Efficient evaluation of highly oscillatory acoustic scattering surface integrals

Reading University Numerical Analysis Report 6/05

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Abstract

We consider the approximation of highly oscillatory weakly singular surface integrals, arising from boundary integral methods for solving high frequency acoustic scattering problems in three dimensions. As the frequency of the incident wave increases, the performance of standard quadrature schemes deteriorates. Naive application of asymptotic schemes also fails due to the weak singularity. We present here a new scheme based on a combination of an asymptotic approach and exact treatment of singularities in an appropriate coordinate system. We demonstrate that the computational cost of evaluating many integrals over the surface of a scatterer does not increase significantly as the frequencw28(ord6n)1(ls)-37694 iw3.549549Tds:25455.805Td[(Abs60.7

1 Introduction

This paper is concerned with the approximation of integrals of the form

$$M\psi(\mathbf{x}) := \int_{\partial D} \frac{m(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} e^{ik[|\mathbf{x}-\mathbf{y}|+\hat{\mathbf{d}}.(\mathbf{y}-\mathbf{x})]}\psi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D, \tag{1}$$

where $m(\mathbf{x},\mathbf{y})$, $\psi(\mathbf{y})$ are smooth and slowly oscillating functions, $\hat{\mathbf{d}}$ is a fixed unit vector (the incident wave direction), and ∂D is the surface of a three dimensional convex obstacle D. Such integrals arise in boundary integral methods for acoustic scattering problems, and if the acoustic size kA is large (where A is the size of obstacle), corresponding to the high frequency problem, the

In particular, for scattering by convex three dimensional obstacles with smooth boundaries a number of very efficient high order boundary integral schemes have recently been proposed [2,4]. These schemes exhibit extremely fast (superalgebraic or even exponential) convergence rates for frequencies starting from the resonance region to a medium level (size of the obstacle is about a hundred times the wavelength). But they break down for shorter wavelengths. One of the main reasons for this is the expense of evaluating many highly oscillatory integrals of the form (1) in the scheme.

The unique radiating solution u of (2) can be represented as [3, p.59]

$$u(\mathrm{x}) = -\int_{\partial D} \Phi(\mathrm{x},\mathrm{y}) v(\mathrm{y})\, \mathrm{d}s(y), \quad \mathrm{x} \in \mathbb{R}^3 \setminus \overline{D},$$

where $\Phi(\mathrm{x},\mathrm{y}) := e^{ik|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathrm{x}-\mathrm{y}|)$, and the unknown density $v \in C(\partial D)$ is the unique solution of the boundary integral equation

$$\frac{1}{2}v(\mathrm{x}) + \int_{\partial D}\left[\frac{\partial \Phi(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})} - \mathrm{i}\Phi(\mathrm{x},\mathrm{y})\right] v(\mathrm{y})\, \mathrm{d}s(\mathrm{y}) = \frac{\partial u^i}{\partial \mathrm{n}}(\mathrm{x}) - \mathrm{i}u^i(\mathrm{x}), \ \mathrm{x} \in \partial D. \quad (3)$$

We write $v(\mathrm{x}) = \varphi(\mathrm{x}) e^{ik\mathbf{x}.\hat{d}}$, where φ is slowly oscillating compared to $e^{ik\mathbf{x}.\hat{d}}$ (see e.g. [1]). This reduces (3) to the second kind boundary integral equation

$$\varphi(\mathrm{x}) + \int_{\partial D} \frac{m(\mathrm{x},\mathrm{y})}{|\mathrm{x}-\mathrm{y}|} e^{ik[|\mathbf{x}-\mathbf{y}| + \hat{\mathbf{d}}.(\mathbf{y}-\mathbf{x})]} \varphi(\mathrm{y})\, \mathrm{d}s(\mathrm{y}) = 2\mathrm{i}(k\mathrm{n}(\mathrm{x}) \cdot \hat{\mathrm{d}} - 1), \quad (4)$$

