM. Therafter, in Section 2, we formulate the main theorem concerning the local error behavior of boundary concentrated FEM and in Section 3 we present some numerical examples. In Section 4, we introduce a new *hp*-interpolation operator, which is an essential tool for our local analysis. The remainder of the paper is devoted to the proof of auxiliary results that were used in the proof of our main theorem, and we conclude the paper with ourv

Now we are in position to make precise statements concerning the regularity of the solution corresponding to Problem 1.1 and to measure the blow-up of the higher order derivatives:

Lemma 1.6. Let be a Lipschitz domain. Let f be analytic on $\bar{}$ and assume $^{1+}(\ )$ solves (1). Then u is analytic on , and there exist C 0 such that

$$\tilde{}_1^2 \quad (C_{\mathbf{u}} \ {}_{\mathbf{u}})$$

Proof. See [9, Thm. 1.4]. □

he geo etric esh the line r degree ector nd the FE sp ce

We will restrict our considerations to -shape-regular triangulations of consisting of affine triangles. That is, each element is the image $F_{\mathbf{K}}(\ \hat{}$

De nition 1.9. (linear degree vector) **Let be a geometric mesh with boundary mesh size h in the sense of De nition 1.7. A polynomial degree vector p** $= (p_{\mathbf{K}})_{\mathbf{K}\,\mathbf{2T}}$ **is said to be a linear degree vector with slope** $\quad 0$ **if**

$$1 + c_1 \log \frac{h_{\mathbf{K}}}{h} \quad p_{\mathbf{K}} \quad 1 + c_2 \log \frac{h_{\mathbf{K}}}{h} \tag{2}$$

for some $c_1 \; c_2 \quad 0$**.**

We furthermore associate with each edge e of the triangulation a polynomial degree

$$p_{\mathbf{e}} := \min \;\; p_{\mathbf{K}} \;\; \text{e is an edge of element} \tag{3}$$

and denote by

$$\mathbf{p}(\quad) := (p_{\mathbf{e}1} \; p_{\mathbf{e}2} \; p_{\mathbf{e}3} \; p_{\mathbf{K}}) \tag{4}$$

the vector containing the polynomial distribution of the triangle with edges $e_{\mathbf{i}}$ $i = 1 \; 2 \; 3$. An important property of a linear degree vector **p** is:

Lemma 1.10. **Let be a geometric mesh and p a linear degree vector. Then there exists a constant** $C \quad 0$ **such that**

$$C^{\;1} p_{\mathbf{K}'} \quad p_{\mathbf{K}} \quad C p_{\mathbf{K}'} \qquad\qquad {}^{0} \quad \text{with} \; \overline{} \cap \overline{}^{0} \neq$$

Proof. See [9]. □

Now we are in the position to define our *hp*-FEM spaces:

De nition 1.11. (FEM spaces) **Let be**

2 Local error analysis

This section is devoted to the main result of the paper, the analysis of the local error of Problem 1.12 in the framework of the boundary concentrated finite element method. The main theorem is:

Theorem 2.1. (local error bound) Let $\mathbb{R}^2$ be a polygonal domain and ^{0} be a compact subset. Let Assumptions 1.2, 1.3 be valid. Let $_h$ be the solution of Problem 1.12 for a geometric mesh with boundary mesh size h and linear degree vector p with slope . Then there exists a $(0\ _0]$ such that for su ciently large slope and all elements with ^{0} we have

$$h\ _{L^2(K)} \quad Ch^{+} \quad CN \tag{5}$$

$$h\ _{W^{k,2}(K)} \quad Cp_K^{2k}h^{+} \quad C(\log N)^{2k}N \tag{6}$$

$$h\ _{W^{k,\infty}(K)} \quad Cp_K^{2k+2}h^{+} \quad C(\log N)^{2k+2}N \tag{7}$$

Here,

or in weak formulation

$$\int_{\Omega} z \quad \mathrm{d} = \int_{\dot{K}} (\quad {}_{h}) \ \mathrm{d} \qquad {}^{1}_{0}(\ ) \tag{8}$$

