CONSERVATIVE MULTIDIMENSIONAL UPWINDING FOR THE STEADY TWO-DIMENSIONAL SHALLOW WATER EQUATIONS

M.E.Hubbard and M.J.Baines

Th Univ rsity of R ading

D partm nt of Math matics

P.O.Box 220

Whit knights

R ading

RG6 6AX

Unit d Kingdom

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Address for proofs:

Dr. M.E.Hubbard

Th Univ rsity of R ading

D partm nt of Math matics

P.O.Box 220

Whit knights

R ading

RG6 6AX

Unit d Kingdom

Phon : +44 (0)118 9875123 x4009

Fax: +44 (0)118 9313423

Email: M.E.Hubbard@r ading.ac.uk

Abstract

In r c nt y ars upwind diff r ncing has gain d acc ptanc as a robust and accurat t chniqu for th num rical approximation of th on -dim nsional shallow wat r quations. In two dim nsions th b n fits hav b n l ss mark d du to th r lianc of th m thods on standard op rator splitting t chniqu s. Two cons rvativ g nuin ly multidim nsional upwind sch m s ar pr s nt d which hav b n adapt d from flux balanc distribution m thods r c ntly propos d for th approximation of st ady stat solutions of th Eul r quations on unstructur d triangular grids. A m thod for d aling with sourc t rms, such as thos introduc d by mod- lling b d slop and friction, is also sugg st d and r sults ar pr s nt d for two-dim nsional st ady stat chann l flows to illustrat th accuracy and robustn ss of th n w algorithms.

1 Introduction

In recent years, many advances have been made in the numerical solution of hyperbolic systems of conservation laws in one and more dimensions [14, 12, 1]. Of particular interest has been the prediction of discontinuous solutions to the equations, which can occur when the system is nonlinear.

In the case of the numerical solution of the shallow water equations traditional methods, such as those of Preissmann, Abbott [5] and McCormack [9] rely on central differencing and are well known to require special treatment before a realistic numerical approximation of discontinuous flows can be obtained. More recently, the concept of upwinding has been adopted from the field of gas dynamics for the modelling of shallow water flows [11, 3]. This has proved to be highly successful, particularly in one dimension, in which high order upwind schemes have been constructed which capture discontinuities sharply and smoothly. This is achieved without the addition of artificial viscosity which is normally required to stabilise central difference schemes in the vicinity of high flow gradients. Furthermore, the upwind discretisation arises naturally from the physical interpretation of hyperbolic systems of equations, also giving a framework in which boundary conditions can be applied easily. The upwinding approach is therefore ideal for the modelling of transcritical and supercritical flows.

The practical advantages of upwind schemes in higher dimensions are less clear. Historically, they have been applied to the two-dimensional shallow water equations via the use of standard operator splitting techniques, *e.g.* [3], which by implication involves the application of one-dimensional methods to a multi-dimensional system of equations, albeit in two independent directions. Recently

In this pap r a cons rvativ formulation is pr s nt d, tog th r with two alt rnativ d compositions of th syst m of shallow wat r quations and a m thod of incorporating sourc t rms such as thos arising from th consid ration of b d slop and friction. R sults ar pr s nt d to illustrat th quality of th num rical solutions obtain d for st ady stat probl ms.

2 The Governing Equ tions

Th shallow wat r quations can b us d to d scrib th motion of 'shallow' fr -

ar th cons rvativ variabl s and th corr sponding flux v and

$$\mathbf{B}_{\mathrm{U}} \;=\; \frac{\partial \underline{\mathrm{G}}}{\partial \underline{\mathrm{U}}} \;=\; \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + c^2 & 0 & 2v \end{pmatrix} , \tag{2.9}$$

in which $c = \sqrt{gh}$ is th gravity wav sp d or wav c l rity. Furth r d tails about th math matical asp cts of th shallow wat r quations can b found in [23].

3 A Conserv tive Line ris tion

An appropriat lin arisation of th shallow wat r quations is r quir d so that th d composition and distribution stag s of th algorithm giv ris to a cons rvativ sch m . Many diff r nt cons rvativ lin arisations hav b n construct d for th Eul r quations, s   for xampl [7, 2], but it is th most robust of th s , bas d on Ro 's on -dim nsional lin arisation using a s t of param t r v ctor variabl s [19], which is g n rally us d for practical calculations. This lin arisation is adapt d h r to giv an analogous discr t form of th shallow wat r quations.

Consid r th two-dim nsional homog n ous syst m,

$$\underline{\mathrm{U}}_t + \underline{\mathrm{F}}_x + \underline{\mathrm{G}}_y \;=\; \underline{0} \,, \tag{3.1}$$

in which th cons rvativ variabl s $\underline{\mathrm{U}}$ and flux s $\underline{\mathrm{F}}$, $\underline{\mathrm{G}}$ ar giv n by (2.2) and (2.3) r sp ctiv ly. For a giv n c ll in a triangular discr tisation of th computational domain th flux balanc is d fin d by

$$\begin{aligned} \underline{\Phi}_{\mathrm{U}} \;&=\; -\int\!\!\int_{\triangle} \left(\underline{\mathrm{F}}_x + \underline{\mathrm{G}}_y \right) \,\mathrm{d}x\,\mathrm{d}y \\ &=\; \oint_{\partial\triangle} (\underline{\mathrm{F}} \,,\, \underline{\mathrm{G}}) \cdot \mathrm{d}\vec{n} \,, \end{aligned} \tag{3.2}$$

in which d$\vec{n}$ r pr s nts th inward pointing normal to th c ll boundary. Th num rical appro by assuming that th compon nts of th param t r v ctor

$$\underline{Z} = \sqrt{h}\begin{pmatrix} 1 \\ u \\ v \end{pmatrix} \tag{3.5}$$

vary lin arly in spac within ach c ll, *cf.* Ro 's param t r v ctor for th Eul r

(3.4) [7]. Th nonlin ar t rms in (3.7) and (3.8) m an that th lin arisation of th shallow wat r quations cannot b construct d in pr cis ly th sam mann r. In pr vious work [10, 17] non-cons rvativ lin arisations hav b n us d, in which th flux balanc (3.3) is valuat d consist ntly from an appropriat av rag stat , but in th pr s nt work a cons rvativ form is sought.

