Abstract

Hamilton's principle is used to create a variational principle which has as its natural conditions the equations of irrotational motion in an incompressible, homogeneous fluid with a free surface.

By applying the shallow water approximation to the flow variables, this variational principle is reduced to one whose natural conditions are the corresponding shallow water equations of motion. For unsteady shallow water flow four functionals, whose integrands are related by Legendre transformations, are generated. Boundary terms are added to these functionals to give variational principles whose natural conditions include boundary conditions and initial conditions as well as the equations of motion. These variational principles are reduced to corresponding variational principles for steady shallow water flows by assuming that the flow variables are independent of time. The natural conditions of the variational principles derived by this method then include the steady state equations of motion in shallow water and boundary conditions on certain flow variables.

Constrained variations are made on the unsteady and steady shallow water functionals and some reciprocal variational principles are established.

The purpose of this report is to identify variational principles for non-linear, irrotational, free surface flows of a homogeneous, incompressible fluid over a fixed, prescribed bed profile. The application of such principles to the computation of approximations to flows will be considered in a later report.

Luke [1] showed that a variational principle in which the integrand is taken to be the fluid pressure, as given by Bernoulli's energy integral, has as its natural conditions the equations governing a free surface flow. These are Laplace's equation for the velocity potential, in the fluid domain, the no-flow condition across

t_2

t_1

i i

n

$-$ i i i

$i=1$

$= \ (\ _1 \ldots \ _n)$ is a given function and therefore $= \ (\ _1 \ldots \ _n \ _1 \ \ _n$

$$\sum_{i=1}^{n} \ _i \ _i$$

$_i$ $_i$

$_i$ $\frac{}{_i}$ $_i$ $\frac{}{_i}$

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[1] at this stage in ignoring boundary and initial conditions, and show that, with this limitation, the ulerian version of Hamilton's principle, constructed as we have indicated, is the same as Luke's 'pressure' principle. Thus Luke's principle is merely a disguised version of Hamilton's principle.

The same interpretation of Hamilton's principle is equally successful for shallow flows, as is shown in Section 3. Moreover, we can implement the shallow water approximation in Hamilton's principle for general free surface flows, and show that the resulting principle is identical to that obtained by an *ab initio* approach to the shallow flow problem.

Alternative representations of the variational principle for shallow flow are available, based on the notion of a closed quartet of Legendre transformations introduced by Sewell [6]. In particular one version of the principle involving the pressure and another version identifiable as Hamilton's principle are connected by a Legendre transformation. These two principles are Luke's principle and Hamilton's principle for a general free surface flow, as modified by the provisions of shallow water theory. Referring to the construction (1.2), the appearance of a Legendre transformation here is not surprising.

In Section 3 we complete the shallow flow principles by including boundary terms. The modified variational principles then have, as natural conditions, boundary and initial conditions in addition to the field equations. There is some latitude in the variables which need to be assigned on lateral boundaries and initially, which is a matter of significance from a practical viewpoint. We do not explore here how to overcome one particular undesirable feature of these principles, that conditions are given on both time boundaries, $t = t_1$ and $t = t_2$, say. This is a long-standing difficulty, which is present in the classical Hamilton principle (1.1). One simple theoretical remedy is to consider only variations which vanish at $t = t_2$, but it is not clear at present how this device can be translated into an approximation method.

The latter issue is not present, of course, in steady flows which are considered in Section 4. We also give some constrained principles which lead to the notion of reciprocity, for both unsteady and steady flows. In the latter case, Bateman's [7] principles emerge. Other points of contact with existing literature are mentioned in Section 5.

This report is not intended to give an exhaustive account of all possible aspects of variational principles for free surface flows. Its main aim is to provide a clear theoretical base for the development of numerical approximation methods.

In this section the equations of irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are shown to be the natural conditions of two variational principles derived from different viewpoints. It is shown that these principles are, in fact, closely related.

