# A collocation method for high frequency scattering by convex polygons

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The kernel, right hand side, and solution of (5) all oscillate rapidly when is large, and thus it is well known that the computational cost of solving (5) by standard schemes, with piecewise polynomial approximation spaces, grows at least linearly with respect to the frequency (see e.g. [8,21] and the references therein). However, by removing the high frequency asymptotics and solving a modified integral equation whose solution approaches zero almost everywhere as $\rightarrow \infty$, it is possible to devise numerical schemes for solving integral equations such as (5) with computational costs that grow at a sublinear rate as increases (see e.g. [1,6,8,19]).

In particular, in [8] Chandler-Wilde and Langdon recently proposed a novel Galerkin boundary element method for solving (5) for which it was demonstrated via both a rigorous error analysis and numerical simulations that the number of degrees of freedom required to solve (5) (and thus (1)–(3)) to a prescribed level of accuracy grows only logarithmically with respect to . This appears to be the best result to date for problems of scattering by bounded obstacles, and was achieved by removing the leading order high frequency asymptotic behaviour from (5) and using a consideration of a related set of half plane problems to demonstrate that for $s \in [0\ L]$, (where $\mathrm{x}(s)$, $s \in [0\ L]$, parametrises Γ)

$$\frac{1}{\ }\frac{\ }{\mathrm{n}}(\mathrm{x}(s)) = \text{known leading order terms} + \mathrm{e}^{iks}\,{}_{+}(s) + \mathrm{e}^{-iks}\,{}_{-}(s) \qquad (6)$$

with $_{\pm}$ and all its derivatives highly peaked near the corners of the polygon, and rapidly decaying away from the corners. The oscillatory nature of $/\ \mathrm{n}$ is thus represented exactly in (6) by the known leading order terms and the terms $\mathrm{e}^{\pm iks}$, and to approximate $/\ \mathrm{n}$ all that is required is to approximate the smooth functions $_{\pm}$. These functions decay su□ciently quickly that the number of degrees of freedom required to maintain the accuracy of their best $L^2$ approximation from a space of piecewise polynomials supported on a graded mesh, with a higher concentration of mesh points closer to the corners of the polygon, grows only logarithmically with respect to as $\rightarrow \infty$.

The question then arises of how we might go about selecting our best $L^2$ approximation to $_{\pm}$ from the approximation space. In [8] a Galerkin scheme is used, for which both stability and convergence are proved. However, the implementation of this scheme requires the evaluation of many 1

This is the integral equation we will solve numerically, with existence and boundedness for $( + )^{-1}$ following immediately from [8, theorem 2.5].

We now define more precisely our approximation space $_{N,\nu}$. We begin by defining the graded mesh we will use, which is the same as in [8].

**Definition 1** *For* $A$ , $N = 2\ 3$ , *the mesh*

$$\Lambda_{N,A,\lambda,q} := \{y_0 \quad y_{N+\hat{N}_A \quad q}\}$$

*consists of the points*

$$y = \left(\frac{}{N}\right)^q \qquad = 0 \qquad N$$

*together with the points*

$$y_{N+j} := \left(\frac{A}{}\right)^{j/\hat{N}_A \quad q} \qquad = 1 \qquad \hat{N}_{A,\lambda,q} \tag{13}$$

*where* $\hat{N}_{A,\lambda,q} = \lceil N \rceil$, *the smallest integer greater than or equal to* $N$, *with*

$$N = \frac{-\log(A/\ )}{q\log(1-1/N)} \tag{14}$$

For $= 1 \quad n$, we define $q_j := (2 + 3)/(2 /\Omega_j - 1)$, and the two meshes

$$\Gamma_j^+ := \tilde{L}_{j-1} + \Lambda_{N,L\ ,\lambda,q} \qquad \Gamma_j^- := \tilde{L}_j - \Lambda_{N,L\ ,\lambda,q\ +1}$$

