A Hybrid Central/Upwind Approach to the Solution of the One Dimensional Euler Equations[1]

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Abstract

It is a fact of life that you rarely get something for nothing. In CFD there is a balance which needs to be struck between the speed and the accuracy of a method. To get results quickly and cheaply it is inevitable that the solution will be of questionable quality and budgets restrict the accuracy which can be achieved, so a compromise is often necessary.

However, it is usually the case that the faster methods are able to model smooth flows quite adequately and only fail when confronted with more complex flow features such as shocks or vortices, which occur in relatively small regions of the entire flow domain. Therefore, it should be possible to use a simple and fast scheme for the majority of the flow, and more appropriate ones for local phenomena.

This report documents a preliminary study in which such a method is used to model the one dimensional Euler equations. It is demonstrated that a domain decomposition approach can be used to gain significant savings with little or no loss of accuracy.

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a reasonable amount of success, using an operator splitting technique. However, it has long been known that these methods are incapable of properly modelling certain flow features, such as shear and shocks not aligned with the mesh. There is much work being carried out at present to produce a truly multidimensional wave decomposition model [3, 7, 10, 11], but it is by no means clear which will prove to be the best. It may even be that in this domain decomposition method, different models need to be used for different flow features. The work will be done solely on unstructured triangular or tetrahedral meshes.

The result will hopefully be a 'hybrid' scheme incorporating the best of both worlds; having accurate modelling of the flow where it is needed but producing answers quickly and cheaply.

In the next section, a brief description is given of the three schemes used in this study, Roe's scheme, central differencing and Lax-Wendroff, and their application to the one dimensional uler equations. Section 3 describes the results obtained for a number of test cases, used initially to validate the individual schemes, and then to evaluate the domain decomposition approach, *ie.* using different schemes in different regions. Finally, section 4 gives a summary of the results and conclusions drawn from the study.

The work discussed in this report is a brief study of the effects of domain decomposition in one dimension. This is being used to gain an insight into the work, in a simplified environment, before it is extended to more dimensions. In this way, problems related to the use of two different schemes in a single flow domain can be solved, before the introduction of the fluctuation distribution scheme and the wave decomposition model necessary for multidimensional upwinding, neither of which can be considered to be perfected. This section gives a brief description of the three schemes used in this study.

The first of the three schemes used here is upwinding and, in particular, Roe's scheme is used for solving the uler equations. The scheme is most easily described in conjunction with a single, scalar conservation law, the one dimensional linear advection equation

$$ {}_t + \quad {}_x = 0 \qquad (1)$$

and the upwinding scheme applied to this gives

$$ {}^{n+1}_{i} = {}^{n}_{i} \quad \frac{\Delta}{\Delta}\left({}^{n}_{i} \quad {}^{n}_{i-1}\right) \qquad (2)$$

It is explicit and first order accurate, and for 0 it is stable for CFL numbers

$$ = \frac{\Delta}{\Delta} \quad 1 \qquad (3)$$

For 0 it is unconditionally unstable. In this case, a 'left hand' version of the scheme is required

$$ {}^{n+1}_{i} \quad {}^{n}_{i} \quad \overline{\qquad} \quad {}^{n}_{i} \quad {}^{n}_{i+1}$$

$$ \_{t} \quad \_ \; \_ \; {}_{x} \quad \_ $$

$$ \_ \qquad \_ \qquad {}^{2} $$

$$ \overline{\qquad} \quad \_ \quad {}^{2} $$

The method used here to solve this system is Roe's scheme, which is described in detail from a theoretical point of view in [8, 9], and from a computational point of view in [12]. This method involves the decomposition of the fluctuation, $(\_{R} \quad \_{L})\Delta \quad \Delta$, in each cell, on to the eigenvectors of the Jacobian matrix, $\tilde{}$, of $\_(\_)$, where $\_{R}$ and $\_{L}$ are the flux vectors at the right and left hand nodes of the cell respectively, Δ is the time step and Δ is the cell width. The fluctuation can then be distributed to increment the nodal values of $\_$, being subtracted from the left node if the corresponding eigenvalue is negative, or from the right node if it is positive. Thus, the nodal update formula can be written

$$\_{i}^{n+1} = \_{i}^{n} \quad \frac{\Delta}{\Delta} \quad \left(\sum_{k=1}^{3} {}_{k}^{+} \; {}_{k}\_{k} \right)_{i-1/2}^{n} + \left(\sum_{k=1}^{3} {}_{k}^{-} \; {}_{k}\_{k} \right)_{i+1/2}^{n} \qquad (8)$$

where ${}_{k}^{+}$ and ${}_{k}^{-}$ are the positive and negative eigenvalues of the local Jacobian matrix, $\tilde{}$, $\_{k}$ is the corresponding eigenvector and ${}_{k}$ is the wave strength. In effect, the uler equations have been locally linearised and decoupled leaving three independent linear advection problems to be solved in each cell.