where $\mathrm{n}(\mathrm{x})$ is the unit outward normal vector to the surface ∂D at x and $m(\mathrm{x},\mathrm{y})$ is a smooth function, given by

$$m(\mathrm{x},\mathrm{y}) := \frac{1}{4\pi}$$

2 Singularity-free formulation

Under the assumption that the surface ∂D of the convex scatterer can be described globally in spherical coordinates, we write $\mathrm{x} \in \partial D$ as

$$\mathrm{x} = \mathrm{q}(\hat{\mathrm{x}}) = r(\theta, \varphi)\mathrm{p}(\theta, \varphi), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi], \tag{6}$$

where $\hat{\mathrm{x}} \in \partial B$ (the unit sphere), is given by

$$\hat{\mathrm{x}} = \mathrm{p}(\theta, \varphi) := (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^{\mathrm{T}}, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

With J being the Jacobian of q, we get for any integrable ψ on ∂D,

$$\int_{\partial D} \psi(\mathrm{x})\, \mathrm{d}s(\mathrm{x}) = \int_{\partial B} \psi(\mathrm{q}(\hat{\mathrm{x}}))J(\hat{\mathrm{x}})\, \mathrm{d}s(\hat{\mathrm{x}}), \tag{7}$$

and using (6) and (7), we rewrite (1) as

$$M\psi(\mathrm{q}(\hat{\mathrm{x}})) = \int_{\partial B} \frac{m(\mathrm{q}(\hat{\mathrm{x}}), \mathrm{q}(\hat{\mathrm{y}}))}{|\mathrm{q}(\hat{\mathrm{x}}) - \mathrm{q}(\hat{\mathrm{y}})|} \mathrm{e}^{\mathrm{i}k[|\mathbf{q}(\hat{\mathbf{x}}) - \mathbf{q}(\hat{\mathbf{y}})| + \hat{\mathbf{d}}.(\mathbf{q}(\hat{\mathbf{y}}) - \mathbf{q}(\hat{\mathbf{x}}))]} \psi(\mathrm{q}(\hat{\mathrm{y}}))J(\hat{\mathrm{y}})\, \mathrm{d}s(\hat{\mathrm{y}}). \tag{8}$$

Recalling that $\hat{\mathrm{d}}$ is a fixed unit direction vector, $|\hat{\mathrm{d}}| = 1$, we write $\hat{\mathrm{d}} = \mathrm{p}(\theta_d, \varphi_d)$, for some $\theta_d \in [0, \pi]$, $\varphi_d \in [0, 2\pi]$. To simplify the weak singularity in (8), we then use the same transformation matrix as in [4]. For each $\hat{\mathrm{x}}$ or each;

With $\hat{\mathbf{z}} = \mathrm{p}(\theta', \varphi')$ (and noting that θ' is then the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{z}}$, equivalently the angle between $\hat{\mathrm{x}}$ and $\hat{\mathrm{y}}$), and recalling (9) and that $T_{\hat{\mathbf{x}}}$ is an orthogonal transformation, it is straightforward to show that $T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{z}} = \mathrm{p}(\theta'_x, \varphi'_x)$, where θ'_x and φ'_x are functions of $\theta, \varphi, \theta', \varphi'$ satisfying

$$\sin\theta'_x \cos\varphi'_x = \sin\theta'(\cos\theta\cos\varphi\cos(\varphi-\varphi') + \sin\varphi\sin(\varphi-\varphi')) + \cos\theta'\sin\theta\cos\varphi,$$

$$\sin\theta'_x \sin\varphi'_x = \sin\theta'(\cos\theta\sin\varphi\cos(\varphi-\varphi') - \cos\varphi\sin(\varphi-\varphi')) + \cos\theta'\sin\theta\sin\varphi,$$

$$\cos\theta'_x = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\varphi-\varphi').$$