Let be cartesian coordinates, with measured vertically upwards, and let be the time. Consider the domain, $\tilde{\Omega}$, extending over a fixed region, , in the plane and enclosed by the surfaces = ( ) and = ( ), where

$- \quad - \quad t$

t

$t \quad -$

$1 \quad t_2 \quad t_1 \quad D \quad \eta \quad -h \quad t \quad -$

$1 \quad 1$

$1 \quad 2 \quad 1 \quad 2$

$1 \quad t_2 \quad t_1 \quad D \quad t \quad - \quad z{=}\eta$

$t \quad x \quad x \quad y \quad y \quad z \quad z{=}\eta \quad x \quad x \quad y \quad y \quad z \quad z{=}{-h}$

$\eta \quad 2 \quad -h$

which yields the natural conditions

~ 2

t —

t x x y y z

x x y y z

1 2

t_2 η

t_1 D $-h$ t —

1 2 3

t —

t 1 x 2 y 3

1 x 2 y 3

2

2 3

η

D $-h$ —

The condition that the functional is stationary with respect to variations is then related to Hamilton's principle in the sense described in the Introduction. Although Luke [1] mentions this more traditional form of the Lagrangian and notices that the difference between 'Hamilton's principle' and (2.3) is related to conservation of mass he does not pursue this observation.

In the fixed domain $\tilde{\Omega}$ conservation of mass,

$$\tilde{} \qquad = 0 \qquad (2\ 13)$$

must be enforced (see Introduction). The kinematic free surface condition, which ensures zero mass flow across this surface, in the form

$$(\ _t + \ _x + \ _y \ )_{z=\eta} = 0 \qquad (2\ 14)$$

and the condition of zero flow through the bed,

$$(\ _x + \ _y + \ )_{z=-h} = 0 \qquad (2\ 15)$$

must also be enforced. These requirements are met by adding (2.13)–(2.15) to

$$_t \quad _x \quad _y \quad _{z=\eta} \quad _x \quad _y \quad _{z=-h}$$

$$2 \quad \int_{t_1}^{t_2} \int_D \int_{-h}^{\eta} \quad - \quad _t \quad _x \quad _y \quad _{z=\eta} \quad _x \quad _y \quad _{z=-h}$$

$$2 \quad 2$$

$$_t \quad -$$

$$_t \quad _x \quad _y$$

$$_x \quad _y$$

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=

1 2

^{2}

$_{1}$ = =

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Shallow water theory offers an approximation to free surface flows in circumstances where the water depth is much less than some other characteristic length scale of the motion, such as the radius of curvature of the surface. To lowest order, this theory can be generated by assuming that the fluid pressure is hydrostatic. That is,

$$\tilde{}(\quad) = \ (\quad) \qquad (3\ 1)$$

taking the assumed constant surface pressure to be zero as a matter of convenience. The hypothesis (3.1) implies that the horizontal velocity components, and , are independent of and that the vertical velocity component is negligible compared with and . This can be summarised as

$$_z = 0 \qquad _z = 0 \qquad = 0 \qquad (3\ 2)$$

Details may be found in, for example, Stoker [8].

We can use (3.1) to determine the vertically averaged pressure = ( ) defined by

rtDrhegOFC[X5fUxennt

$$= \frac{1}{\ } \int_{-h}^{\eta} \tilde{}$$

from which it follows that

$$= \frac{1}{2}\ ^2 \qquad (3\ 3)$$

where (a5fFfizz)k[]X5fi(F]((XflfiFfi][X5fi(F](wvqfizFmatqF'wpq[)k[dX5fi(F](5fwSrgflfiflzFkhwonez(

t

t

$\frac{1}{2}$

— —

terms evaluated at these levels. The result is the functional $_2 = \;_2(\quad)$ given by

$$_2 = \int_{t_1}^{t_2} \int_D \frac{1}{2} \quad \frac{1}{2} \;^2 + \;(\;_t + \;(\;)) + \tag{3 8}$$

Assuming that variations vanish on the space and time boundaries then the natural conditions of

$$_2 = 0$$

are

$$\begin{aligned} _t + \quad + \quad \tfrac{1}{2} \quad &= 0 \\ &= \qquad \text{in } \Omega \\ _t + \;(\;) &= 0 \end{aligned} \tag{3 9}$$

t t 1

$$1 \int_{t_1}^{t_2} \int_D \; - \;^2 \quad _t \quad -$$

1

$_t \quad \tfrac{1}{2}$

$_t$

$_{1,2}$

t_2

t_1 D t

t_2

t_1 D t

D $^{2}_{1}$ $^{2}_{1}$ Σ

2 $^{2}_{1}$ $-$ $-$ 2

1 2

$\frac{1}{2}$

2 2

which allows the definition of a new function $(\quad)$ by substituting for in the pressure ^{2} 2, namely

$$(\quad) = \frac{1}{2} \qquad \frac{1}{2} \qquad ^{2}$$

which has the values of pressure.