An interesting point in the solution of this problem is the introduction of a parameter vector

$$\_\_ = {}^{1/2} \quad 1 \qquad (9)$$

where is the enthalpy. This has the useful property that each component of $\_$ and $\_$ is merely quadratic in the components of $\_\_$, so transforming into these variables simplifies the algebra a great deal. It has also been suggested that these parameter vector variables might have other desirable properties, and that it may be advantageous to solve the uler equations in these variables rather than the conserved variables.

There are in fact two ways of constructing Roe's linearisation. The first, mentioned above, uses the analogy of solving an approximate Riemann problem within each cell, assuming the variables to be piecewise constant in space, and develops the linearisation from there. A second derivation, developed afterwards

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which illustrates how central differencing can be thought of as a fluctuation distribution scheme where half the fluctuation is sent to each of the left and right nodes of the cell. It also shows that the artificial diffusion term is dealt with just as simply. The only slight problem is the calculation of the time step, which involves finding the value of the largest eigenvalue, λ_{max}, of the local Jacobian matrices of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$, so

$$\Delta t = \frac{\nu \Delta x}{\lambda_{max}} \tag{12}$$

where ν is the CFL number. However, this only requires the calculation of u, the velocity of the flow, and c, the local speed of sound, since the eigenvalues of the Jacobian are known to be $u - c$, u and $u + c$.

2.3 Lax-Wendroff

The third and final scheme used here is Lax-Wendroff. It is being considered as an alternative cheap scheme for modelling the smooth parts of the flow, despite the seemingly greater computational expense, because it doesn't have the problems of central differencing mentioned above. Applied to the linear advection equation the scheme looks like

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{13}$$

It is again explicit and second order accurate but is stable for CFL numbers

$$-1 \leq \nu \leq 1, \tag{14}$$

a great improvement on central differencing. Lax-Wendroff also has the advantages of not having the extra tuning parameter, ϵ.

The extension of the scheme to the uler equations is again relatively straightforward. As with the derivation of quation 13, the 'Lax-Wendroff trick' is used to express the time derivatives in terms of space derivatives

$$\underline{\mathbf{u}}_{tt} = -\underline{\mathbf{f}}_{xt} = -\underline{\mathbf{f}}_{tx} = -(\tilde{A}\underline{\mathbf{u}}_t)_x = (\tilde{A}\underline{\mathbf{f}}_x)_x \tag{15}$$

where $\tilde{A}$ is the Jacobian matrix of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$. This can then be substituted into the Taylor series expansion for $\underline{\mathbf{u}}_i^{n+1}$, giving

$$\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n - \Delta t \underline{\mathbf{f}}_x + \frac{(\Delta t)^2}{2}(\tilde{A}\underline{\mathbf{f}}_x)_x. \tag{16}$$

The appropriate central differences can now be chosen to approximate the space derivatives to give the Lax-Wendroff scheme. Using simple algebraic manipulation the scheme can be written

$$\begin{aligned}\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n &- \frac{\Delta t}{2\Delta x}\left(I - \frac{\Delta t}{\Delta x}\tilde{A}_{i+1/2}\right)(\underline{\mathbf{f}}_{i+1}^n - \underline{\mathbf{f}}_i^n) \\ &- \frac{\Delta t}{2\Delta x}\left(I + \frac{\Delta t}{\Delta x}\tilde{A}_{i-1/2}\right)(\underline{\mathbf{f}}_i^n - \underline{\mathbf{f}}_{i-1}^n).\end{aligned} \tag{17}$$

This shows how the scheme can be formulated in terms of the distribution of a fluctuation within each cell, with each node getting a contribution from the left and the right cells. This makes the scheme compatible with Roe's scheme at

max

max

$\frac{PS_x}{S}$

ing a two-step Lax-Wendroff scheme. This, in fact, makes Lax-Wendroff slightly faster than central differencing overall. It should be noted that all these times have been obtained using a very general code in which computational overheads become increasingly significant as the schemes become simpler. Also, in two and three dimensions the difference in speed will become far more marked, as the multidimensional upwinding scheme is much more complex than its one dimensional counterpart.