Using (6), we then get the following equalities;

$$\tilde{f}_1(\hat{\mathrm{x}}, \hat{\mathrm{z}}) = \sqrt{[r(\theta'_x, \varphi'_x) - r(\theta,\varphi)\cos\theta']^2 + [r(\theta,\varphi)]^2\sin^2\theta'} =: f_1(\theta, \varphi, \theta', \varphi'), \tag{10}$$

$$\begin{aligned}\tilde{f}_2(\hat{\mathrm{x}}, \hat{\mathrm{d}}, \hat{\mathrm{z}}) &= r(\theta'_x, \varphi'_x) h(\theta_d, \varphi_d, \theta'_x, \varphi'_x) - r(\theta,\varphi) h(\theta_d, \varphi_d, \theta, \varphi), \\ &=: f_2(\theta, \varphi, \theta_d, \varphi_d, \theta', \varphi').\end{aligned} \tag{11}$$

where $h(a, b, c, d) := \sin a \sin c \cos(b-d) + \cos a \cos c$. Hence using the notation

$$f(\theta, \varphi, \theta_d, \varphi_d, \theta', \varphi') := f_1(\theta, \varphi, \theta', \varphi') + f_2(\theta, \varphi, \theta_d, \varphi_d, \theta', \varphi'), \tag{12}$$

$$H(\theta, \varphi, \theta', \varphi') := m(r(\theta,\varphi)\mathrm{p}(\theta,\varphi), r(\theta'_x, \varphi'_x)\mathrm{p}(\theta'_x, \varphi'_x)) \frac{2\sin(\theta'/2)}{f_1(\theta, \varphi, \theta', \varphi')} J(\mathrm{p}(\theta'_x, \varphi'_x)), \tag{13}$$

and noting the 2π periodicity of the integrand with respect to φ', we get

$$\begin{aligned}&(M\psi)(r(\theta,\varphi)\mathrm{p}(\theta,\varphi)) = \\ &\int_{\varphi}^{2\pi+\varphi}\int_0^{\pi} H(\theta, \varphi, \theta', \varphi') e^{ikf(\theta,\varphi,\theta_d,\varphi_d,\theta',\varphi')} \psi(r(\theta'_x, \varphi'_x)\mathrm{p}(\theta'_x, \varphi'_x)) \cos\frac{\theta'}{2}\, \mathrm{d}\theta'\, \mathrm{d}\varphi'.\end{aligned} \tag{14}$$

Recalling the smoothness of $m(\cdot, \cdot)$ and $J(\cdot)$, and noting that

$$\left|\frac{2\sin(\theta'/2)}{f_1(\theta, \varphi, \theta', \varphi')}\right| \le \min\left(\frac{1}{|r(\theta,\varphi)\cos(\theta'/2)|}, \frac{1}{|r(\theta'_x, \varphi'_x) - r(\theta,\varphi)\cos\theta'|}\right),$$

we have that $H(\theta, \varphi, \theta', \varphi')$ is a smooth (analytic) function in $\theta, \varphi, \theta', \varphi'$.

3 Evaluation of critical points

It is well known (see e.g. [6]) that the main contribution to the generalized Fourier integral (14) comes only from the values of the integrand at three types of *critical points*:

(i) Stationary points, at which $\nabla f := \left(\frac{\partial f}{\partial \theta'}, \frac{\partial f}{\partial \varphi'}\right)^T = 0.$

(ii) Points on the boundary, at which one of the following equations holds.

$$\frac{\partial f(0, \varphi')}{\partial \varphi'} = 0;$$

In general, the nonlinear system (16)–(17) cannot be solved analytically. For notational simplicity and analytical calculations, in the remainder of this paper we will assume $r \equiv 1$ and $\hat{d} = [0, 0, 1]^T$. Using (11)–(13), we get

$$H(\theta, \varphi, \theta', \varphi') = H(k, \theta') := \frac{1}{4\pi}\left(-\frac{1}{2} + \mathrm{i}\left(k \sin\frac{\theta'}{2} - 1\right)\right), \tag{18}$$

$$f(\theta, \varphi, \theta', \varphi') = 2\sin\frac{\theta'}{2} + \cos\theta(\cos\theta' - 1) - \sin\theta\sin\theta'\cos(\varphi - \varphi'), \tag{19}$$

$$\frac{\partial f}{\partial \theta'} = \cos\frac{\theta'}{2} - \cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi - \varphi'), \tag{20}$$

$$\frac{\partial f}{\partial \varphi'} = -\sin\theta\sin\theta'\sin(\varphi - \varphi') \tag{21}$$

In the following theorem, we describe critical points of type (i).