Letting $e_\pm(s) := e^{\pm iks}$, $s \in [0\ L]$, we then define

$$_{\Gamma^+,\nu} := \{\ e_+ : \ \in \Pi_{\Gamma^+,\nu}\} \qquad _{\Gamma^-,\nu} := \{\ e_- : \ \in \Pi_{\Gamma^-,\nu}\}$$

for $= 1 \quad n$, where

$$\Pi_{\Gamma^+,\nu} := \{\ \in L^2(0\ L) : \ |_{(\tilde{L}\ _{-1}+y\ _{-1},\tilde{L}\ _{-1}+y\ )} \text{ is a polynomial of degree } \le$$
$$\text{for } = 1 \quad N + \hat{N}_{L\ ,\lambda,q} \text{ and } |_{(0,\tilde{L}\ _{-1})\cup(\tilde{L}\ ,L)} = 0\}$$
$$\Pi_{\Gamma^-,\nu} := \{\ \in L^2(0\ L) : \ |_{(\tilde{L}\ -\tilde{y}\ ,\tilde{L}\ +\tilde{y}\ _{-1})} \text{ is a polynomial of degree } \le$$
$$\text{for } = 1 \quad N + \hat{N}_{L\ ,\lambda,q\ +1} \text{ and } |_{(0,\tilde{L}\ _{-1})\cup(\tilde{L}\ ,L)} = 0\}$$

**with the points of the mesh** $\boldsymbol{\Lambda}_{N,L\ ,\lambda,q}$ **given by** $y_0 \quad y_{N+\hat{N}_{\boldsymbol{L}}\ \ q}$**, and the points of the mesh** $\boldsymbol{\Lambda}$

combination of the basis functions of $_{N,0}$, we have

$$\varphi_N(s) := \sum_{j=1}^{M} c_j \; _j(s) \tag{19}$$

where $_j$ is the th basis function and $M_N$ is the dimension of $_{N,0}$. For $p = 1 \quad n$, where $n$ is the number of sides of the polygon, we define $n_p^{\pm}$ to be the number of elements of $\Gamma_p^{\pm}$, so

$$n_p^+ := N + \hat{N}_{L_p,\lambda,q_p} \qquad n_p^- := N + \hat{N}_{L_p,\lambda,q_{p+1}}$$

and we denote the elements of $\Gamma_p^{\pm}$ by $s_{p,l}^{\pm}$, for $l = 1 \quad n_p^{\pm}$. Denoting further the total number of elements supported on $\Gamma_p$ by $\hat{p} := \sum_{=1}^{p-1} n_p^+ + n_p^-$, we then have for $p = 1 \quad n$,

$$_{\hat{p}+j}(s) := \begin{cases} e^{ik(s-x_p^+\;)} \; _{[s_p^+\;\; -1, s_p^+\;)}(s) & = 1 \quad n_p^+ \\ e^{-ik(s-x_p^-} & \end{cases}$$

will almost match, leading to ill-conditioned systems (see also [9] where a related problem was solved using a mesh of this type).

(2) This approach leads to a much larger number of degrees of freedom than is necessary, with $_{-}$ being approximated by far more basis functions than necessary on $\Gamma_j$ near $P_j$, and $_{+}$ being approximated by far more basis functions than necessary on $\Gamma_j$ near $P_{j+1}$.

Instead we use the mesh described above, as for the Galerkin method of [8]. In general, it is hard to say much about the spacing of the collocation points, and hence about the conditioning of the linear system (22) for a general polygon. However, considering for simplicity the side $\Gamma_1$ we remark that the collocation points $^{+}_{1,j}$ will be very dense on [0 ], and sparse on ( $L_1$], whilst the collocation points $^{-}_{1,j}$ will be very dense on [$L_1 -$ $L_1$], and sparse on [0 $L_1 -$ ). So, provided there are no collocation points $^{-}_{1,j}$ in [0 ] or $^{+}_{1,j}$ in [$L_1 -$ $L_1$], then there is a better chance for the system to be well conditioned. Considering the points of $^{+}_{1,j}$, we require $s^{+}_{1,n_p^+}$ −

The Galerkin approximation (16) leads to a linear system of the form

$$\sum_{j=1}^{N} c_j [( \ _j \ _m) + ( \ _j \ _m)] = (f \ _m) \qquad \text{for } = 1 \ 2 \qquad N_G$$