A cons rvativ lin arisation of th shallow wat r quations is achi v d by valuating th int grals in (3.6) xactly. This do s not imm diat ly giv ris to lin aris d flux Jacobians of th form (3.4), so inst ad a compon nt of (3.6) is isolat d which do s hav this form and which th r for can b d compos d using th s cond stag of th algorithm, d scrib d in S ction 4. H nc th num rical flux balanc (3.3) is split into two parts, taking th form

$$\widehat{\underline{\Phi}}_{\mathrm{U}} \;=\; - \underbrace{S_{\triangle} \left(\overline{\frac{\partial \underline{\mathrm{F}}}{\partial \underline{\mathrm{Z}}}}\, \overline{\underline{\mathrm{Z}}_x} + \overline{\frac{\partial \underline{\mathrm{G}}}{\partial \underline{\mathrm{Z}}}}\, \overline{\underline{\mathrm{Z}}_y} \right)}_{(1)} \;-\; \underbrace{S_{\triangle} \left(\mathbf{S}_{\mathrm{Z}}\, \overline{\underline{\mathrm{Z}}_x} + \mathbf{T}_{\mathrm{Z}}\, \overline{\underline{\mathrm{Z}}_y} \right)}_{(2)} . \qquad (3.9)$$

Th ov rbar indicat s th consist nt valuation of a quantity sol ly from th c ll-av rag stat giv n by

$$\overline{\underline{\mathrm{Z}}} \;=\; \frac{1}{3} \sum_{i=1}^{3} \underline{\mathrm{Z}}_i \;, \qquad (3.10)$$

as w ll as th corr sponding discr t gradi nt (valuat d und r th assumption of lin arly varying $\underline{\mathrm{Z}}$)

variabl s sinc

$$\frac{\partial \underline{\mathrm{U}}}{\partial \underline{\mathrm{Z}}} = \begin{pmatrix} 2\sqrt{h} & 0 & 0 \\ \sqrt{h}u & \sqrt{h} & 0 \\ \sqrt{h}v & 0 & \sqrt{h} \end{pmatrix} \tag{3.12}$$

is lin ar in th compon nts of $\underline{\mathrm{Z}}$. It th n follows that th discr t gradi nt of th

cons rvativ v

ly d coupl d. Th corr sponding analysis of th shallow wat r quations clos ly follows that of [18, 16, 25] and th r sulting pr condition rs ar d scrib d h r .

For th sak of simplifying th alg bra, th homog n ous part of th sys- t m (2.1) is consid r d in t rms of th str amwis variabl s, ξ and η, and th symm trising variabl s $\underline{\mathrm{Q}}$, d fin d by

$$\partial \underline{\mathrm{Q}} = \begin{pmatrix} \frac{c}{h}\,\partial h \\ \partial q \\ q\,\partial\theta \end{pmatrix}, \qquad (4.1)$$

wh r $q = \sqrt{u^2+v^2}$ is th sp d of th flow and $\theta = \tan^{-1}\left(\frac{v}{u}\right)$ its dir ction. Th symm tris d form of th shallow wat r quations ar now pr condition d by a matrix $\mathbf{P}$, and th r sulting syst m writt n in th form

$$\underline{\mathrm{Q}}_t + \mathbf{P}\left(\mathbf{A}^{\xi}_{\mathrm{Q}}\,\mathrm{Q}\right.$$

4.1 Decomposition 1 (HELW)

Following th analysis of M saros and Ro for th Eul r quations [16], th first pr conditioning matrix sugg st d h r is giv n by

$$\mathbf{P} \;=\; \frac{1}{q}\begin{pmatrix} \frac{\varepsilon F^2}{\beta\kappa} & -\frac{\varepsilon F}{\beta\kappa} & 0 \\ -\frac{\varepsilon F}{\beta\kappa} & \frac{\varepsilon}{\beta\kappa}+\varepsilon & 0 \\ 0 & 0 & \frac{\beta}{\kappa} \end{pmatrix}, \tag{4.4}$$

wh r $F = \frac{q}{c}$ is th local Froud numb r of th flow,

$$\beta \;=\; \sqrt{|F^2-1|}\,, \qquad \kappa \;=\; \max(F,1) \tag{4.5}$$

and ε is a function of th Froud numb r such that $\varepsilon(0) = \frac{1}{2}$ and $\varepsilon(F) = 1$ for $F \geq 1$. Th s r strictions on ε nsur that th d composition is not s nsitiv to th flow angl in th limit as $F \to 0$ [26] and that th transition of th pr condition r through th transcritical r gion is smooth. H r , as in [26] ε is tak n to b th C_1 function

$$\varepsilon(F) \;=\; \begin{cases} \frac{1}{2} & \text{for} \qquad F \;\leq\; \frac{1}{3} \\ -27F^3 + \frac{81}{2}F^2 - 18F + 3 & \text{for} \quad \frac{1}{3} \;<\; F \;<\; \frac{2}{3} \\ 1 & \text{for} \qquad F \;\geq\; \frac{2}{3} \end{cases} \tag{4.6}$$

so that th first d rivativ also vari s smoothly. Th matrix $\mathbf{P}$ in (4.4) is, in fact, pr cis ly that of [26] with th Mach numb r r plac d by th Froud numb r and without th involv m nt of th   ntropy quation. Th variabl κ has simply b n introduc d so that (4.4) is corr ct for both subcritical and sup rcritical flow.