The integrand of the functional being constructed is now

$$(\quad) \qquad (\ + \ _t) + \qquad (\qquad\quad) \tag{3 17}$$

which is the integrand of (3.10) modified in the manner outlined above. For the

1

$- \_\_\_\_ \quad - \quad ^{2}$

t

$\phi \qquad Q$

conditions for ϕ are sought on Σ_ϕ and for $\mathbf{Q}$ on Σ_Q for $t \in [t_1, t_2]$. Similarly the domain is divided into two by $\quad = \quad _d + \quad _\phi$ and conditions at the time boundaries t_1 and t_2 are sought for d in $\quad _d$ and for ϕ in $\quad _\phi$.

By this method a new functional is constructed — one which includes boundary terms. Let the functional $I_1(E, \mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$\begin{aligned}
I_1 = & \int_{t_1}^{t_2} \iint_D \left(p(\mathbf{v}, E) - d\,(E + \phi_t) + \mathbf{Q}.\,(\mathbf{v} - \boldsymbol{\nabla}\phi)\right)\, dx\, dy\, dt \\
& + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi \, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\
& + \iint_{D_\phi} \left((d\,(\phi - h_2))|_{t_2} - (d\,(\phi - h_1))|_{t_1} \right) dx\, dy \\
& + \iint_{D_d} \left(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \right) dx\, dy
\end{aligned} \tag{3.18}$$

where $f = f(x, y, t)$, $C = C(x, y, t)$ are given functions on Σ_ϕ and Σ_Q respectively and $g_i = g_i(x, y)$, $h_i = h_i(x, y)$ $(i = 1, 2)$ are given functions on $\quad _d$ and $\quad _\phi$ respectively.

The natural conditions of the revised 'p' principle

$$\delta I_1 = 0$$

are

$$\left.\begin{aligned}
p_\mathbf{v} + \mathbf{Q} &= \mathbf{0} \\
p_E - d &= 0 \\
d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \\
E + \phi_t &= 0 \\
\mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0}
\end{aligned}\right\} \qquad \text{in } \Omega,$$

$$\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\
\phi - f &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\
d|_{t_i} - g_i &= 0 && \text{in } \quad _d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 && \text{in } \quad _\phi \text{ for } i = 1, 2,
\end{aligned}$$

where

$$p_\mathbf{v} \equiv \frac{\partial p}{\partial \mathbf{v}} = -\frac{\mathbf{v}}{\mathbf{g}} \left(E - \frac{1}{2}\mathbf{v}.\mathbf{v} \right) \quad \text{and} \quad p_E \equiv \frac{\partial p}{\partial E} = \frac{1}{g} \left(E - \frac{1}{2}\mathbf{v}.\mathbf{v} \right).$$

Thus the first two conditions in the domain, $\quad$, are

$$-\frac{\mathbf{v}}{\mathbf{g}} \left(E - \frac{1}{2}\mathbf{v}.\mathbf{v} \right) + \mathbf{Q} = \mathbf{0} \quad \text{and} \quad \frac{1}{g} \left(E - \frac{1}{2}\mathbf{v}.\mathbf{v} \right) - d = 0,$$

which together give

$$\mathbf{Q} = d\mathbf{v} \quad \text{and} \quad E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v},$$

so that the last three natural conditions in Ω are the conservation laws and the irrotationality condition.

Consider now the 'r' principle. The domain and domain boundary are again divided into two, as for the 'p' principle, to provide a choice of boundary and initial conditions. Using the same functions, $= (\quad)$, $= (\quad)$, $_i =$ $_i(\quad)$, $_i =$ $_i(\quad)$ $(\; = 1 \; 2)$, a second functional $_2(\quad)$ can be derived, namely,

