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Very Large Inverse Problems in Atmosphere and Ocean Modelling

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Very Large Inverse Problems in Atmosphere and Ocean Modelling

C. Johnson, N.K. Nichols and B.J. Hoskins

Abstract

For the very large nonlinear dynamical systems that arise in a wide range of physical, biological and environmental problems, the data needed to initialize a numerical forecasting model are seldom available. To generate accurate estimates of the expected states of the system, both current and future, the technique of 'data assimilation' is used to combine the numerical model predictions with observations of the system measured over time. Assimilation of data is essentially an ill-posed inverse problem. In four dimensional variational assimilation schemes, the dynamical model equations provide constraints that act to spread information into data sparse regions, enabling the state of the system to be reconstructed accurately. The mechanism for this is not well understood. Singular value decomposition techniques are applied here to analyse the critical features in this process. Simplified models are used to demonstrate how information is propagated from observed regions into unobserved areas. The impact of the size of the observational noise and the temporal position of the observations is examined. The best signal-to-noise ratio needed to extract the most information from the observations is estimated using Tikhonov regularization theory.

Keywords: Large-scale inverse problems; variational data assimilation; nonlinear dynamical systems; weather, ocean and climate models; singular vectors; Tikhonov regularization

1 INTRODUCTION

Accurate prediction of the behaviour of very large evolutionary systems requires both accurate numerical models for simulating the system dynamics and accurate data for initializing the forecast. In practice, precise data describing the current state of a system are not available, and uncertainties in the initial data lead to significant errors between the predicted states

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information from the observations can be determined.

In the next section we present the variational data assimilation method.

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$$\blacktriangledown = \frac{1}{2}({}_0 - {}^b_0)^T B_0^{-1}({}_0 - {}^b_0) + \frac{1}{2}\sum_{j=0}^{N-1}(\mathbf{h}_j({}_j) - {}_j)^T R_j^{-1}(\mathbf{h}_j({}_j) - {}_j) \quad (3)$$

[illegible] *(1).*

In practice the constrained minimization problem is solved iteratively by a gradient method. The problem is first reduced to an unconstrained problem using the method of Lagrange. Necessary conditions for the solution to the unconstrained problem then require that a set of adjoint equations together with the system equations must be satisfied. The adjoint equations are given by

$$\boldsymbol{\lambda}_N = 0, \quad (4)$$

$$\boldsymbol{\lambda}_k = F_k^T({}_k)\boldsymbol{\lambda}_{k+1} - H_k^T R_k^{-1}(\mathbf{h}_k({}_k) - {}_k), \qquad k = N-1, \ldots, 0, \quad (5)$$

where $\boldsymbol{\lambda}_k \; \mathbb{R}^n$, $j = 0, \ldots, N$, are the adjoint variables and $F_k \; \mathbb{R}^{n\times n}$ and $H_k \; \mathbb{R}^{n\times p_k}$ are the Jacobians of $\mathbf{f}_k$ and $\mathbf{h}_k$ with respect to ${}_k$.

The gradient of the objective function (3) with respect to the initial data ${}_0$ is then given by

$$_{\mathbf{x}_0}\blacktriangledown = B_0^{-1}({}_0 - {}^b_0) - \boldsymbol{\lambda}_0. \quad (6)$$

At the optimal, the gradient (6) is required to be equal to zero. Otherwise this gradient provides the local descent direction needed in the iteration procedure to find an improved estimate for the optimal initial states. Each step of the gradient iteration process requires one forward solution of the model equations, starting from the current best estimate of the initial states, and one backward solution of the adjoint equations. The estimated initial conditions are then updated using the computed gradient direction. This process is expensive, but it is operationally feasible, even for very large systems, such as weather and ocean systems, which may involve as many as 10^7 state variables.

3 APPLICATION TO THE EADY MODEL

The success of the 4DVar assimilation technique is largely due to the action of the dynamical model equations, which spread information from the boundaries. The density, static stability and Coriolis parameter are taken to be constants, and it is assumed that the interior quasi-geostrophic potential vorticity is zero.