Th pr condition d syst m (4.2) is d compos d by transforming it into a s t of charact ristic quations,

$$\underline{\mathrm{W}}_t + \mathbf{A}^{\xi}_{\mathrm{W}}\,\underline{\mathrm{W}}_\xi + \mathbf{B}^{\xi}_{\mathrm{W}}\,\underline{\mathrm{W}}_\eta \;=\; \underline{0}\,, \tag{4.7}$$

wh r th charact ristic variabl s $\underline{\mathrm{W}}$ ar d fin d by

$$\partial\underline{\mathrm{W}}_{\mathrm{sb}} = \begin{pmatrix} \frac{g\beta}{q}\,\partial h \\ q\,\partial\theta \\ \frac{g}{c}\,\partial h + F\,\partial q \end{pmatrix} \quad \text{and} \quad \partial\underline{\mathrm{W}}_{\mathrm{sp}} = \begin{pmatrix} \frac{g\beta}{c}\,\partial h + Fq\,\partial\theta \\ \frac{g\beta}{c}\,\partial h - Fq\,\partial\theta \\ \frac{g}{c}\,\partial h + F\,\partial q \end{pmatrix}, \tag{4.8}$$

for subcritical and sup rcritical flow r sp ctiv ly. Th corr sponding transformation matric s ar giv n by

$$\frac{\partial\underline{\mathrm{W}}_{\mathrm{sb}}}{\partial\underline{\mathrm{Q}}} = \begin{pmatrix} \frac{\beta}{F} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & F & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial\underline{\mathrm{W}}_{\mathrm{sp}}}{\partial\underline{\mathrm{Q}}} = \begin{pmatrix} \beta & 0 & F \\ \beta & 0 & -F \\ 1 & F & 0 \end{pmatrix} \tag{4.9}$$

and th ir inv rs s, and th r sulting charact ristic flux Jacobian matric s can b calculat d asily using

$$\mathbf{A}^{\xi}_{\mathrm{W}} = \frac{\partial\underline{\mathrm{W}}}{\partial\underline{\mathrm{Q}}}\mathbf{P}\mathbf{A}^{\xi}_{\mathrm{Q}}\frac{\partial\underline{\mathrm{Q}}}{\partial\underline{\mathrm{W}}} \quad \text{and} \quad \mathbf{B}^{\xi}_{\mathrm{W}} = \frac{\partial\underline{\mathrm{W}}}{\partial\underline{\mathrm{Q}}}\mathbf{P}\mathbf{B}^{\xi}_{\mathrm{Q}}\frac{\partial\underline{\mathrm{Q}}}{\partial\underline{\mathrm{W}}} \tag{4.10}$$

for both subcritical and sup rcritical flows.

Not that th choic of variabl s giv n by (4.8) chang s across th transcritical r gion. Wh n th flow is sup rcritical th st ady quations ar hyp rbolic and th choic of variabl s d fin d by $\partial\underline{\mathrm{W}}_{\mathrm{sp}}$ in (4.8) uniqu ly l ads to th syst m b ing compl t ly d coupl d into scalar compon nts. How v r, in th subcritical cas only on quation can b d coupl d, l aving a s cond compon nt which manif sts its lf as a 2×2 lliptic subsyst m, th form of which d p nds on th choic of charact ristic variabl s, d fin d h r by $\partial\underline{\mathrm{W}}_{\mathrm{sb}}$ in (4.8). Th shallow wat r quations cannot b d coupl d furth r in subcritical flow.

Th compl t d coupling of th quations in sup rcritical flow allows th syst m (4.7) to b writt n in th form of thr scalar adv ction quations, *i.e.*

$$\mathrm{W}_t^k + \vec{\lambda}_\xi^k \cdot \vec{\nabla}_\xi \mathrm{W}^k \;=\; 0\,, \qquad k = 1,2,3, \tag{4.11}$$

in which th adv ction v lociti s in th str amwis coordinat syst m ar

$$\vec{\lambda}_\xi^1 \;=\; \begin{pmatrix} \frac{\beta}{F} \\ \\ \frac{1}{F} \end{pmatrix}_\xi , \qquad \vec{\lambda}_\xi^2 \;=\; \begin{pmatrix} \frac{\beta}{F} \\ \\ -\frac{1}{F} \end{pmatrix}_\xi \qquad \text{and} \qquad \vec{\lambda}_\xi^3 \;=\; \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}_\xi . \tag{4.12}$$

H nc th first compon nt of th flux balanc in (3.14) tak s th form

$$\overline{\underline{\Phi}_{\mathrm{U}}} \;=\; -\,S_\triangle \sum_{k=1}^{3} \left(\vec{\lambda}_\xi^k \cdot \vec{\nabla}_\xi \mathrm{W}^k \right) \underline{\mathrm{r}}_{\mathrm{U}}^k\,, \tag{4.13}$$

wh r v ry t rm on th right hand sid of (4.13) is valuat d consist ntly from th c ll-av rag stat d fin d by (3.10) and (3.11), and $\underline{\mathrm{r}}_{\mathrm{U}}^k$ is th k^{th} column of th matrix

$$\mathbf{R}_{\mathrm{U}} \;=\; \frac{\partial \underline{\mathrm{U}}}{\partial \underline{\mathrm{Q}}}\, \mathbf{P}^{-1}\, \frac{\partial \underline{\mathrm{Q}}}{\partial \underline{\mathrm{W}}}\,. \tag{4.14}$$

This matrix transforms th compon nts of th flux balanc corr sponding to th charact ristic quations back into compon nts of th cons rvativ flux balanc . H nc (4.13) r pr s nts a consist nt d composition of $\overline{\underline{\Phi}_{\mathrm{U}}}$ of (3.14), th compo- n nts of which may ach b distribut d using a simpl *scalar* sch m such as that d scrib d in S ction 5.1 b low.