Find $z_h \quad {}^{p}_{0}($

reference element be marked by a hat. Then, since we assume shape regularity and since implies $h_{\dot{K}}$ C, we have

$$\mathrm{h} \; W^{k,2}(\dot{K}) \qquad Ch^{1}_{\dot{K}} \;\mathrm{k}\; \hat{} \quad \hat{}\mathrm{h} \; W^{k,2}(\hat{K})$$
$$C_{\mathrm{k}} \;\hat{}\quad \hat{q} \; W^{k,2}(\hat{K}) + C_{\mathrm{k}} \; \hat{q} \quad \hat{}\mathrm{h} \; W^{k,2}(\hat{K})$$

for arbitrary

Numerical examples

In this section, we present some numerical examples to confirm the theoretical results of Theorem 2.1. In all examples we start with a coarse grid $_0$ of the given domain , and we create a sequence of hierarchically nested geometric meshes $_{l\ l=0;1;:::}$ with boundary mesh sizes h_l 2 $^l h_0$ by applying a suitable mesh refinement strategy (see Figure 1 for an example). Furthermore, we define for each mesh $_l$ and a common slope parameter 0 the polynomial degree distribution via

$$p_{K;l} := \left\lfloor \frac{3}{2} + \ln\left(\frac{h_K}{\underline{h_l}}\right)\right\rfloor \qquad \underline{h_l} := \min\ \text{length}(e)\quad e \text{ is an edge in }_l$$

and compute the finite element solution $_l$ $^p_0($ $_l)$.
In order to check the statements of Theorem 2.1, we choose an arbitrary point $\mathbf{x}$ e e is an edge of $_l$ for some 0 and consider the sequence $_l$, where $_l$ denotes the triangle uniquely determined by the conditions $_l$ $_l$ and $\mathbf{x}$ $_l$. Since it is not difficult to show that there always exists an integer L such that $_m =$ $_n$ for all n L, we can use $_{l\ l=0;1;:::}$ to compute a sequence of local errors $_{l\ H^1(K_l)\ l=0;1;:::}$ which is well suited for pointing out the dependence of the local error on the boundary mesh size h.

Example 3.1. We consider the L-shaped domain $= (1\ 1)^2\ ([0\ 1]\ [\ 1\ 0])$ **as shown in Figure 1 together with the model problem**

$$\triangle = f \ \textbf{ on } \qquad = 0 \ \textbf{ on}$$

where the right-hand side f **is chosen in such a way that the exact solution given by**

$$= r^{\frac{2}{3}} \sin\left(\frac{2}{3}\right)\left(1 \quad r^2\cos^2\right)\left(1 \quad r^2\sin^2\right)$$

According to [5, Thm. 1.4.5.3]), we have $^{\frac{5}{3}}$ "$( )$ 0**. Furthermore, we choose** $\mathbf{x}_1 = (0\ 4\ 0\ 3)$ **and** $\mathbf{x}_2 = (0\ 1\ 0\ 2)$**.**

Our computations are performed with $= 1$ and the results are collected in Table 1 and plotted in Figure 2. Since we have $^{\frac{5}{3}}$ "$( )$, we achieve a global convergence rate of $O(N^{\ \frac{2}{3}})$ measured in the energy norm. As Figure 2 shows, the local convergence rates are about twice the rates of the global error, which confirms our theoretical result of an increased local convergence rate. In the second example we want to verify our theoretical results for a domain with a more complicated boundary. To that end, we consider a domain looking like a snow flake (see Figure 3) together with the following Dirichlet problem:

Example 3.2.