An alternative to the domain decomposition method used above would be to use central differencing or Lax-Wendroff to obtain a reasonably converged solution, and then use this as the starting point for a second calculation using Roe's scheme everywhere. This removes the problem of the two schemes giving different answers, but unfortunately saves little time and is not a viable alternative to the domain decomposition method. Although it decreases the time taken for the second calculation to reach convergence by about 10%, most of this is used in gaining the partially converged solution. The same problem will be found in two dimensions so it is not worth pursuing any further.

4 onclusions

The work presented in this report shows that it is possible to obtain savings in one dimension by using a cheap and simple scheme in most of the flow domain and only using a more accurate, but expensive, scheme (Roe's upwinding scheme) in the small regions where it is necessary.

In this study two 'cheap' schemes were considered, central differencing and Lax-Wendroff. Unfortunately, the savings were not as large as might be expected for central differencing, due to a restrictive CFL limit, and both schemes suffered from using a very small number of nodes in the test case considered, which caused the iteration times to be swamped by overheads. These problems will be of less significance in higher dimensions where many more nodes are used and Roe's scheme is relatively far more expensive.

The domain decomposition method has been successfully used to produce sharp shocks, while using Roe's scheme in only a very small part of the domain. Shocks were found to be the only one dimensional flow feature which required special attention.

However, for central differencing, the interfaces between the two schemes produced wiggles which could not be completely removed by moving the interfaces away from the shock. Also, the sharpened shock was too far upstream. This seems to be due to the fact that central differencing with artificial diffusion and Roe's scheme do not converge to precisely the same steady state solution and, most importantly, predict different shock positions. If Lax-Wendroff is used instead, these problems disappear and an extremely good comparison can be made between the 'hybrid' scheme and Roe's scheme. This strongly suggests that a Lax-Wendroff type scheme and not central differencing should be used as the simple scheme when this work is extended to two dimensions. The Lax-Wendroff 'hybrid' scheme in one dimension takes only half the time to reach a converged solution that Roe's scheme does, a speed advantage which will be increased greatly in two dimensions due to the greater complexity of Roe's multidimensional scheme and the reduced significance of the computational overheads.

This work has also given a valuable insight into the domain decomposition approach, highlighting problems to be solved before the extension to two dimensions is attempted. This should make it simpler to tackle the essentially multidimensional problems related to the fluctuation distribution scheme and the wave decomposition model.