In th cas of subcritical flow th choic of charact ristic variabl s d fin d by $\partial\underline{\mathrm{W}}_{\mathrm{sb}}$ in (4.8) l ads to Jacobian matric s in th syst m (4.7) giv n by

$$\mathbf{A}_{\mathrm{W}}^\xi \;=\; \begin{pmatrix} -\varepsilon\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}_{\mathrm{W}}^\xi \;=\; \begin{pmatrix} 0 & \varepsilon & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.15}$$

H nc th charact ristic quations tak th form of a singl scalar adv ction quation, which is pr cis ly th sam as th $k = 3$ quation d fin d by (4.11) and (4.12), tog th r with a 2×2 lliptic subsyst m, so $\overline{\underline{\Phi}_{\rm U}}$ of (3.14) is writt n

$$
\begin{aligned}
\overline{\underline{\Phi}_{\rm U}} \;\; &= \;\; -\,S_{\triangle}\left(\underline{\rm r}^1_{\rm U},\, \underline{\rm r}^2_{\rm U}\right) \left[\begin{pmatrix} -\varepsilon\beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} {\rm W}^1 \\ {\rm W}^2 \end{pmatrix}_{\xi} + \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {\rm W}^1 \\ {\rm W}^2 \end{pmatrix}_{\eta} \right] \\
&\qquad\qquad\qquad\qquad - S_{\triangle}\left(\vec{\lambda}^3_{\underline{\xi}} \cdot \vec{\nabla}_{\underline{\xi}} {\rm W}^3\right) \underline{\rm r}
\end{aligned}
$$

[18] and tak s th form

$$\mathbf{P} = \frac{1}{q}\begin{pmatrix} \frac{\chi F^2}{{\beta_\epsilon}^2} & -\frac{\chi F}{{\beta_\epsilon}^2} & 0 \\ -\frac{\chi F}{{\beta_\epsilon}^2} & \frac{\chi}{{\beta_\epsilon}^2}+1 & 0 \\ 0 & 0 & \chi \end{pmatrix}, \tag{4.17}$$

wh r

$$\beta_\epsilon = \sqrt{\max\left(\,\epsilon^2,\, |F^2-1|\,\right)}\,, \qquad \chi = \frac{\beta_\epsilon}{\max\left(F,1\right)} \tag{4.18}$$

and ϵ is a nonz ro constant which typically tak s a valu of 0.05. This matrix is again d riv d by following th analysis of th Eul r quations [25], th r sult b ing that th Mach numb r is r plac d by th Froud numb r and th ntropy quation disapp ars.

Th d coupling of th syst m proc ds as in th pr vious d composition, l ading to a s t of charact ristic quations (4.7) in n w variabl s $\underline{\mathrm{W}}$, now d fin d by

$$\partial\underline{\mathrm{W}} = \begin{pmatrix} \frac{g\beta_\epsilon}{c}\,\partial h + F q\,\partial\theta \\ \frac{g\beta_\epsilon}{c}\,\partial h - F q\,\partial\theta \\ \frac{g}{c}\,\partial h + F\,\partial q \end{pmatrix}, \tag{4.19}$$

ind p nd nt of th flow sp d, *cf.* (4.8). Th corr sponding transformation matrix is giv n by $\frac{\partial \underline{\mathrm{W}}_{\mathrm{sp}}}{\partial \underline{\mathrm{Q}}}$ in (4.9).

Th diff r nc b tw n th two d compositions li s in th tr atm nt of th syst m for subcritical and transcritical flows ($F^2 \le 1+\epsilon^2$). Th d cision to k p th sam charact ristic variabl s in both subcritical and sup rcritical flow l ads to Jacobian matric s in th transform d syst m (4.7) which ar giv n by

$$\mathbf{A}^{\xi}_{\mathrm{W}} = \begin{pmatrix} \chi\nu^+ & \chi\nu^- & 0 \\ \chi\nu^- & \chi\nu^+ & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}^{\xi}_{\mathrm{W}} = \begin{pmatrix} \frac{\chi}{\beta_\epsilon} & 0 & 0 \\ 0 & -\frac{\chi}{\beta_\epsilon} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.20}$$

wh r

$$\nu^+ \; = \; \frac{F^2 - 1 + {\beta_\epsilon}^2}{2{\beta_\epsilon}^2} \qquad \text{and} \qquad \nu^- \; = \; \frac{F^2 - 1 - {\beta_\epsilon}^2}{2{\beta_\epsilon}^2} \; . \tag{4.21}$$

It is asy to s that in th sup rcritical r gion $\nu^- = 0$, th syst m is compl t - ly d coupl d, and th d composition (and subs qu nt distribution) r duc s to pr cis ly that giv n for sup rcritical flow in S ction 4.1.

In th subcritical cas th syst m is again d compos d into a singl , ind p n- d nt scalar compon nt and a pair of coupl d quations, but rath r than r garding th latt r as a 2×2 subsyst m it is inst ad tr at d as in [18], as two s parat scalar quations with sourc t rms. As a cons qu nc , th d composition of $\overline{\underline{\Phi}_{\mathrm{U}}}$ of (3.14) tak s th form

$$\overline{\underline{\Phi}_{\mathrm{U}}} \; = \; -\, S_\triangle \sum_{k=1}^{3} \left(\vec{\lambda}^k_{\underset{\sim}{\xi}} \cdot \vec{\nabla}_{\underset{\sim}{\xi}} \mathrm{W}^k + q^k_{\underset{\sim}{\xi}} \right) \underline{\mathrm{r}}^k_{\mathrm{U}} \; , \tag{4.22}$$

in which $\underline{\mathrm{r}}^k_{\mathrm{U}}$ is th k^{th} column of th matrix $\mathbf{R}_{\mathrm{U}}$ (4.14), n wly d fin d from th $\mathbf{P}$ of (4.17) and th $\underline{\mathrm{W}}$ of (4.19),