$$_2 \int_{t_1}^{t_2} \int_{D} \quad t$$

$$\int_{t_1}^{t_2} \int_{\Sigma_Q} \qquad \int_{t_1}^{t_2} \int_{\Sigma_\phi}$$

$$\int_{D_d} \quad {}_2 \; {}_{t_2} \qquad {}_1 \; {}_{t_1}$$

$$\int_{D_\phi} \; {}_{t_2} \; {}_2 \qquad {}_{t_1} \; {}_1$$

$$2$$

$$d \qquad t$$

$$\mathbf{Q}$$

$$t$$

$$Q \qquad 1 \quad 2$$

$$\phi \qquad 1 \quad 2$$

$$t_i \quad i \qquad d$$

$$t_i \quad i \qquad \phi$$

$$-\frac{\quad}{2} \qquad t \qquad \frac{}{}$$

$$1$$

A sequence of Legendre transformations can be used to generate a quartet of functionals which have as natural conditions of their first variations the unsteady shallow water equations. Two such functionals — based on the p and r functions — have already been described and were independently derived. Two further functionals are now sought.

By applying the divergence theorem and integration by parts, the 'p' functional (3.18) can be expressed in the form

$$I_1 = \int_{t_1}^{t_2} \int_D (p(\ , E) \quad Ed + \quad . \quad + \phi(d_t + \quad . \quad)) \, dx \, dy \, dt$$
$$+ \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi(C \qquad . \ ) \, d\Sigma \, dt \qquad \int_{t_1}^{t_2} \int_{\Sigma_\phi} f \quad . \quad d\Sigma \, dt$$
$$\int_{D_d} (\ (\qquad _2)) \ _{t_2} \quad (\ (\qquad _1)) \ _{t_1}$$
$$\int_{D_\phi} \quad _{t_2} \ ^2 \qquad _{t_1} \ ^1 \qquad (3.20)$$

Comparing this with $_2$, as given by (3.19), suggests that there is a relationship between the two functions ( ) and ( ) such that

$$(\quad) = \ (\quad) \qquad +$$

in value, which can be confirmed directly using (3.5). The relation is in fact a Legendre transformation as we will now show.

The Legendre transform ( ) of ( ), with and as dual active variables and passive, is defined by

$$(\quad) = \qquad (\quad) \qquad (3\ 21)$$

and is such that

$$_{\mathbf{v}} = \quad \mathbf{v} \qquad _d =$$

Using $(3\ 5)_2$ can be constructed from (3.21) and is

$$(\quad) = \frac{1}{2} \quad ^2 + \frac{1}{2}$$

Notice that is equal to the total energy of a fluid particle. The function is also a Legendre transform of ( ), with active and passive, in that, using $(3\ 5)_1$, we may write

$$(\quad) = \qquad (\quad) \qquad (3\ 22)$$

having first derivatives

$$_d = \quad _d \qquad _{\mathbf{v}} =$$

This implies the required connection, that

$$(\quad) = \qquad (\quad) = \ (\quad) \qquad +$$

in value.

We can of course bypass the intermediate function R and connect p and r directly by a Legendre transformation. Since $p_{\mathbf{V}} = -$ and $p_E = d$, then if and E are both active variables, the transformation of p is

$$r(\ , d) = \quad . \quad - Ed + p(\ , E)$$

and

$$r_{\mathbf{Q}} = \quad , \quad r_d = -E.$$

A fourth function $P(\ , E)$ completes a closed quartet of functions related by Legendre transformations and is derivable from p, r and R by using appropriate active variables. P cannot be given explicitly, but is defined by eliminating and d from

$$P(\ , E) = \frac{1}{2}gd^2 + d \ . \quad , \qquad = d \quad , \quad E = gd + \frac{1}{2} \ . \ .$$

The function P is related to p and r by

$$p(\ , E) - P(\ , E) \ = \ - \ . \tag{3.23}$$

$$r(\ , d) - P(\ , E) \ = \ -Ed. \tag{3.24}$$

We can now formulate two further functionals, the natural conditions of the first variations of which are expected to include the equations of motion in shallow water. The process is to use (3.23) to substitute for p in the integrand of (3.18) and (3.22) to substitute for r in the integrand of (3.19) by what is essentially a change of variables using (3.5). We note here that although (3.21) could be used to substitute for p in (3.18) and (3.24) could be used to substitute for r in (3.19) this would not change the nature of the functionals being generated. For instance integration by parts and the divergence theorem can be used on the 'P' functional generated by substituting (3.24) into (3.19) to give the functional formed by substituting (3.23) into (3.18).