Theorem 3.1 *The stationary points $(\theta', \varphi') \in [0, \pi] \times [\varphi, 2\pi + \varphi)$ of the phase function in (19) are as follows:*

- *If $\theta = 0$ then $\nabla f = 0$ for $(\theta', \varphi') = (\pi/3, \varphi'), (\pi, \varphi')$.*
- *If $\theta \in (0, \pi/2)$ then there are five solutions of $\nabla f = 0$, given by $(\theta', \varphi') = (\pi - 2\theta, \varphi)$, $((\pi - 2\theta)/3, \varphi)$, $((\pi + 2\theta)/3, \varphi + \pi)$, $(\pi, \varphi + \pi/2)$ and $(\pi, \varphi + 3\pi/2)$.*
- *If $\theta = \pi/2$*

i.e. if $\theta' = (\pi(1 + 4n)$

$$(M_j\psi)(\mathrm{p}(\theta,\varphi)) := \frac{\int_{\varphi}^{2\pi+\varphi}\int_0^{\pi} G_j(\theta,\varphi,\theta',\varphi')\mathrm{e}^{\mathrm{i}kf(\theta,\varphi,\theta',\varphi')}\,\mathrm{d}\theta'\,\mathrm{d}\varphi'}{\int_{\varphi}^{2\pi+\varphi}\int_0^{\pi} g(\theta,\varphi,\theta',\varphi')\mathrm{e}^{\mathrm{i}kf(\theta,\varphi,\theta',\varphi 0}}, \quad j = 1,\ldots,N(\theta), \quad j$$

$$
\begin{aligned}
(\tilde{M}_{N(\theta)+1}\psi)(\mathrm{p}(\theta,\varphi)) ={}& \frac{2\pi \mathrm{i} H(k,0)\psi(\mathrm{p}(\theta,\varphi))}{k|\cos\theta|} - \frac{2\pi H(k,0)\psi(\mathrm{p}(\theta,\varphi))}{k^2\cos^4\theta}\left(1+\frac{1}{2}\sin^2\theta\right) - \frac{2\pi\frac{\partial H}{\partial\theta'}(k,0)\psi(\mathrm{p}(\theta,\varphi))}{k^2|\cos^3\theta|} \\
&- \frac{H(k,0)}{k^2}\int_{\varphi}^{2\pi+\varphi} \left.\frac{\partial\psi(\mathrm{p}(\theta'_x,\varphi'_x))}{\partial\theta'}\right|_{\theta'=0} \frac{\mathrm{d}\varphi'}{(1-\sin\theta\cos(\varphi-\varphi'))^2}.
\end{aligned}
$$

where

$$J(n) := \sum_{s=0}^{n-1} \left(\frac{\mathrm{i}}{k}\right)^{s+1} \int_\Gamma (\mathbf{u}_s.\mathbf{n}) \mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma, \quad g_{s+1} := (\nabla.\mathbf{u}_s), \quad \mathbf{u}_{s+1} := \frac{\nabla f}{|\nabla f|^2} g_{s+1}, \tag{35}$$

Γ is the positively oriented (anticlockwise) boundary of supp(g), σ is the arc length of Γ, and $\mathbf{n} := (n_1, n_2)$ is the unit outward normal vector to Γ. We immediately deduce that for $n = 1, 2, \ldots,$

$$|(M_{N+1}\psi)(\mathbf{p}(\theta,\varphi)) + J(n)| \le \frac{1}{k^n}\left|\int\int_{\mathrm{supp}(g)} g_n \mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\theta'\,\mathrm{d}\varphi'\right| \le \frac{C(\theta,\varphi)}{k^{n+1}}\|g_n\|_\infty. \tag{36}$$