$$\vec{\lambda}^1_{\underset{\sim}{\xi}} \; = \; \begin{pmatrix} \chi\nu^+ \\ \\ \frac{\chi}{\beta_\epsilon} \end{pmatrix}_{\underset{\sim}{\xi}} \; , \qquad \vec{\lambda}^2_{\underset{\sim}{\xi}} \; = \; \begin{pmatrix} \chi\nu^+ \\ \\ -\frac{\chi}{\beta_\epsilon} \end{pmatrix}_{\underset{\sim}{\xi}} \qquad \text{and} \qquad \vec{\lambda}^3_{\underset{\sim}{\xi}} \; = \; \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}_{\underset{\sim}{\xi}} \; , \tag{4.23}$$

and

$$q^1_{\underset{\sim}{\xi}} \; = \; \chi\nu^- \mathrm{W}^2_\xi \; , \qquad q^2_{\underset{\sim}{\xi}} \; = \; \chi\nu^- \mathrm{W}^1_\xi \qquad \text{and} \qquad q^3_{\underset{\sim}{\xi}} \; = \; 0 \; . \tag{4.24}$$

Th distribution of th d coupl d compon nt of this s cond d composition is onc again carri d out using th scalar upwind sch m of S ction 5.1 b low. It is possibl to us th sam m thod for th coupl d compon nts, with an appro- priat modification (d scrib d in S ction 5.2) to nsur that th sch m r mains lin arity pr s rving [18] in th pr s nc of sourc t rms. How v r positivity is lost as a cons qu nc , so wh n th Froud numb r is clos to unity and th ad- v ction v lociti s associat d with ach compon nty v ry clos ly align d, th distribution provid s v ry littl cross-str am diffusion and as a r sult lacks robustn ss. Th actual scalar sch m us d h r is th SUPG sch m sugg st d in [18] and d scrib d at th nd of S ction 5.2. As in S ction 4.1 d∆∆TD5ion stat solutions of (5.1) ar calculat d by r p ating this updat it rativ ly in ord r to approximat th solution in th limit as $t \to \infty$.

Th v ctor $\vec{\lambda}$ in (5.2) may not b constant, in which cas a cons rvativ lin arisation of th scalar adv ction quation (5.1) can oft n b construct d by tr ating it as a sp cial cas of th syst m lin arisation discuss d in S ction 3 [8].

to a nodal updat of th form

$$u_i^{n+1} \ = \ u_i^n + \frac{\Delta t}{S_i} \sum_{\cup\triangle_i} \alpha_i^j \phi_j \ , \tag{5.6}$$

wh r S_i is th ar a of th m dian dual c ll for nod i (on third of th total ar a of th triangl s with a v rt x at i), α_i^j is th distribution co ffici nt which indicat s th proportion of th fluctuation ϕ_j to b s nt from c ll j to nod i, and $\cup\triangle_i$ r pr s nts th s t of c lls with v rtic s at nod i. It can b s n from th s cond xpr ssion for ϕ

- Lin arity pr s rvation - th xact st ady stat solution is pr s rv d wh n this vari s lin arly in spac , so no updat is s nt to th nod s wh n a c ll fluctuation is z ro and th sch m is s cond ord r accurat at th st ady stat on a r gular m sh with a uniform choic of diagonals [8].

- Continuity - th contributions to th nod s, $\alpha_i^j \phi_j$ (5.6), d p nd continuously on th data, avoiding limit cycling as conv rg nc is approach d and improving th robustn ss of th sch m .

Lin arity pr s rvation should also b satisfi d by th d composition, so that no updat is s nt to th v rtic s of a c ll wh n its flux balanc is z ro and th high r ord r accuracy poss ss d by th lin arity pr s rving scalar sch m is r tain d by th ov rall algorithm. Th prop rty is obviously satisfi d by th two d compositions d scrib d h r b caus th columns of th matrix $\mathbf{R}_{\mathrm{U}}$ (4.14) ar lin arly ind p nd nt.

A simpl distribution sch m with all of th abov prop rti s is th so-call d PSI sch m [8]. It is most asily d scrib d by consid ring a singl triangular c ll in isolation. If, according to th lin aris d adv ction v locity, $\widehat{\vec{\lambda}}$ of (5.5), th triangl has a singl downstr am v rt x, at nod i say, th n that nod r c iv s th whol fluctuation, so

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{S_i} \widehat{\phi} , \qquad (5.8)$$

whil th valu s of u at th oth r two v rtic s r main unchang d. In th cas of a triangl with two downstr am v rtic s, at nod s i and j for xampl , th fluctuation is divid d b t fluctuation can th r for b writt n

$$
\begin{aligned}
u_i^{n+1} &= u_i^n + \frac{\Delta t}{S_i}\phi_i^* \,, \\
u_j^{n+1} &= u_j^n + \frac{\Delta t}{S_j}\phi_j^* \,,
\end{aligned}
\tag{5.9}
$$

wh r $\phi_i^* + \phi_j^* = \widehat{\phi}$ for cons rvation. In th PSI sch m [22]

$$
\begin{aligned}
\phi_i^* &= \phi_i - L(\phi_i, -\phi_j) \\
\phi_j^* &= \phi_j - L(\phi_j, -\phi_i) \,,
\end{aligned}
\tag{5.10}
$$

wh r

$$
\phi_i = -\frac{1}{2}\widehat{\vec{\lambda}} \cdot \vec{n}_i \,(u_i^n - u_k^n) \,, \qquad \phi_j = -\frac{1}{2}\widehat{\vec{\lambda}} \cdot \vec{n}_j \,(u_j^n - u_k^n) \,, \tag{5.11}
$$

and L d not s th minmod limit r function,

$$
L(x, y) = \frac{1}{2}(1 + \mathrm{sgn}(xy))\,\frac{1}{2}(\mathrm{sgn}(x) + \mathrm{sgn}(y))\,\min(|x|, |y|) \,. \tag{5.12}
$$

Th PSI sch m is positiv for a r striction on th tim -st p at a nod i giv n by

$$
\Delta t \le \frac{S_i}{\sum_{\cup\triangle_i} \max(0, \frac{1}{2}\widehat{\vec{\lambda}}^j \cdot \vec{n}_i^j)} \,, \tag{5.13}
$$

and is us d in th ov rall algorithm for th distribution of th homog n ous scalar compon nts which aris from th d compositions of S ction 4.