Let the functional $I_3(E, \ , d, \phi)$ be defined by

$$\begin{aligned} I_3 = & \int_{t_1}^{t_2} \int_D (P(\ , E) - \ . \ \phi - d\,(E + \phi_t))\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f) \ . \ d\Sigma\, dt \\ & + \int_{D_\phi} (d\,(\phi \quad h_2))_{t_2} \quad (d\,(\phi \quad h_1))_{t_1} \, dx\, dy \\ & + \int_{D_d} \phi_{t_2} g_2 \quad \phi_{t_1} g_1 \, dx\, dy. \end{aligned} \tag{3.25}$$

The natural conditions of this 'P' principle

$$\delta I_3 = 0$$

are

$$\left.\begin{array}{rcl} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ P_E - d &=& 0 \\ E + \phi_t &=& 0 \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

$$\begin{array}{rcll} C - \mathbf{Q}.\mathbf{n} &=& 0 & \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\ \phi - f &=& 0 & \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\ d|_{t_i} - g_i &=& 0 & \text{in } \quad {}_d \text{ for } i = 1, 2, \\ \phi|_{t_i} - h_i &=& 0 & \text{in } \quad {}_\phi \text{ for } i = 1, 2. \end{array}$$

The first condition in Ω is

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0}.$$

Thus if equations (3.5) are assumed, the 'P' principle yields the conservation laws and the irrotationality condition as natural conditions in Ω and gives boundary conditions on ϕ and $\mathbf{Q}$ at space boundaries and on d and ϕ at time boundaries.

The 'R' Principle

Now consider a principle based on the function R. Let the functional $I_4(\mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$\begin{aligned} I_4 = & \int_{t_1}^{t_2} \iint_D \left(-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\; d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & - \iint_{D_d} \left((\phi\,(d - g_2))|_{t_2} - (\phi\,(d - g_1))|_{t_1} \right)\, dx\, dy \\ & - \iint_{D_\phi} \left(d|_{t_2}\, h_2 - d|_{t_1}\, h_1 \right)\, dx\, dy. \end{aligned} \tag{3.26}$$

The natural conditions of this 'R' principle

$$\delta I_4 = 0$$

are

$$\left.\begin{array}{rcl} -R_{\mathbf{v}} + \mathbf{Q} &=& \mathbf{0} \\ -R_d - \phi_t &=& 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

$$\begin{aligned}
C \quad . &= 0 && \text{on } \Sigma_Q \text{ for } t \quad [t_1, t_2],\\
\phi \quad f &= 0 && \text{on } \Sigma_\phi \text{ for } t \quad [t_1, t_2],\\
d_{\ t_i} \quad g_i &= 0 && \text{in } {}_d \text{ for } i = 1, 2,\\
\phi_{\ t_i} \quad h_i &= 0 && \text{in } {}_\phi \text{ for } i = 1, 2.
\end{aligned}$$

The first two conditions in Ω may be written

$$d \ + \quad = \quad , \quad gd \quad \frac{1}{2} \ . \qquad \phi_t = 0.$$

Thus the natural conditions of the 'R' principle include the equations of motion in shallow water and boundary conditions on the variables.

So there exists a quartet of functionals (3.18), (3.19), (3.25) and (3.26), based on the four functions p, r, P and R, from which the shallow water equations can be derived as the natural conditions of the first variations. Notice that the statements of the natural boundary conditions of all of the variational principles are identical and that the natural conditions in the domain are the same equations expressed in different variables.

Variational principles can be constrained by assuming that the variations are made subject to the requirement that the variables satisfy one or more of the natural conditions. The principles constrained in this way will have the remaining natural conditions as natural conditions ([9]).

The functional used in the 'p' principle (3.18) has an integrand which contains the integrated conservation of momentum equation and the irrotationality condition explicitly. It seems natural to constrain the 'p' principle to satisfy these two conditions. This can be done by specifying

$$\begin{aligned} E &= \quad \phi_t \\ &= \quad \phi \end{aligned} \quad , \tag{3.27}$$

which results in the functional I_1 reducing to a functional $\tilde{I}_1(\ , d, \phi)$. The constrained principle is given by

$$\begin{aligned}
&\tilde{L}Q\!\!\!=\!\!d\, S\,\delta\,\delta \int_{t_1}^{t_2} \int_D C \ , \ \delta f\,\hat{}\,(\imath\ \imath)\delta\,\beta \qquad + \int_{t_1}^{t_2} \int_{\Sigma_Q} \qquad \Sigma \\
&+ \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\qquad) \qquad \Sigma \ + \int_{D_\phi} (\ (\qquad {}_2)) \ {}_{t_2} \quad (\ (\qquad {}_1)) \ {}_{t_1} \\
&+ \int_{D_d} \quad {}_{t_2} \ {}^2 \qquad {}_{t_1} \ {}^1 \qquad = 0
\end{aligned} \tag{3.28}$$

where'XflCF[

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t_1 D t_1 Σ_Q t

t_2

t_1 Σ_ϕ t D_d 2 t_2 1 t_1

D_ϕ t_2 2 t_1 1

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Reciprocal 'P' and 'R' Principles