Recalling (9), (10), this leaves many double integrals of the form

$$( \ _j \ _m) = \int_{\mathrm{supp}_\rho} \int_{\mathrm{supp}_\rho} \left( \frac{\Phi}{\mathrm{n}} + \ \Phi \right) \ _j (s) \ _m ( \ ) \, \mathrm{d}s \, \mathrm{d} \qquad (24)$$

to evaluate (see [8] and also [18] for details). This is a double integral over the support of each of the basis functions of an oscillatory function, since the term ( $\Phi/$ n + i $\Phi$) is oscillatory as are the basis functions $_j$ and $_m$. Using the Riemann-Lebesgue Lemma, and as described in [16], in principal at least an integral should become easier to evaluate as it becomes more oscillatory, as due to cancellation of oscillating terms the exact value will tend to zero more quickly as the oscillations increase. However, using this information to construct an accurate numerical scheme for highly oscillatory integrals of the form (24) is a di□cult task, and most schemes presented recently in the literature for the evaluation of highly oscillatory integrals focus on one-dimensional integrals.

However, for the linear system (22) the single integrals

$$_j(s_m) = 2 \int_{y}^{y+1} (s_m \ ) \mathrm{e}^{\pm ik(t-s \ )} \, \mathrm{d} \qquad (25)$$

are a little easier to evaluate, where here $s_m$, $= 1$ $M_N$ represent the collocation points and $[y_j \ y_{j+1}]$ the support of $_j$.

If the collocation point lies on the same side as the support of the basis function then

$$(s_m \ ) = - \ _0^{(1)} ( \ |s_m - \ |) \qquad (26)$$

and using the identity [20, equation (12.31)]

$$_0^{(1)} (s) = -\frac{2}{} \int_0^\infty \frac{e^{( -t)s}}{^{\frac{1}{2}} ( \ - 2 \ )^{\frac{1}{2}}} d \qquad s > 0 \qquad (27)$$

we can write (25) as

$$\frac{}{2\ e^{iks}} \int_0^\infty \frac{(r)}{r^{\frac{1}{2}}(r-2i)^{\frac{1}{2}}}\, dr \tag{28}$$

where

$$(r) := \int_{y}^{y\ +1} e^{(i-r)k|s\ -t|+\sigma\ ikt}\, d \tag{29}$$

with $_j = \pm 1$. It is shown in [3] that

$$(r) = \begin{cases} \frac{e\ (\ -\ )s\ \left(e^{-}\ (\ -\ (1\ ))\ -e^{-}\ +1(\ -\ (1\ ))\right)}{k(r-(1+\sigma\ ))} & s_m < y_j \\ \frac{e^{-}\ (\ -\ )s\ \left(-e\ (\ +\ (\ -1))\ +e\ +1(\ +\ (\ -1))\right)}{k(r+(\sigma\ -1))} & s_m > y_{j+1} \\ \frac{e\ s\ -e\ (\ -s\ )+\ (s\ +\ (\ -1))}{ik((\sigma\ -1))} + \frac{e\ s\ -e\ (s\ -\ +1)+\ (\ +1(1\ )-s\ )}{ik(r-(1+\sigma\ ))} & y_j < s_m < y_{j+1} \end{cases}$$

and then to evaluate (29) we make the substitution $r = s^2/(1-s^2)$, to reduce the inter of

$$_{+} := {}_{a}^{b} (s)\mathrm{e}^{\mathrm{i}k(s+\sqrt{s^2+c^2})}\,\mathrm{d}s$$

with the method for the evaluation of $_{-} := {}_{a}^{b} (s)\mathrm{e}^{\mathrm{i}k(-s+\sqrt{s^2+c^2})}\,\mathrm{d}s$ following analogously. Making the substitution $= s + \sqrt{s^2 + c^2}$ we have

$$_{+} = {}_{a+\sqrt{a^2+c^2}}^{b+\sqrt{b^2+c^2}} \left(\frac{{}^2 - c^2}{2}\right) \frac{\sqrt{{}^2 + c^2}}{2\,{}^2}\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d} \tag{30}$$

and methods for evaluating this type of integral are well established. In par-

| | $N$ | $M_N$ | $\| \quad - \quad _{NC}\|_2 \cdot \| \quad \|_2$ | OC |
|---|---|---|---|---|
| [illegible] | | | [illegible] $\times$ [illegible] $^{-1}$ | |
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