$$\triangle = 1 \quad \textbf{on} \qquad = 0 \quad \textbf{on}$$

Figure 1: Coarse grid and refinement level 7

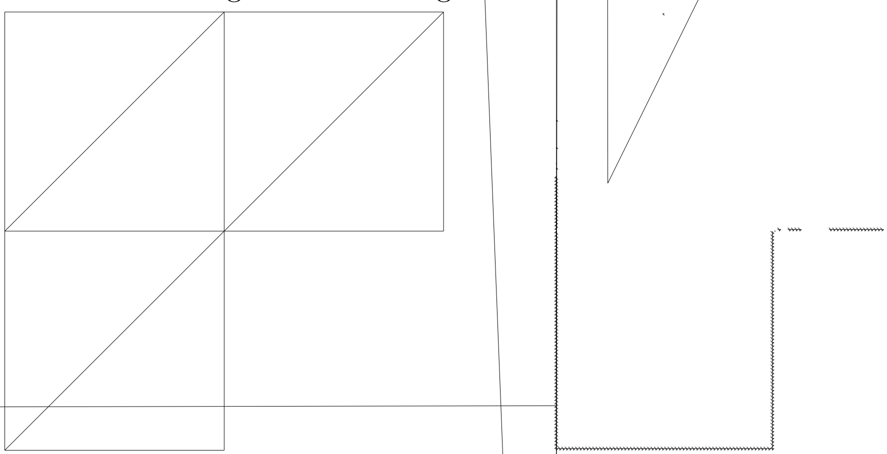
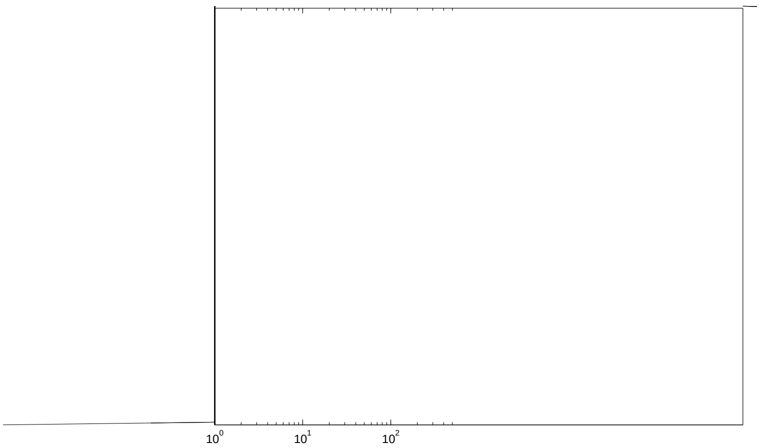


Figure 2: Results corresponding to Example 3.1



We do not know the exact solution of Example 3.2, but extrapolation leads to the results shown in Figure 3. As in the previous example and according to Theorem 2.1, we obtain local convergence rates that are significantly better than the rate of $O(N^{0.6})$ observed for the global energy norm.

An hp interpolation operator

In this section, we present a new variable order hp-interpolation operator. The operator is based on Gauss-Lobatto interpolation and is a very useful tool for our local analysis.

Properties of the Gauss Lobatto interpolation operator

In order to define our hp-interpolation operator, we start with recalling some facts about the one-dimensional Gauss-Lobatto interpolation operator i_p:

Lemma 4.1. On the interval $= (\ 1\ 1)$ **let** i_p **be the Gauss-Lobatto interpolation operator. Then for every** 1 **and** r $[0\ 1]$ **there exists** C 0 **depending solely on and** r **such that for every** $^{k}(\ )$

$$i_p \quad {}_{H^r(I)} \qquad Cp^{(k\ r)} \qquad {}_{H^k(I)} \tag{10}$$

$$\frac{1}{\sqrt{1\ \ ^2}}(\quad i_p\ )\ {}_{L^2(I)} \qquad Cp^{\ k}$$

the η-variable. Since $i_p = (i_p^x\,)|_\Gamma$, we get with the trace theorem and the one-dimensional stability results (14), (12)

$$\|i_p\|_{H^{1/2}(I)} \leq C\|i_p^x\|_{H^1(S)} \leq C\left[(1+p^0\,p)\|\ \|_{L^2(S)}+\|\ \|_{L^2(S)}\right] \leq C(1+p^0\,p)\|\ \|_{H^1(S)} \leq C(1+p^0\,p)\|\ \|_{H^{1/2}(I)}$$

For the last bound, estimate (16), we employ (11) with $= 1$ and the inverse estimate $\|\ \|_{H^1(I)} \leq p^0\|\ \|_{H^{1/2}(I)}$, which is valid for all polynomials p':

$$\left\|\frac{1}{\sqrt{1-\eta^2}}(\ -i_p\ )\right\|_{L^2(I)} \leq Cp^{-1}\|\ \|_{H^1(I)} \leq C\frac{p^0}{p}\|\ \|_{H^{1/2}(I)}$$

This estimate together with (15) implies (16). $\square$

By tensorization, the one-dimensional results can be generalized to results on the square:

Lemma 4.2. Let $= (-1, 1)^2$**. For** $p \in \mathbb{N}$ **denote by** $i_p^x\, i_p^y : C(\overline{\ }) \to\ _p$ **the tensor pr**