Perturbations to the basic state are advected zonally by the basic shear flow. The system dynamics are described by the non-dimensional equations

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)b = \frac{\partial \psi}{\partial x}, \qquad \text{for } z = \quad \frac{1}{2}, \quad x \quad [0, X], \tag{7}$$

where b is the buoyancy and the geostrophic streamfunction ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \qquad z \quad [-\frac{1}{2}, \frac{1}{2}], \quad x \quad [0, X], \tag{8}$$

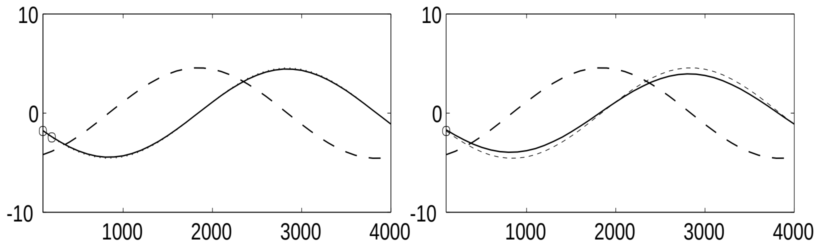
with boundary conditions

$$\frac{\partial \psi}{\partial z} = b, \qquad \text{for } z = \quad \frac{1}{2}, \quad x \quad [0, X]. \tag{9}$$

The buoyancy and streamfunction are assumed to be periodic in x on $[0, X]$.

3.1 Experiments

The aim of the experiments is to reconstruct the buoyancy wave on the upper boundary of the region from observations of the buoyancy on the lower boundary at the beginning and end of the assimilation interval using 4DVar. The model equations are obtained by discretizing the system equations (7) - (9) using a leap-frog advection scheme with 11 vertical levels and 40 grid points in the horizontal. Perfect observations representing the 'truth' are generated by model runs over a time interval corresponding to 6 hours, initiated with the most rapidly growing (or decaying) normal mode of the system. The initial fields have a tilt with height that is associated with vertical coupling between the upper and lower waves, leading to exponential growth (or decay) of the solution. Uncorrelated random noise with variance $\sigma_o = 1$ is added to the observations. The prior estimate of the state at time t_0



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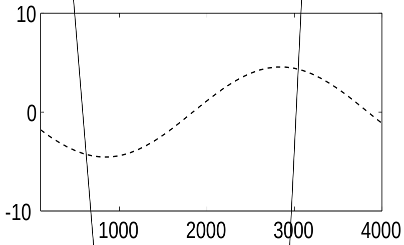
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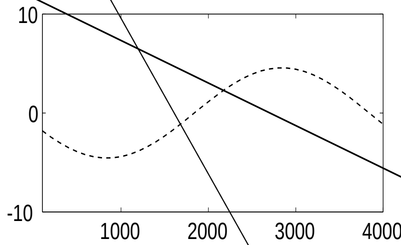
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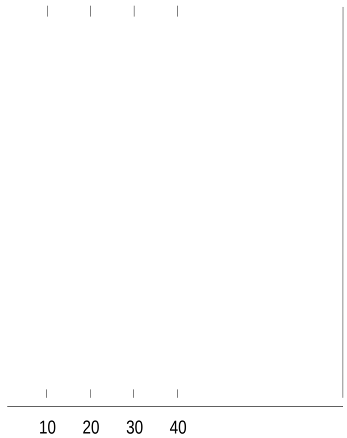
similation interval are shown in Figure 3. The observations are assimilated over an interval of length t_f, corresponding to twelve hours, and a forecast over a time interval corresponding to 24 hours is produced from the analysis at time t_f, the end of the assimilation interval. Observations at the end time $t = t_f$

where

$$\hat{\mathbf{H}} = \left[H_0^T, (H_1 M(t_1, t_0))^T, \; \ldots, \; (H_{N-1} M(t_{N-1}, t_0))^T\right.$$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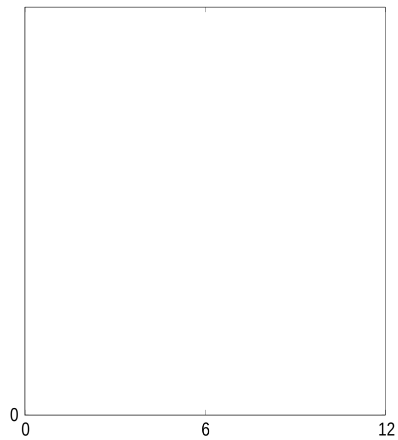
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5 TIKHONOV REGULARIZATION