M. . Hubbard is supported financially by a contract funded by DRA Farnbor-

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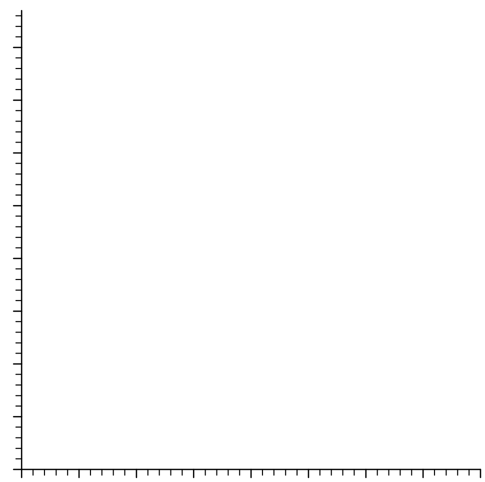
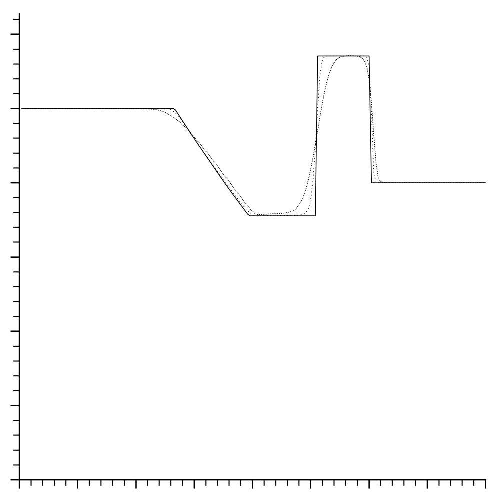
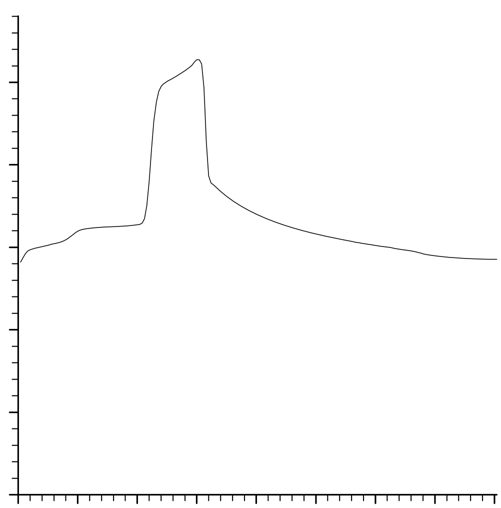
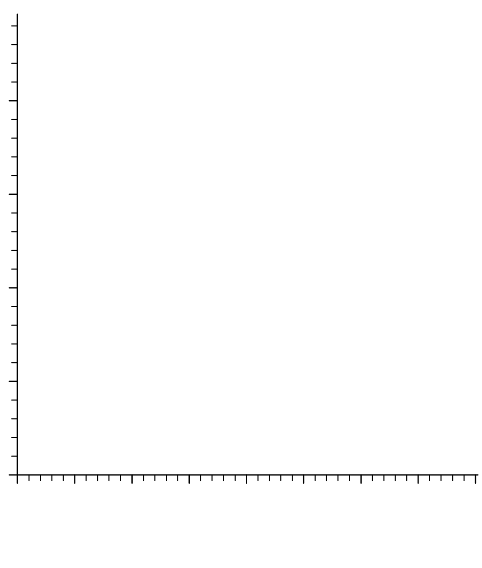
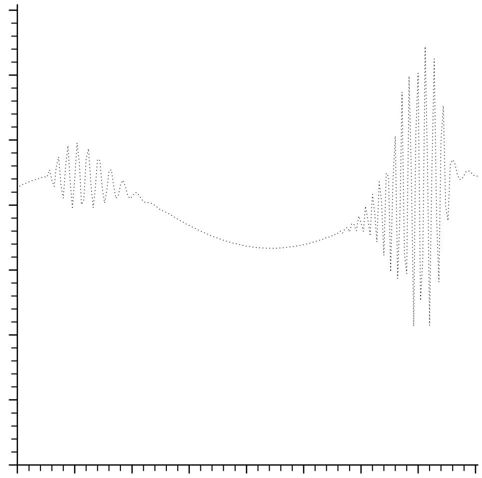
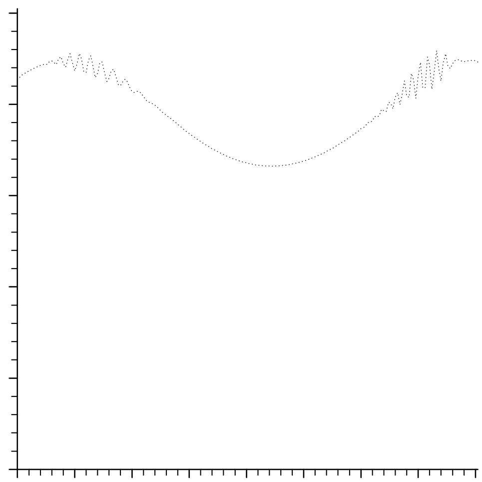
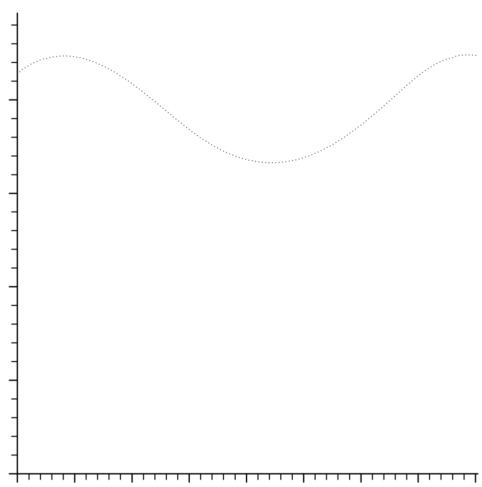
[15] P.Woodward and P.Colella, *The Numerical Simulation of Two-Dimensional Fluid Flow with Strong Shocks.* J. Comp. Phys., **54**, 1, pp. 115-173, 1984.