Now consider the other two variational principles — based on P and R. The integrands of the 'P' and 'R' functionals are not expressible in the form

$$P \text{ or } R \text{ function} + \text{multiplier} \times \text{conservation law}$$

so there is no corresponding way of constraining the variational principles and the functionals cannot in the same way be reduced to depend on one variable. However, the following structure can be deduced.

Consider the 'P' functional (3.25). Let $\Sigma_Q = \Sigma$ and $_d =$, and constrain the variables to satisfy conservation of momentum using the first of (3.27). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D (P(\mathbf{Q}, -\phi_t) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy\, dt + \int_{t_1}^{t_2}\int_\Sigma C\phi\, d\Sigma\, dt \right.$$
$$\left. + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1\right) dx\, dy\right\} = 0, \tag{3.33}$$

where the variables are $\mathbf{Q}$ and ϕ. The natural conditions are given by

$$\left.\begin{array}{rcl} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ (P_{-\phi_t})_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

$$\begin{array}{rcll} C - \mathbf{Q}.\mathbf{n} &=& 0 & \text{on } \Sigma \text{ for } t \in [t_1, t_2], \\ g_i - P_{-\phi_t}|_{t_i} &=& 0 & \text{in} \quad \text{for } i = 1, 2. \end{array}$$

The first two conditions may be rewritten as

$$\left.\begin{array}{rcl} \mathbf{v} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

which are the irrotationality condition and the conservation of mass equation.

In the 'R' functional (3.26) let $\Sigma_\phi = \Sigma$ and $_\phi =$ and constrain the variations to satisfy conservation of mass by imposing (3.30). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D (-R(\mathbf{v}, \boldsymbol{\nabla}.\boldsymbol{\psi}) - \boldsymbol{\psi}_t.\mathbf{v})\, dx\, dy\, dt + \int_{t_1}^{t_2}\int_\Sigma f\boldsymbol{\psi}_t.\mathbf{n}\, d\Sigma\, dt \right.$$
$$\left. - \iint_D \left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right) dx\, dy\right\} = 0, \tag{3.34}$$

which involves a functional of $\mathbf{v}$ and $\boldsymbol{\psi}$. The natural conditions are given by

$$\left.\begin{array}{rcl} -R_{\mathbf{v}} - \boldsymbol{\psi}_t &=& \mathbf{0} \\ \boldsymbol{\nabla} R_{\boldsymbol{\nabla}.\boldsymbol{\psi}} + \mathbf{v}_t &=& \mathbf{0} \end{array}\right\} \quad \text{in } \Omega,$$

$$
\begin{aligned}
{}_t &= 0 \qquad \text{on } \Sigma \text{ for } \quad [\ {}_1 \quad {}_2] \\
{}_i \quad {}_{t_i} &= \qquad \text{in} \quad \text{for} \quad = 1 \ \ 2 \\
{}_{t_i} \quad {}_i &= 0 \qquad \text{on } \Sigma \text{ for} \quad = 1 \ \ 2
\end{aligned}
$$

The first two equations are

$$
\begin{aligned}
(\qquad) \qquad {}_t &= \\
{}_t + \qquad\qquad + \tfrac{1}{2} \qquad\qquad &=
\end{aligned}
\qquad \text{in } \Omega
$$

the second of which is conservation of momentum. This, together with the natural condition in for ${}_1$, implies the irrotationality condition in for $[\ {}_1 \quad {}_2]$.

The constrained ‘P’ and ‘R’ principles (3.33) and (3.34) are reciprocal since the constraint of conservation of momentum in (3.33) is a natural condition of (3.34) and the conservation of mass constraint in (3.34) is a natural condition of (3.33). The irrotationality condition is a natural condition of both principles.