Next we evaluate

$$\int_\Gamma (\mathbf{u}_s.\mathbf{n}) \mathrm{e}^{\mathrm{i}kf}\,\mathrm{d}\sigma =$$

where

$$P(\theta', \varphi') := \frac{f_{\theta'\theta'} + f_{\varphi'\varphi'}}{f_{\theta'}^2 + f_{\varphi'}^2} - 2\frac{f_{\theta'}^2 f_{\theta'\theta'} + 2f_{\theta'} f_{\varphi'} + f_{\varphi'}^2 f_{\varphi'\varphi'}}{(f_{\theta'}^2 + f_{\varphi'}^2)^2}.$$

From the definition of χ_j, $j = 1, \dots, N$, g and all its derivatives, and hence g_s, $s = 0, 1, \dots$, then vanish on all other sections of Γ (Γ_7, Γ_8 and Γ_9 in Figure 1, plus four other semicircles in the case $\theta \in (0, \pi/2)$). Thus

$$\int_\Gamma (\mathrm{u}_s.\mathrm{n})e^{ikf}\, d\sigma = \int_\varphi^{2\pi+\varphi} \frac{g_s(\theta, \varphi, 0, \varphi')}{1 - \sin\theta\cos(\varphi - \varphi')}\, d\varphi' + \frac{e^{ik(2-2\cos\theta)}}{\sin\theta}\int_\varphi^{2\pi+\varphi} \frac{g_s(\theta, \varphi, \pi, \varphi')}{\cos(\varphi - \varphi')}\, d\varphi'. \quad (40)$$

Using (39),

$$g_{s+1}(\theta, \varphi, 0, \varphi') = \frac{\cos\theta}{(1 - \sin\theta\cos(\varphi - \varphi'))^2} g_s(\theta, \varphi, 0, \varphi') + \frac{\frac{\partial g_s}{\partial\theta'}(\theta, \varphi, 0, \varphi')}{1 - \sin\theta\cos(\varphi - \varphi')}, \quad (41)$$

$$g_{s+1}(\theta, \varphi, \pi, \varphi') = \left(\frac{1/2 - \cos\theta}{\sin^2\theta\cos^2(\varphi - \varphi')}\right) g_s(\theta, \varphi, \pi, \varphi') + \frac{\frac{\partial g_s}{\partial\theta'}(\theta, \varphi, \pi, \varphi')}{\sin\theta\cos(\varphi - \varphi')}, \quad (42)$$

and since $\mathrm{p}(\theta'_x, \varphi'_x)|_{\theta'=0} = \mathrm{p}(\theta, \varphi)$ and $\mathrm{p}(\theta'_x, \varphi'_x)|_{\theta'=\pi} = \mathrm{p}(\pi - \theta, \varphi)$,

$$g(\theta, \varphi, 0, \varphi') = H(k, 0)\psi(\mathrm{p}(\theta, \varphi)), \qquad g(\theta, \varphi, \pi, \varphi') = 0,$$

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$$+\frac{e^{ik(2-2\cos\theta)}H(k,\pi)\psi(p(\pi-\theta,\varphi))}{2\sin^2\theta}\int_{\varphi}^{2\pi+\varphi}\frac{\left(1-\sum_{j=1}^{N}\chi_j(\theta,\varphi,\theta',\varphi')\right)}{\cos^2(\varphi-\varphi')}\,d\varphi'. \qquad (44)$$

Applying Lemma 4.2 and the fact that $H(k,\pi)$ is of order k in (43) and (44), the result (33) follows from (35) (with $n = 2$), (34) and (32). □