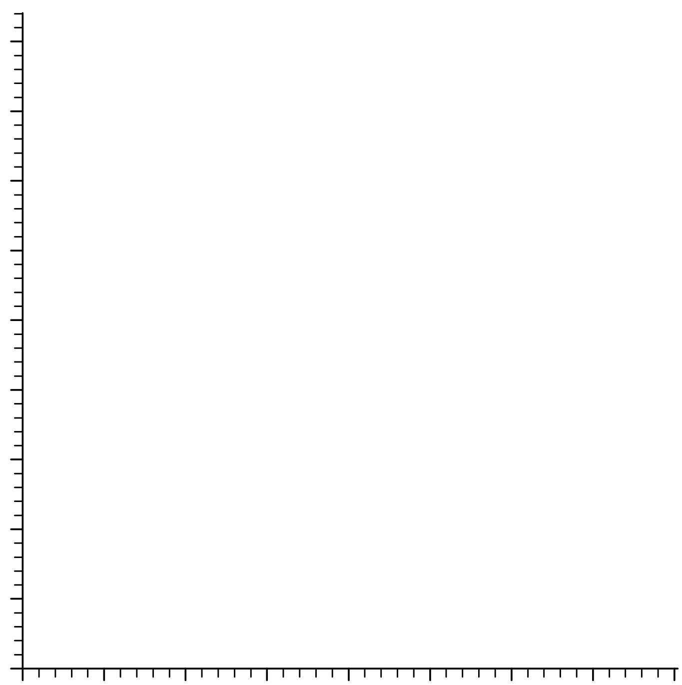










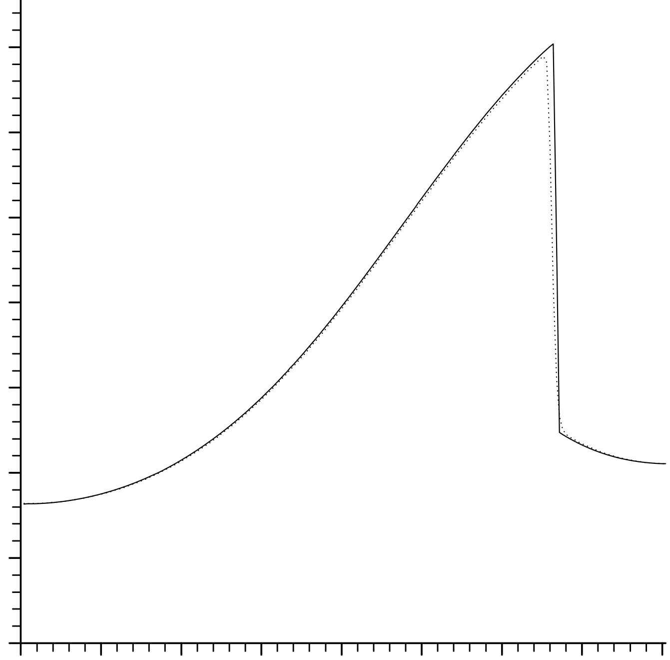


Figure 9: Comparison of Roe's scheme (solid line) and central differencing with $\epsilon = 0.006$ (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

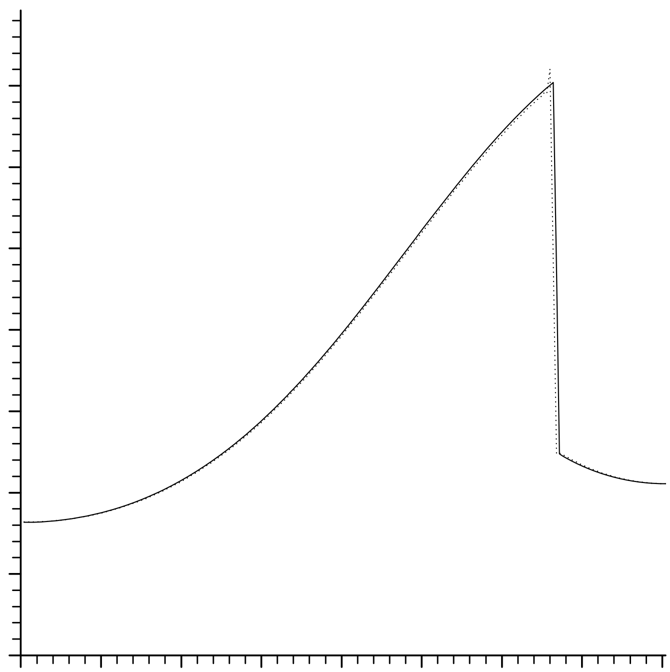


Figure 10: Comparison of Roe's scheme (solid line) and the central differencing hybrid scheme with only the two cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.