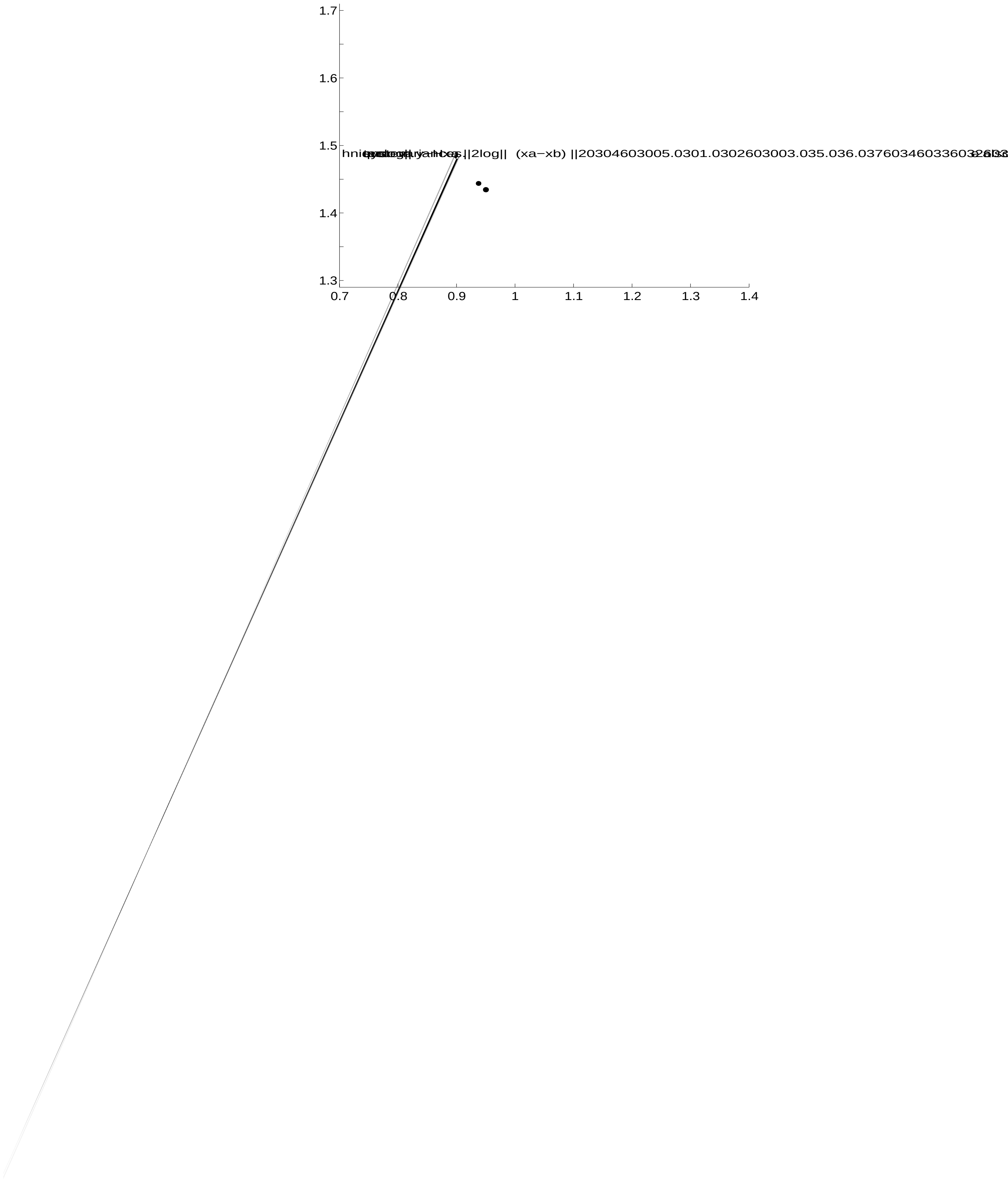
In Section 4 we have demonstrated the importance of the value of the variance ratio μ^2, between the variances of the background and observational errors, in maximizing the information that can be extracted from the observations. Good choices for μ^2 can be determined by using Tikhonov regularization theory [8].

We first reformulate the objective function (3) for the variational assimilation problem by making a change of variable. We let C_B and $\mathbf{C}_R$ be such that $B_0 = \sigma_b^2 C_B$, $\hat{\mathbf{R}} = \sigma_o^2 \mathbf{C}_R$, and define $\chi = C_B^{1/2}(\ _0 - \ ^b_0)$. For the linear model (11), minimizing the objective function (3) is then equivalent to minimizing the function

$$\tilde{J}(\chi) = \mu^2 \ \chi \ ^2_2 + \ \mathbf{C}_R^{-1/2}\hat{\mathbf{d}} - \mathbf{C}_R^{-1/2}\hat{\mathbf{H}} C_B^{1/2}\chi \ ^2_2, \qquad (19)$$

where $\mu^2 = \sigma_o^2/\sigma_b^2$.

We see that if $\mu^2 = 0$, that is, if there is no background constraint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