The discussion so far has concerned derivation of variational principles whose natural conditions include the unsteady shallow water equations of motion developed both from principles whose natural conditions are the equations of motion of a free surface flow and independently. We now seek to apply these principles to steady state conditions.

The steady state equations of motion in shallow water for a domain with constant undisturbed depth can be deduced from the unsteady equations (3.4). The steady state condition assumes that all of the flow variables — mass flow, energy, depth and velocity — are independent of time. The potential is not a flow variable and cannot therefore be assumed to be independent of time although its general form can be deduced.

The irrotationality condition is

$$=$$

$$2$$

$${}_t$$

$${}_t \qquad\qquad {}_t$$

and thus

$$_{tt} = 0 \qquad _{t} =$$

Therefore is of the form

$$(\quad) = \hat{} + \tilde{}(\quad) \tag{4 1}$$

where the energy $\hat{}$, the steady counterpart of , is a constant. The expression (4.1) will prove useful in reducing functionals for unsteady motion to functionals for steady motion.

The steady state equations are given by

$$= \ 0 \qquad \text{conservation of mass} \tag{4.2}$$

$$\hat{} \ = \qquad \text{conservation of momentum} \tag{4.3}$$

$$= \qquad \text{irrotationality} \tag{4.4}$$

where and $\hat{}$ are given by

$$= \tag{4.5}$$

$$\hat{} \ = \quad + \frac{1}{2} \tag{4.6}$$

i i

Q

ϕ

1 1 2

d

1 2 1

2 ϕ

t_2 t_1 2 1

2 1 1 2 2 1

1

t_2

s 1 D Σ_Q t_1

t_2

Σ_ϕ t_1 D_d

s 1 s 1

1 $-$ 1 2

t_2 t_2

1 $-$ 2 1 $-$ 1 2

t_1 t_1

$-$ 1 2

t_2 t_2

$-$ 2 $-$ 1 2

t_1 t_1

s 1 D Σ_Q

Σ_ϕ D_d

1 D Σ_Q Σ_ϕ

1 1

2 D

Σ_Q Σ_ϕ

3 D

$$+\int_{\Sigma_Q}\phi\,(C-\mathbf{Q}.\mathbf{n})\,d\Sigma-\int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.10)$$

$$L_4 = \iint_D(-R(\mathbf{v},d)+\mathbf{Q}.\mathbf{v}+Ed+\phi\boldsymbol{\nabla}.\mathbf{Q})\,dx\,dy$$
$$+\int_{\Sigma_Q}\phi\,(C-\mathbf{Q}.\mathbf{n})\,d\Sigma-\int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.11)$$

where $L_2 = L_2(\mathbf{Q}, d, \phi)$, $L_3 = L_3(\mathbf{Q}, \phi)$ and $L_4 = L_4(\mathbf{Q}, d, \mathbf{v}, \phi)$.

The natural conditions of the steady state variational principles

$$\delta L_1 = \delta L_2 = \delta L_3 = \delta L_4 = 0$$

are expected to include the shallow water equations of motion (4.2) and (4.4) and possibly (4.5) or (4.6). quation (4.3) is satisfied exactly since the energy E is regarded as a given constant.

The natural conditions of $\delta L_1 = 0$, the 'p' principle, are

$$\left.\begin{aligned} p_{\mathbf{v}}+\mathbf{Q} &= \mathbf{0} \\ \mathbf{v}-\boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$\begin{aligned} C-\mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ \phi - f &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$-\frac{\mathbf{v}}{\mathbf{g}}\left(E-\frac{1}{2}\mathbf{v}.\mathbf{v}\right)+\mathbf{Q}=\mathbf{0} \qquad \text{in} \quad .$$

The natural conditions of $\delta L_2 = 0$, the 'r' principle, are

$$\left.\begin{aligned} r_{\mathbf{Q}}-\boldsymbol{\nabla}\phi &= \mathbf{0} \\ r_d+E &= 0 \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$\begin{aligned} C-\mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ \phi - f &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} \frac{\mathbf{Q}}{\mathbf{d}}-\boldsymbol{\nabla}\phi &= \mathbf{0} \\ -\tfrac{1}{2}\tfrac{\mathbf{Q}.\mathbf{Q}}{d}-gd+E &= 0 \end{aligned}\right\} \qquad \text{in} \quad .$$

The natural conditions of $\delta L_3 = 0$, the 'P' principle, are

$$\left.\begin{aligned} P_{\mathbf{Q}}-\boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in} \quad ,$$

$$= 0 \qquad \text{on } \Sigma_Q$$

$$= 0 \qquad \text{on } \Sigma_\phi$$

the first equation being

$$= \qquad \text{in}$$

where is a function of and using (4.5) and (4.6).