Next, we define $_{\mathbf{p}}(\widehat{\ })$ as

$$_{\mathbf{p}}(\widehat{\ }) := \qquad _{\mathbf{p}_{int}}(\widehat{\ }) \quad \Gamma_i \quad _{\mathbf{p}_i} \; i = 1 \quad n \tag{20}$$

For the edges $\Gamma_{\mathbf{i}}$ of $\widehat{\ }$, we denote by $i_{\mathbf{p};\Gamma_i}$ the Gauss-Lobatto interpolation operator of degree p on that edge.
Before coming to the construction of the interpolation operator, we recall the following polynomial lifting result:

Lemma 4.3. Let $\widehat{\ }$ be the reference square or the reference triangle. Then there exists a bounded linear operator E : $^{1=2}(\widehat{\ })$ $^{1}(\widehat{\ })$ such that $(E\ )_{@\widehat{\mathbf{K}}} =$ with the following property: if $^{1=2}(\widehat{\ })$ is a polynomial of degree p on each edge, then E $_{\mathbf{p}}(\widehat{\ })$.

Proof. See, e.g., [1, 13]. ☐

Theorem 4.4. Let $\widehat{\ }$ be the reference square or the reference triangle. Let $3\ 2$. Let $p_{\mathbf{i}}$, $i = 1 \quad n$, $p_{\mathbf{int}}$ $\mathbb{N}$ satisfy (19) and set

$$\underline{p} := \min_{\mathbf{i}=1;:::;\mathbf{n}} p_{\mathbf{i}} \qquad \overline{p} := \max_{\mathbf{i}=1;:::;\mathbf{n}} p_{\mathbf{i}} \quad p_{\mathbf{int}}$$

Then there exists a generic constant C 0 and a linear operator : $^{\mathbf{k}}(\widehat{\ })$ $_{\mathbf{p}}$ such that

- **(i)** $(\ )_{\Gamma_i} = i_{\mathbf{p}_i;\Gamma_i}$ **for** i $1 \quad n$ **;**
- **(ii)** $=$ **for all** $_{\mathbf{p}}(\widehat{\ })$**;**
- **(iii)** $_{\mathbf{H}^1(\mathbf{T})}$ $C(1 + p^0\ \underline{p})$ $_{\mathbf{H}^1(\mathbf{T})}$ **for all** $_{\mathbf{p}'}$**;**
- **(iv)** $_{\mathbf{H}^1(\mathbf{T})}$ $C(1 + p^0\ \underline{p})$ $_{\mathbf{H}^1(\mathbf{T})}$ **for all** $_{\mathbf{p}'}$**.**

Furthermor **erator**

the function is extended to via the universal extension operator of [15,

Auxiliary Results

This sections is devoted to the proof of all the auxiliary results that were used in the proof of Theorem 2.1.

he eight f nction $\omega_{\beta,\mathcal{T}}$

We start with studying the most important properties of the weight function $\omega_{\beta,\mathcal{T}}$ introduced in Definition 2.2.

Lemma 5.1. (properties of $\omega_{\beta,\mathcal{T}}$) Let be a geometric mesh and let $\omega_{\beta,\mathcal{T}}$ be given by De nition 2.2. Then there exist constants $C_1, \ldots, C_4$ 0 depending only on the shape-regularity constant and the constants of De nition 1.7 such that for all and arbitrary (0 1]

1. $\inf_{x \in K} \omega_{\beta,\mathcal{T}}(x) \quad C_2 \quad h$

Next, since [illegible];

Next, using property 3 of Lemma 5.1, we obtain:

$$\left\| \frac{f}{\overline{\quad ;\mathrm{T}}} \quad ;\mathrm{T} \right\|^2_{\mathbf{L}^2(\Omega)} \quad (C_2 \quad)^2 \int_\Omega \left(\frac{f}{r} \quad \overline{\quad ;\mathrm{T}} \right)^2 \mathrm{d} \qquad (33)$$

$$= \quad (C_2 \quad)^2$$

and by choosing $q_{\mathbf{K}} := \ ;\mathbf{T};\mathbf{K}\, g := \ ;\mathbf{T}\,(\mathbf{x_K})g(\mathbf{x})$ for $\mathbf{x_K}$ arbitrary, we arrive at

$$\frac{1}{\overline{\ ;\mathbf{T}}} \quad (\ ;\mathbf{T}\, g \quad (\ ;\mathbf{T}\, g))^{2}_{\mathbf{L}^2(\Omega)}$$