Remark 4.4 *Using (40), (41) and Lemma 4.2, we see that for* $\theta = \pi/2 \pm \delta$, *the leading order term of* $\int_\Gamma (\mathbf{u}_s.\mathbf{n})e^{ikf}\,d\sigma$ *for fixed* k *as* $\delta \to 0$ *is of order*

$$\int_{\varphi}^{2\pi+\varphi}\frac{\cos^s\theta}{(1-\sin\theta\cos(\varphi-\varphi'))^{2s+1}}\,d\varphi' \quad \frac{\cos^s\theta}{|\cos\theta|^{4s+1}} \quad \frac{1}{\quad}$$

We computed the exact value of $(M\psi)(\mathbf{p}(\theta, 0))$ for $\theta \in S_{1001}$, where the set S_{1001} consists of 1001 equally spaced points on $[0, \pi]$, including the end points. We computed the approximation $(M_{app}\psi)(\mathbf{p}(\theta, 0))$ with $\theta \in S_{1001} \setminus \{0, \pi/2, \pi\}$.

In Figures 2 and 3, we plot the exact value $(M\psi)(\mathbf{p}(\theta, 0))$ for $\theta \in S_{1001} \setminus \{0, \pi\}$, and approximate solution $(M_{app}\psi)(\mathbf{p}(\theta, 0))$ for $\theta \in S_{1001} \setminus \{0, \pi\}$, $|\theta - \pi/2| > 10k^{-1/3}$, for $k = 1,310,720$ and $k = 2,621,440$. Our approximations are seen to be qualitatively correct (in the sense that the crosses for the approximate values lie inside the circles representing the exact values) outside a region of width of the order of $k^{-1/3}$ around $\theta = \pi/2$. Evaluation of just the one dimensional inner integral for the exact solution of $(M\psi)(\mathbf{p}(\theta, 0))$ with $\theta \in S_{1001}$, $k = 1,310,720$ and $k = 2,621,440$ took over 44 hours and 94 hours of CPU time respectively on a AMD Opteron 2.0Ghz computer, while our approximation $(M_{app}\psi)(\mathbf{p}(\theta, 0))$ with $\theta \in S_{1001} \setminus \{0, \pi/2, \pi\}$ was computed for both the cases in less than 0.03 seconds.

In Figure 4 we plot the error $E(k, \theta)$ for $|\theta - \pi/2| > 10k^{-1/3}$, for $k = 10240$, $k = 40960$, $k = 163840$, $k = 655360$ and $k = 1,310,720$, to demonstrate efficiency of our formula for computing (1) within a few seconds of CPU time.

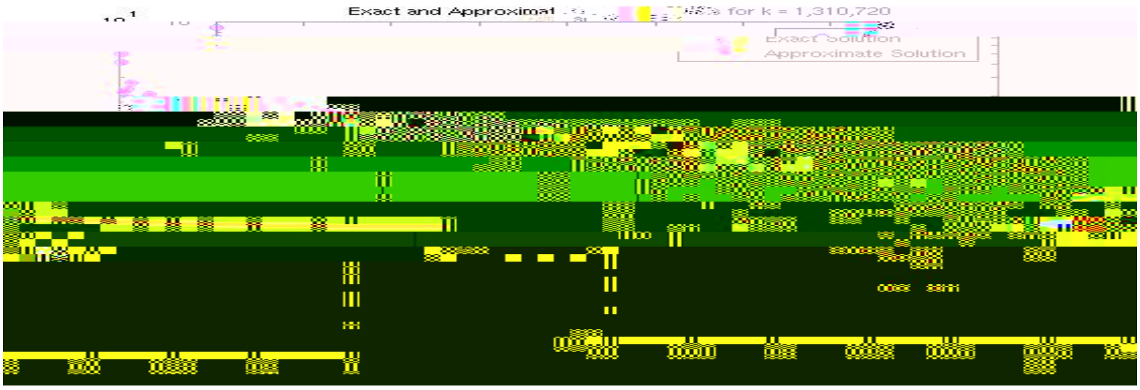

Figure 2: Exact and approximate solutions for $k = 1,310,720$.