5.2 Distribution of Coupled Components/Subsystems

Th lliptic natur of th 2×2 subsyst m which r sults from th d composition of th shallow wat r quations in subcritical flow sugg sts that an upwind distribution strat gy is l ss appropriat than for th scalar compon nts. Two sch m s ar d scrib d h r for th distribution of this compon nt, on for ach d composition, following th diff r nt distributions sugg st d for th corr sponding d compositions of th Eul r quations [16, 18].

In th first d composition (HELW) th two coupl d quations ar mod ll d by th syst m

uu

Th s cond approach (HESUPG) quat s th coupl d subsyst m with a pair of scalar adv ction quations with sourc t rms of th form

$$u_t + \vec{\lambda} \cdot \vec{\nabla} u \;=\; q \;, \tag{5.20}$$

in which u, $\vec{\lambda}$ and q ar d fin d by th first two ntri s in (4.19), (4.23) and (4.24) r sp ctiv ly. In [18] th quantity

$$\widehat{\phi_q} \;=\; -S_{\triangle}\,(\hat{\vec{\lambda}} \cdot \widehat{\vec{\nabla} u} - \hat{q}) \;, \tag{5.21}$$

is distribut d for ach of th two quations using a sch m which is quival nt to a mass-lump d str amlin upwind P trov-Gal rkin (SUPG) finit l m nt sch m with additional artificial viscosity [4].

Th distribution co ffici nts for this lin arity pr s rving and continuous but non-positiv sch m ar giv n by

$$\alpha_i^j \;=\; \frac{1}{3} + \tau \frac{\vec{\lambda} \cdot \vec{n}_i}{2 S_{\triangle_j}} + \kappa \frac{\vec{\nabla} u \cdot \vec{n}_i}{2 S_{\triangle_j}} \;, \tag{5.22}$$

in which

$$\tau \;=\; C_1 \frac{h}{|\vec{\lambda}|} \;, \qquad \kappa \;=\; C_2 \frac{h \,\mathrm{sgn}(\hat{\phi})}{|\vec{\nabla} u| + h} \;. \tag{5.23}$$

Th constants C_1 and C_2 ar both tak n to b 0.5 [13], h is a typical local l ngth scal , *e.g.* th l ngth of th long st dg of th c ll, and $\hat{\phi}$ is d fin d in (5.3). This sch m is us d h r for th distribution of th coupl d quations which r sult from th subcritical HESUPG d composition.

6 Source Terms

Sourc t rms app ar in th lin aris d shallow wat r quations both as a r sult of mod lling b d slop and friction (2.1) and from th lin arisation (3.14), and th s t rms must b includ d in th updating of th solution.

Th simpl st m thod of tr ating th mom ntum sourc s, $\underline{q}$ in (2.1), is to calculat th m pointwis at ach nod and th n add th m to th cons rvativ variabl s onc th flux balanc distribution has b n compl t d, so

$$\underline{U}_i^{n+1} \;=\; \underline{U}_i^n + \delta\underline{U}_i + \Delta t\,\underline{q}_i\;, \tag{6.1}$$

in which $\delta\underline{U}_i^n$ is th updat indicat d by th distribution of th d compos d flux balanc . How v r, it is mor appropriat to th sch m s pr s nt d h r for all of th sourc s to b incorporat d within th flux balanc distribution its lf. This is th obvious way to tr at th lin arisation sourc t rms sinc th y ar inh r ntly c ll-bas d quantiti s.

On way of achi ving this is to includ th sourc t rms within th d composition, so th charact ristic quations of (4.7) b com

$$\underline{W}_t + \mathbf{A}_{\mathrm{W}}^{\xi}\underline{W}_\xi + \mathbf{B}_{\mathrm{W}}^{\xi}\underline{W}_\eta \;=\; \mathbf{R}_{\mathrm{U}}^{-1}\underline{q}_{\mathrm{tot}}\;, \tag{6.2}$$

wh r $\underline{q}_{\mathrm{tot}}$ is th sum of th mom ntum and lin arisation sourc t rms consist ntly valuat d from th c ll-av rag stat $\overline{\underline{Z}}$. Th two typ s of sourc t rm can b consid r d s parat ly but ar combin d h r for simplicity.

Th ff ct of $\underline{q}_{\mathrm{tot}}$ on th flux balanc distribution can b illustrat d simply by consid ring a scalar compon nt of th d composition. A charact ristic quation tak n from (4.11) now has th form

A positiv distribution sch m do s not r main positiv und r this modification but th lin arity pr s rvation prop rty is r tain d by calculating th distribution co ffici nts pr cis ly as in th homog n ous cas but th n using th m to distribut th quantity $\widehat{\phi_q}$. Th modifi d updat s ar th n transform d into incr m nts of th cons rvativ variabl s using th matrix $\mathbf{R}_{\mathrm{U}}$ (4.14) as b for . Th sourc t rms which now app ar in th   lliptic subsyst m can also b tr at d in this mann r for both th matrix and scalar distributions.

A third m thod of tr ating th sourc t rm $\underline{q}_{\mathrm{tot}}$ is to distribut it s parat ly from $\underline{\Phi}_{\mathrm{U}}$, and th simpl st way to do this is via a symm tric distribution in which on third of $\underline{q}_{\mathrm{tot}}$ within a c ll is s nt to ach of its v rtic s. All thr   ways of incorporating th sourc t rms ar consid r d in th following s ction.