$$C \sum_{\mathbf{K}\,2\mathbf{T}} \frac{1}{\ ;\mathbf{T};\mathbf{K}} (\ ;\mathbf{T} \quad ;\mathbf{T};\mathbf{K})g^{\ 2}_{\mathbf{H}^1(\mathbf{K})}$$

$$C \sum_{\mathbf{K}\,2\mathbf{T}} \frac{1}{\ ;\mathbf{T};\mathbf{K}} \quad g \ ;\mathbf{T}\ ^{2}_{\mathbf{L}^2(\mathbf{K})} + \ (\ ;\mathbf{T} \quad ;\mathbf{T};\mathbf{K}) \ g\ ^{2}_{\mathbf{L}^2(\mathbf{K})}$$

From Lemma 5.1 we deduce $\overline{\ } ;\mathbf{T};\mathbf{K} \quad C \ ;\mathbf{T};\mathbf{K}$ for all . Thus, repeated use of Lemma 5.1 leads to

$$\frac{1}{\overline{\ ;\mathbf{T}}} \quad (\ ;\mathbf{T}\, g \quad (\ ;\mathbf{T}\, g))^{2}_{\mathbf{L}^2(\Omega)}$$

$$C\sum_{\mathbf{K}\,2\mathbf{T}} \frac{1}{\ ;\mathbf{T};\mathbf{K}} \left({}^{2} \ g\frac{\ ;\mathbf{T}}{r}\ ^{2}_{\mathbf{L}^2(\mathbf{K})} + h^2_{\mathbf{K}} \ ;\mathbf{T}\ ^{2}_{\mathbf{K}} \quad g\ ^{2}_{\mathbf{L}^2(\mathbf{K})}\right)$$

$$C\ {}^{2}\sum_{\mathbf{K}\,2\mathbf{T}} \frac{1}{\ ;\mathbf{T};\mathbf{K}} \left(\ g\frac{\ ;\mathbf{T}}{r}\ ^{2}_{\mathbf{L}^2(\mathbf{K})} + \overline{\ }^{2} ;\mathbf{T};\mathbf{K} \quad g\ ^{2}_{\mathbf{L}^2(\mathbf{K})}\right)$$

$$C\ {}^{2}\sum_{\mathbf{K}\,2\mathbf{T}} \left(g\frac{\overline{\ ;\mathbf{T}}}{r}\ ^{2}_{\mathbf{L}^2(\mathbf{K})} + \ \overline{\ ;\mathbf{T}} \quad g\ ^{2}_{\mathbf{L}^2(\mathbf{K})}\right)$$

Finally exploiting Lemma 5.2 gives the desired result. ☐

Appro i tion of $\mathcal{B}_{-\delta}$ f nctions fro $S^{\mathbf{p}}$ in ω eighted nor

In this subsection we will use the results of [9, Section 2.3.2] to deduce an approximation result for the -weighted ^{1}-seminorm. We start with recalling from [9] the following approximation result:

Lemma 5.4. Let be a geometric mesh with boundary mesh size h as de ned in De nition 1.7. Let p be a linear degree vector with slope 0. Let $\tilde{\mathcal{F}}^2_1 \ (C_{\mathbf{u}} \ \mathbf{u})$, $C_{\mathbf{u}} \ \mathbf{u}$ 0. Then there exists an element $^{\mathbf{p}}(\ )$ such that

$$_{\mathbf{H}^1(\mathbf{K})} \quad \begin{cases} CC_{\mathbf{K}} h_{\mathbf{K}} & \textbf{for all} \qquad \textbf{abutting on} \\ CC_{\mathbf{K}} h_{\mathbf{K}}\ ^{\mathbf{b}'} h\ ^{\mathbf{b}'} & \textbf{otherwise} \end{cases}$$

where C, $^{\mathbf{0}}$ 0 depend only on the shape-regularity constant , the constants of De nitions 1.7, 1.9, and $_{\mathbf{u}}$; C depends additionally on . The constants $C_{\mathbf{K}}$ are given by

$$C^2_{\mathbf{K}} := \sum_{\mathbf{n}=0}^{\mathbf{1}} \frac{1}{(2\ \ \mathbf{u})^{2\mathbf{n}}(n!)^2} \ r^{\mathbf{n}+1} \quad {}^{\mathbf{n}+2}\ {}^{2}_{\mathbf{L}^2(\mathbf{K})} \ \textbf{and we have} \sum_{\mathbf{K}\,2\mathbf{T}} C^2_{\mathbf{K}} \quad \frac{4}{3}C^2_{\mathbf{u}}$$