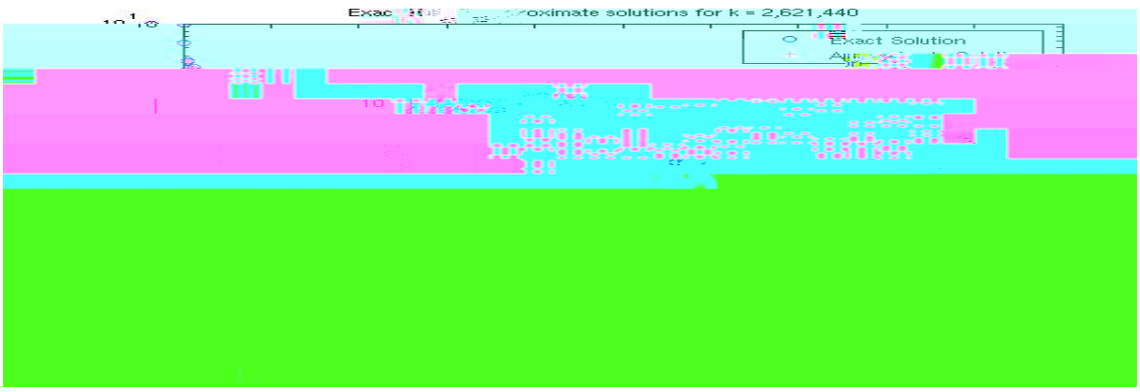

Figure 3: Exact and approximate solutions for $k = 2,621,440$.

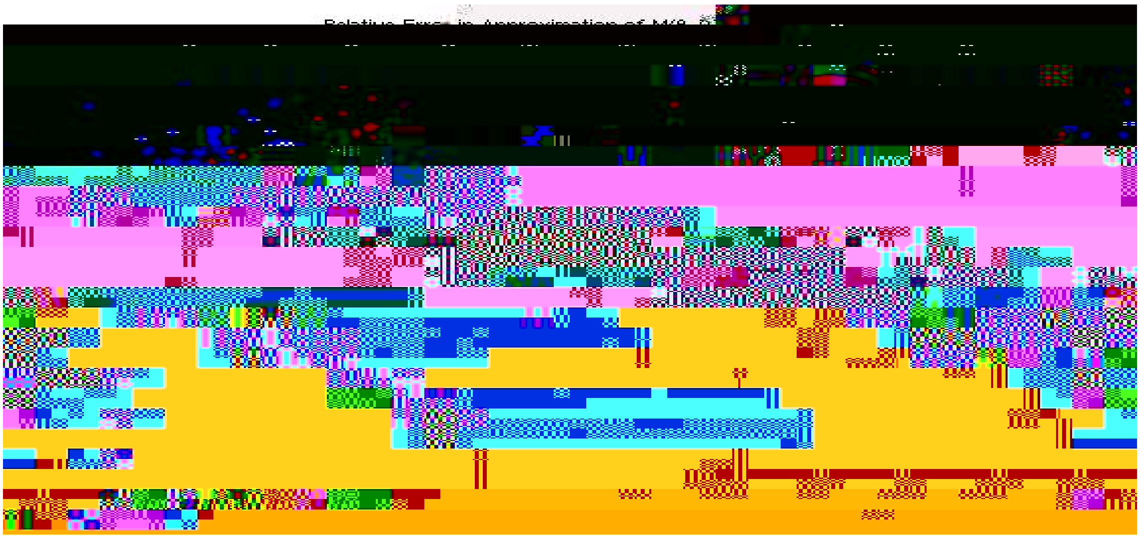

Figure 3: Errors $E(k, \theta)$ for $|\theta - \pi/2| > 10k^{-1/3}$, various $k > 10000$.

6 Conclusions

Outside of a band of width $Ck^{-1/3}$