7 Results

Both algorithms d scrib d in th pr vious s ctions (HELW and HESUPG) hav b n us d to solv num rically a wid vari ty of st ady stat t st cas s for th two-dim nsional shallow wat r quations. In all cas s th lin arisation sourc t rms ar distribut d s parat ly from th r st of th flux balanc by a simpl c ntral sch m sinc this strat gy prov s to b mor robust than an upwind distribution and th r is n gligibl diff r nc b tw n th r sults. Th mom ntum sourc s, wh n th y app ar, ar distribut d in an upwind mann r as part of th flux balanc for th purpos s of accuracy, xc pt wh n robustn ss b com s an issu in which cas th y ar consid r d s parat ly and valuat d on a pointwis basis.

Th boundary conditions ar appli d v ry simply by r f rring to th th ory of charact ristics. This d t rmin s th num which should b impos d at a chos n point on th boundary. On condition must b appli d for ach positiv ig nvalu of th matrix

$$\mathbf{C}_{\mathrm{U}} = \mathbf{A}_{\mathrm{U}}\, n_x + \mathbf{B}_{\mathrm{U}}\, n_y \ , \tag{7.1}$$

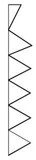
wh r $\vec{n} = (n_x, n_y)^{\mathrm{T}}$ is th inward pointing normal to th boundary of th computational domain. In th cas of th shallow wat r quations th s ig nvalu s ar giv n by

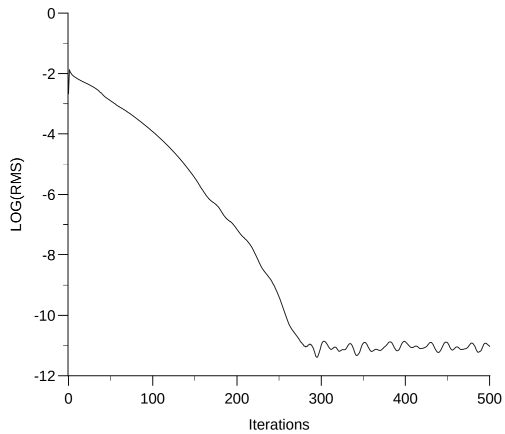
$$\lambda_1 = \vec{u} \cdot \vec{n} \ , \qquad \lambda_2 = \vec{u} \cdot \vec{n} + c \qquad \text{and} \qquad \lambda_3 = \vec{u} \cdot \vec{n} - c \ . \tag{7.2}$$

Thus, wh n th compon nt of th flow normal to th boundary is sup rcritical ith r th whol solution is sp cifi d (at inflow) or non of it (outflow). For subcritical inflow two conditions ar sp cifi d (total h ad and tang ntial v locity compon nt) whil for subcritical outflow a singl pi c of information, th d pth of th flow, is s t to a pr sp cifi d fr str am valu . At a solid wall only λ_2 is positiv and this is accommodat d by s tting th normal v locity compon nt to z ro.

7.1 Oblique Hydraulic Jump

F w standard st ady stat t st cas s xist for th homog n ous two-dim nsional shallow wat r quations, but th r ar som simpl probl ms for which xact solutions hav b n calculat d. On such xampl [3] is sup rcritical flow through a frictionl ss chann l with a flat b d containing a w dg inclin d at an angl θ to th dir ction of th flow at which an obliqu hydraulic jump is induc d by th int raction of th flow with th front of th w dg . Th angl β which this





Figur 7.2: Conv rg nc history for th obliqu hydraulic jump t st cas .

walls. This figur also shows th local Froud numb r contours of th st ady stat solution calculat d for this t st cas (both th HELW and th HESUPG sch m s ar th sam for sup rcritical flow). Th hydraulic jump can b s n to b captur d sharply at th corr ct angl and a discontinuous wat r surfac d void of oscillations is obtain d. Th valu s of th flow variabl s downstr am of th jump (sampl d on th outflow boundary at th point indicat d by th ast risk in Figur 7.1) ar $h_d = 1.5001$m and $|\vec{u}_d| = 7.9506$ms^{-1} ($F_d = 2.073$), v ry clos to th xact v

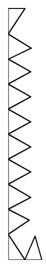
Eul r quations. N ith r t chniqu is us d h r but it is xp ct d that both could b us d to similar advantag .

Not that a CFL numb r of 0.7 has b n us d h r but in th subs qu nt t st cas s, all of which hav r gions of subcritical flow, th CFL numb r is tak n to b 0.2 which prov d to b th high st valu which could b tak n which was stabl for all of th s cas s. This s ms to b b caus of th discontinuity in th distribution at th critical lin and th nonorthogonality of th ig nv ctors of th pr condition d syst m at low Froud numb rs (d scrib d in mor d tail in [6]).

7.2 Symmetric Constricted Channel Flows

Th domain for th s t st cas s r pr s nts a chann l of l ngth 4 m tr s and width 1 m tr with bumps of th sam shap and siz in th c ntr of ith r wall of th chann l. Th bumps ar on m tr in l ngth and ar d fin d such that th br adth of th chann l is giv n by

$$B \;=\; B_0 - 2B_h \cos^2\left(\frac{\pi(x-x_c)}{x_l}\right) \qquad \text{for} \qquad |x-x_c| \;\leq\; \frac{x_l}{2}\,,$$