Proof. See [9, Proposition 2.10] for the construction of such an element. □

Now, by means of Lemma 5.4, we are able to prove the follo

Remark 5.6. **In the proof of Lemma 5.5 we demand** $^{0}\,{}^{1}(\;+\; 2)$**. Since** $(0\;1]$ **and** $(0\;1]$ **this claim will be ful lled if** $3\;(2\;{}^{0})$**, independent of and .**

□

Properties of z and z_h

In this section we want to point out the most important properties of z and $z_{\mathbf{h}}$ defined in Definition 2.3.

Lemma 5.7. **(basic properties of** z **and** $z_{\mathbf{h}}$**) Let the assumptions of Theorem 2.1 be valid and let** $^{00}\quad{}^{0}$ **. Furthermore, let** z **and** $z_{\mathbf{h}}$ **be given by De nition 2.3 Then for constants** C_Ω**,** $C_{\Omega'}$**,** $_{\mathbf{z}}$ **depending on ,** ^{0}**,** $_{0}$ **we have:**

1. $z\;_{\mathbf{H}^1(\Omega)}\quad z\;_{\mathbf{H}^{1+\delta_0}(\Omega)}\quad C_\Omega\quad \mathbf{h}\;_{\mathbf{L}^2(\dot{\mathbf{K}})}$
2. $z\quad{}^{2}(\;{}^{0})$ **and** $z\;_{\mathbf{H}^2(\Omega')}\quad C_{\Omega'}\quad \mathbf{h}\;_{\mathbf{L}^2(\dot{\mathbf{K}})}$
3. $z\;_{\Omega\mathbf{n}\Omega'}\quad \tilde{r}^2_1\;{}_0(C_{\Omega'}\quad \mathbf{h}\;_{\mathbf{L}^2(\dot{\mathbf{K}})}\;\mathbf{z})$
4. $z\quad z_{\mathbf{h}}\;_{\mathbf{H}^1(\Omega)}\quad c\quad \mathbf{h}\;_{\mathbf{L}^2(\dot{\mathbf{K}})}$.

Proof.

1. This is just a rephrasing of Assumption 1.3.
2. This expresses interior regularity for elliptic problems: From [6, Thm. 9.1.26] we obtain $z\quad{}^{2}(\;{}^{0})$ together with

   $$z\;_{\mathbf{H}^2(\Omega')}\quad C_{\Omega'}\quad \mathbf{h}\;_{\mathbf{L}^2(\dot{\mathbf{K}})}+\;z\;_{\mathbf{H}^1(\Omega)}$$

   The desired bound now follows from this estimate and the preceding one.
3. This follows from [9, Thm. A.1]: Without loose of generality, we may assume ^{00} to be a smooth domain. Since $z\quad{}^{1+}\;{}_0(\quad C\;z_{\mathbf{h}}\;\mathbf{in}\;1\quad 0\;6\quad\Omega(\quad)\;\Omega 23\quad 4\quad 1\;1\;2002\quad\Omega(\;43\;11$

Lemma 5.8. **Let the assumptions of Theorem 2.1 be valid. Furthermore, let z be given by Definition 2.3. Then there exists an element $q \quad Y^{\mathbf{P}}(\quad)$ such that**

$$\, _1 z \quad q \, _{\mathbf{H}^{-\frac{1}{2}}(@\Omega)} \quad C_{\mathbf{s}} h^{\mathbf{s}} \quad \mathbf{h} \, _{\mathbf{L}^2(\dot{\mathbf{K}})} \quad s \quad (0 \quad _0] \tag{35}$$

where $q \quad Y^{\mathbf{P}}(\quad)$ denotes the representation of q given by the Riesz representation theorem.