The natural conditions of $_4 = 0$, the 'R' principle, are

$$\begin{aligned} {}_{\mathbf{v}} + \quad &= \\ {}_{d} + \quad &= 0 \\ &= \\ &= 0 \end{aligned} \qquad \text{in}$$

$$= 0 \qquad \text{on } \Sigma_Q$$

$$= 0 \qquad \text{on } \Sigma_\phi$$

the first two equations being

$$\begin{aligned} + \quad &= \\ \tfrac{1}{2} \quad + \quad &= 0 \end{aligned} \qquad \text{in}$$

Thus the natural conditions of the steady state motion variational principles derived from free surface unsteady motion variational principles include the steady state equations in shallow water — (4.2) and (4.4). In order that the

Consider the integrands of the functionals (4.8)–(4.11). In Section 3 emphasis was placed on the structure of the integrands of the 'p' and 'r' functionals — they were expressed as a function plus a multiple of a conservation law or the irrotationality condition. For steady flows the 'p' and 'P' functionals also exhibit

ϕ

D

$-$ $-$

Q

D

ϕ

In Section 3 four functionals, related to one another by Legendre transformations, are defined. These are the pressure function p, the Lagrangian density r, the flow stress P and the function R which, when integrated over the space domain, gives the total energy of the flow. In [10] Benjamin and Bowman consider conservation laws and symmetry properties of Hamiltonian systems including shallow water, for which they derive four functions. Two of these — identified by them as a Hamiltonian density and a flow force — have the values of the functions R and P respectively, apart from constant multipliers.

That the function R is indeed the Hamiltonian density for shallow water flow can be seen by considering the 'Legendre transformation' of the Lagrangian L. Let

$$L(\mathbf{Q}, d) = \iint_D r(\mathbf{Q}, d)\, dx\, dy \tag{5.16}$$

where $r(\mathbf{Q}, d)$ is the Lagrangian density. Then the functional derivatives $\partial L/\partial d$ and $\partial L/\partial \mathbf{Q}$ can be defined by

$$\delta L = \left(\delta d, \frac{\partial L}{\partial d}\right) + \left(\delta \mathbf{Q}, \frac{\partial L}{\partial \mathbf{Q}}\right),$$

where the inner product (α, β) is given by

$$(\alpha, \beta) = \iint_D \alpha\beta\, dx\, dy.$$

Thus

$$\frac{\partial L}{\partial d} = -\frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d^2} - gd \ , \quad \frac{\partial L}{\partial \mathbf{Q}} = \frac{\mathbf{Q}}{d} = \mathbf{v}.$$

The Hamiltonian $H(\mathbf{v}, d)$ is given, in a form somewhat analogous to (1.2), by

$$H(\mathbf{v}, d) = (\mathbf{v}, \mathbf{Q}) - L(\mathbf{Q}, d),$$

and thus it is easily seen that

$$H(\mathbf{v}, d) = \iint_D R(\mathbf{v}, d)\, dx\, dy$$

where R is the Hamiltonian density.

The analogy with (1.2) would have been closer if we had started with a Lagrangian $L(\mathbf{v}, d)$, with density $r(d\mathbf{v}, d)$, and deduced a Hamiltonian $H(\mathbf{Q}, d)$ with density $R(\mathbf{Q}/d, d)$. This alteration identifies $\mathbf{v}$ as the vector of generalised velocities and $\mathbf{Q} = d\mathbf{v}$ as the vector of conjugate momenta. We can, of course, redefine the Lagrangian and Hamiltonian densities in terms of the variables $\mathbf{v}$, d and $\mathbf{Q}$, d respectively, to achieve the structure mentioned. The redefined densities are themselves linked by a Legendre transformation with $\mathbf{v}$ and $\mathbf{Q}$ active and d passive, but the quartet of densities obtained by taking $r(d\mathbf{v}, d)$ as the starting point does not include a quantity identifiable as the flow stress or the energy E as

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