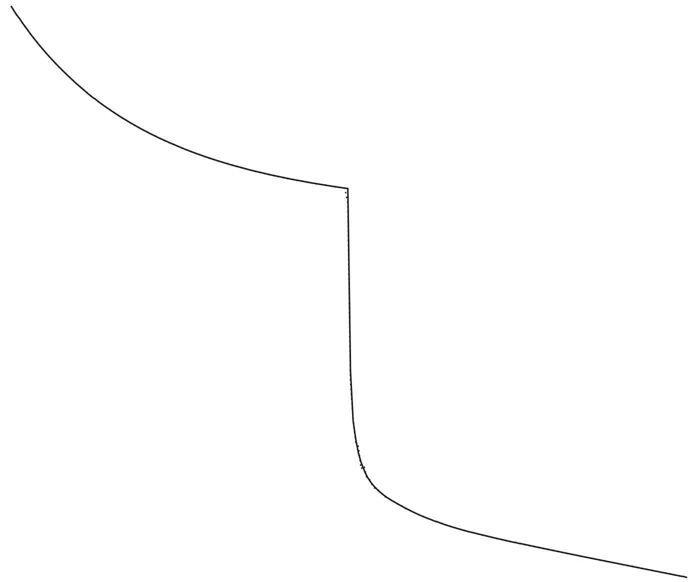


is compl t ly subcritical and th r for symm tric about th c ntr of th con-

th subcritical lliptic subsyst m would r duc th oscillations, but not without sm aring th discontinuity as w ll. Although th HESUPG sch m , with its gr at r inh r nt num rical diffusion, do s this automatically, n ith r tr atm nt is id al and th mod lling of transcritical flows r quir s furth r consid ration.

Th r sults shown in Figur 7.6 illustrat th ff ct of th lin arisation sourc t rms on th solution. Th valu s of th br adth-av rag d local Froud numb r ar plott d along th l ngth of th chann l for th HELW sch m . Th small oscillations downstr am of th jump ar r nd r d almost invisibl by th av raging proc dur and th solutions ar v ry clos to thos produc d by th HESUPG sch m (not plott d h r) although th latt r pr dicts th on -dim nsional discontinuity to b v ry slightly furth r upstr am.

Th num rical r sults shown ar for a cons rvativ and a non-cons rvativ formulation in which th lin arisation sourc s ar simply ignor d. Clos insp ction r v als that th discontinuity is pr dict d to b about half a c ll's width furth r downstr am by th non-cons rvativ sch m . On a grid in which th c ll dg s ar align d with th discontinuity th discr pancy in jump position b tw n th cons rvativ and non-cons rvativ sch m s can b as much as on c ll. Th non-cons rvativ formulation pr dicts th jump to b furth r away from th xact position pr dict d by on -dim nsional th ory for an op n chann l of varying width, th third solution shown in Figur 7.6. Thus it is important to nforc cons rvation for pr cis positioning of th discontinuity v n though an ad quat solution may b obtain d in this cas without cons rvation. Not also that th av rag d cons rvativ num rical approximation pass s from subcritical to sup rcritical flow at th c ntr of th chann l (its narrow st point) as it should



th br adth-av rag d local Froud numb r pr dict d by th two upwind sch m s is shown in Figur 7.8, tog th r with th xact solution. Th thr solutions ar almost indistinguishabl and v n th HELW sch m xhibits no small oscillations on th subcritical sid of th jump.

Both s ts of r sults pr s nt d hav b n obtain d using an upwind distribution of mom ntum sourc s valuat d on a c ll by c ll basis. Clos xamination of th solutions r v als that this m thod of tr ating th s sourc t rms l ads to th b st approximation of th xact solution, although th diff r nc s would not b visibl in th figur . It should b not d though that conv rg nc to th st ady stat is slightly b tt r if th sourc s ar incorporat d at th nodal updat stag , indicating gr at r robustn ss. In actual fact non of th sch m s conv rg to machin accuracy in this transcritical cas so no conv rg nc histori s ar shown.

7.4 Spillway Flow

Th final probl m r pr s nts shallow wat r flow in a spillway and provid s a g nuin ly two-dim nsional t st cas . Th flow is through a chann l 10m wid with a right angl d b nd half way along its l ngth. Th inn r and out r corn rs of th b nd ar tak n to b arcs of conc ntric circl s with radii 10m and 20m r sp ctiv ly, and th r is 30m of straight chann l both upstr am and downstr am of th b nd. Th flow is sup rcritical at both inflow and outflow and th inflow conditions ar such that $h = 1\text{m}$, $u = 0\text{ms}^{-1}$ and $v = -5\text{ms}^{-1}$. Th slop of th chann l is of magnitud 1 in 5 along th straight s ctions and vari s lin arly across th chann l around th b nd from 1 in 5 on th out r curv to 2 in 5 on th inn r curv . Manning's roughn ss co ffici nt for th flow tak s th valu of

8 Conclusions

Two alt rnativ g nuin ly multidim nsional upwind sch m s hav b n pr s nt d for th num rical solution of th two-dim nsional shallow wat r quations on unstructur d triangular grids. T chniqu s which w r originally d v lop d for th solution of th Eul r quations hav b n adapt d for th approximation of th shallow wat r quations and a cons rvativ formulation of th algorithm has b n pr s nt d. A m thod of tr ating g n ral sourc t rms, appropriat for us with multidim nsional upwinding, has also b n sugg st d.

Both sch m s pr s nt d h r hav b n shown to produc high quality r sults for both subcritical and sup rcritical st ady stat flows and hav th ability to captur discontinuiti s v ry sharply. Of th two, th HELW sch m is th l ss diffusiv and as a r sult, slightly l ss robust. It produc s sharp r jumps and mor accurat subcritical approximations, but at th xp ns of small oscillations downstr am of transcritical discontinuiti s. Th s could b smooth d by adjusting th Lax-W ndroff distribution appropriat ly, with a cons qu nt sm aring of th discontinuity, but both sch m s r quir furth r study to improv th mod lling of th transition b tw n sup rcritical and subcritical flows.

Th tr atm nt of th mom ntum sourc s also m rits att ntion sinc it is still uncl ar which is th b st m thod to us for th ir distribution. Th most robust tr atm nt prov d to b to consid r th sourc s on a nodal basis but distributing c ll-av rag d sourc t rms in an upwind mann r app ars to b mor accurat . It is cl ar though that th lin arisation sourc s ar n c ssary for th pr cis positioning of th hydraulic jumps although in many cas s ad quat num rical solutions can b obtain d using a non-cons rvativ formulation.

Multidim nsional up

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