Proof. Assumption 1.3 gives us $_0 \quad 0$ such that $_1 z \qquad ^{1=2+\mathbf{s}}(\quad)$ with

$$\, _1 z \, _{\mathbf{H}^{s-\frac{1}{2}}(@\Omega)} \quad C \quad _{\dot{\mathbf{K}}}(\quad \mathbf{h}) \, _{\mathbf{L}^2(\Omega)} = C \quad \mathbf{h} \, _{\mathbf{L}^2(\dot{\mathbf{K}})}$$

for all $0 \quad s \quad _0$. [9, Lemma 2.8] guarantees the existence of an element $q \quad Y^{\mathbf{P}}(\quad)$ such that

$$\, _1 z \quad q \, _{\mathbf{H}^{-\frac{1}{2}}(@\Omega)} \quad C_{\mathbf{s}} h^{\mathbf{s}} \, _1 z \, _{\mathbf{H}^{s-\frac{1}{2}}(@\Omega)}$$

Combining these two inequalities yields the desired bound (35). ☐

Lemma 5.9. **Let the assumptions of Theorem 2.1 be valid. Furthermore, let z be given by Definition 2.3 and let $\quad _{;T}$ be given by Definition 2.2. Then, for $\quad$ sufficiently large depending only on the**

Exploiting z $^2(\tilde{\ })$ with z $_{\mathbf{H}^2(\tilde\Omega)}$ $C_{\tilde\Omega}$ $\mathbf{h}$ $_{\mathbf{L}^2(\dot{\mathbf{K}})}$ (see Lemma 5.7), pulling back to the reference triangle and making use of Theorem 4.4 we can bound the second sum as follows:

$$\sum_{\mathbf{K}\,2\mathbf{T}_2} \left(\frac{h}{h_{\mathbf{K}}}\right) z \quad z\,^2_{\mathbf{H}^1(\mathbf{K})} \quad C \sum_{\mathbf{K}\,2\mathbf{T}_2} \left(\frac{h}{h_{\mathbf{K}}}\right) \hat{z}\,^2_{\mathbf{H}^2(\hat{\mathbf{K}})}$$
$$C \sum_{\mathbf{K}\,2\mathbf{T}_2} \left(\frac{h}{h_{\mathbf{K}}}\right) h^2_{\mathbf{K}}\ z\,^2_{\mathbf{H}^2(\mathbf{K})}$$
$$Ch \sum_{\mathbf{K}\,2\mathbf{T}_2} z\,^2_{\mathbf{H}^2(\mathbf{K})} \quad Ch \quad \mathbf{h}\ ^2_{\mathbf{L}^2(\dot{\mathbf{K}})} \tag{36}$$

with a constant C independent of h and . In order to bound the first sum, we exploit z $_{\Omega\mathbf{n}\tilde\Omega}$ $\tilde{\mathcal{F}}^2_1$ $_0(C_{\tilde\Omega}$ $\mathbf{h}$ $_{\mathbf{L}^2(\dot{\mathbf{K}})}$ $\mathbf{z})$. Because of Lemma 5.4 and since no $_1$ has a distance less than $ch_{\dot{\mathbf{K}}}$ from we obtain

$$\sum_{\mathbf{K}\,2\mathbf{T}_1} z \quad z\,^2_{\mathbf{H}^1(\mathbf{K})} \quad \sum_{\mathbf{K}\,2\mathbf{T}_1\mathbf{j}\mathbf{K}\,\backslash@\Omega 6=;} C^2_{\mathbf{K};\mathbf{z}} h^{2\ 0} + \sum_{\mathbf{K}\,2\mathbf{T}_1\mathbf{j}\mathbf{K}\,\backslash@\Omega\,=;} C^2_{\mathbf{K};\mathbf{z}} h^{2(\ 0\quad \mathbf{b}')}_{\mathbf{K}} h^{2\ \mathbf{b}'}$$

That is, for sufficiently large, we obtain

$$\sum_{\mathbf{K}\,2\mathbf{T}_1} z \quad z\,^2_{\mathbf{H}^1(\mathbf{K})} \quad h^{2\ 0} C \sum_{\mathbf{K}\,2\mathbf{T}_1} C^2_{\mathbf{K};\mathbf{z}} \quad h^{2\ 0} C \quad \mathbf{h}\ ^2_{\mathbf{L}^2(\dot{\mathbf{K}})} \tag{37}$$

Combining (36) and (37) gives us the Tj R34 7.97011 Tf 7.42812 0 Td (ffi)T7Tj R2tLce.givv

Now, Lemma 5.2 garantees the existence of 0 and C^{0}, 0 such that

$$\int_\Omega e \quad e \quad \mathrm{d} \quad C \int_\Omega \overline{\ ;\mathsf{T}}\,\frac{e}{r} \quad \overline{\ ;\mathsf{T}} \quad e\ \mathrm{d}$$
$$C \quad \overline{\ ;\mathsf{T}}\,\frac{e}{r}\ {}_{\mathbf{L}^2(\Omega)} \quad \overline{\ ;\mathsf{T}} \quad e\ {}_{\mathbf{L}^2(\Omega)}$$
$$CC^{0}, \quad \overline{\ ;\mathsf{T}} \quad e\ {}^{2}_{\mathbf{L}^2(\Omega)}$$

for all (0]. Since C is a monotone increasing function of , we additionally claim CC^{0}, that is, for ε_0 sufficiently small we have

$$\| \overline{\omega_{\varepsilon;T}}\, e \|^2_{L^2(\Omega)} \le C_\varepsilon \| \overline{\omega_{\varepsilon;T}}\, e \|_{L^2(\Omega)} \| \overline{\omega_{\varepsilon;T}}\, (z - z) \|_{L^2(\Omega)};$$

for all $\varepsilon \in (0, \varepsilon_0]$ and finally, Lemma 5.9 yields (38).

□

Outlook

In Theorem 2.1 we proved the existence of some $\beta > 0$ such that the local error estimates (5), (6), and (7) hold. Since all of our numerical experiments achieve $\beta = \alpha$ we assume that it is actually possible to prove an improved version of Theorem 2.1, where $\beta > 0$ is replaced by $\beta = \alpha$. Numerical evidence such as Example 6.1 below indicates that Theorem 2.1 is not necessarily restricted to Dirichlet problems but is also true for other types of boundary conditions such as Neumann or mixed boundary conditions.

We want to mention that the doubling of the convergence rate can be obtained using the "standard" duality approach if a slightly different mesh is considered as proposed in [7] (see also [8]). There, the mesh size is defined according to $h_K \sim \min\{\sqrt{h}, h + \operatorname{dist}(K, \partial\Omega)\}$ and the polynomial degree $\mathbf{p}$ is defined as in Definition 1.9. The key thing to note is that in the interior of the computational domain a quasi-uniform mesh with mesh size $O(\sqrt{h})$ and fixed polynomial degree is employed. Thus, the standard duality arguments can be used to recover a local L_2-convergence rate of $O(h^{1/2+\alpha})$ for $u \in H^{1+\alpha}(\Omega)$. It should be noted that the above choice of meshes and polynomial degree distribution also lead to a problem size $N = O(h^{-1})$.

Example 6.1. (mixed boundary conditions) We consider the L-shaped domain as shown in Figure 1 together with the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0]) \\ \partial_{n_1} u &= 0 \quad \text{on } \partial\Omega_{\Gamma_N} = (\{-1\} \times [-1,1]) \cup ([-1,1] \times \{1\}) \\ u &= 0 \quad \text{on } \partial\Omega_{\Gamma_D} = \partial\Omega \setminus \Gamma_N, \end{aligned}$$

where the right-hand side f is chosen in such a way that the exact solution u is given by

$$u = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right)(1 - r^2\cos^2\varphi)(1 + r\cos\varphi) \ldots$$

Table 2: Example 6.1 with $e =$ **h**

			$\mathbf{x}_1 = (0\ 4\ 0\ 3)$		$\mathbf{x}_2 = (0\ 1\ 0\ 2)$	
$Le\ \ e$	h	$p_{\mathbf{max}}$	$e\ \ _{\mathbf{L}^2(\dot{\mathbf{K}})}$	$e\ \ _{\mathbf{H}^1(\dot{\mathbf{K}})}$	$e\ \ _{\mathbf{L}^2(\dot{\mathbf{K}})}$	$e\ \ _{\mathbf{H}^1(\dot{\mathbf{K}})}$
1	5.000e-01	1	3.0616e-02	2.1547e-01	4.4508e-02	1.7582e-01
2	2.500e-01	2	3.9843e-03	3.4705e-02	9.3600e-03	8.2524e-02
3	1.250e-01	2	9.3002e-04	4.0317e-03	2.0784e-03	2.0144e-02
4	6.250e-02	3	3.4768e-04	2.8333e-03	5.8271e-04	3.4600e-03
5	3.125e-02	4	1.0915e-04	4.3509e-04	2.0533e-04	1.4255e-03

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