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Use of Potential Vorticity for Incremental Data Assimilation

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Abstract

Decomposing the mass and wind fields in a data assimilation scheme into balanced and unbalanced flow is part of the process of defining a covariance model. It is not uncommon to assume that the dynamic balanced part of the flow is approximated solely by the rotational part of the wind, which is obtained from a Helmholtz decomposition of the horizontal momentum (with an associated balanced pressure being diagnostically inferred from a balance equation, for example). The unbalanced flow is then represented by the divergence and the residual unbalanced pressure. The assumption that the rotational part of the momentum is a good approximation to the total balanced flow is only valid in certain regimes. We propose a new approach that incorporates flow regime dependence, where we assume that the balanced part of the flow is approximated instead by a linearised potential vorticity increment. We show the benefit of such a formulation in the context of shallow water equations defined on a hemisphere.

Keywords Potential Vorticity, Data Assimilation, Linearisation, Rossby-Haurwitz Waves, Burger number, Incremental Variational.

1 Introduction

Lorenc (2003, §3(*b*)) points out that physical arguments are frequently used to select a set of variables (so-called `control variables' in Numerical Weather Prediction (NWP)) for use in data assimilation schemes, which decompose atmospheric states into balanced and unbalanced components. Such variables enable the background error covariances to be more readily applied than would be the case if, for example, the usual model variables of momentum and pressure were used. Most numerical weather centres that perform data assimilation represent the balanced and unbalanced parts of the flow in a simplified, flow independent fashion that assumes that the balanced flow is just the rotational part, with the associated balanced pressure field derived using a linear balance equation. The `fast modes' (i.e. the gravity wave activity) are represented by the divergence and a residual unbalanced pressure. While it is true that the balanced flow is predominantly rotational, choosing the rotational wind as the `slow variable' is only applicable in process does not cost significantly more computationally than at present, and this is an important constraint in schemes that may involve solving complicated elliptic boundary value problems.

We propose to use a low order potential vorticity (PV) inversion scheme to select a set of control variables that separate the balanced and unbalanced components of the flow in a more flow dependent manner. We make the assumption that the potential vorticity contains the balanced part of the flow, and that the unbalanced flow lies in the kernel of the PV operator (i.e. that flow with PV=0). McIntyre and Norton (2000) discuss various hierarchical approximations to potential vorticity inversion on a hemisphere within a shallow water context. We propose a PV inversion scheme that is similar to their first order direct inversion scheme, except that we use the linear balance equation instead of Charney balance as our associated balance condition. We also work with a linearisation of the potential vorticity because our scheme is designed to work with incremental data assimilation schemes. Our PV inversion scheme is consistent with the shallow water equations linearised about a resting state.

We discriminate between different flow regimes by using the Burger number. This is a measure of the stable stratification of a fluid: when the Burger number is low, the height, or depth, is an appropriate measure of the balanced component of the flow; but when the Burger number is high, the vorticity is the appropriate measure. Another way of saying this is to note that the PV behaves like the reciprocal of the height at low Burger numbers and like the vorticity at high Burger numbers. Therefore, in our application in data assimilation, the PV should be a better representation of the balanced flow where and when the Burger number is small, while giving approximately similar results to the vorticity when the Burger number is large.

We present theoretical and numerical aspects of this PV inversion and show the benefits of such a scheme when compared to a scheme in which the balanced component is represented by the rotational flow. In section 2 we present the theoretical aspects, giving a rationale for using potential vorticity within a shallow water context. We show how the relative contributions to scaled potential vorticity perturbations vary with Burger number. In section 3 we state the numerical method that is used and section 4 we present the numerical results.

2 Theory

2.1 Introduction

In this section we explain why we want to use a potential vorticity (PV) inversion scheme in data assimilation to separate the key dynamical aspects of the flow. Most operational centres use just the rotational and divergent parts of the flow for this purpose. It is our aim to show that using a PV inversion scheme is a more consistent approach as it takes account of the regime dependence of the flow.

The first step is to establish the standard rationale for using the streamfunction ψ, a scalar quantity representing the rotational wind, as the key variable representing the dominant behaviour of the flow. This is necessary in order to show that using PV inversion is an improvement on the standard method.

We use the nonlinear shallow water equations on a sphere. Most meteorological textbooks (eg Haltiner et al 1980) show that through a scale analysis, key non-dimensional numbers are found whose values characterize the flow. One such dimensional number is the Burger number. Cullen (2002) showed that in a high Burger regime, the shallow water equations (SWE) approximate a balanced model called 2D Euler in the asymptotic limit as the Burger number gets much larger than unity. In this balanced model the absolute vorticity, a scalar quantity describing the amount of rotational wind, is materially conserved by the flow. If we are dealing with shallow water °ow in a regime where it behaves similarly to inviscid incompressible 2D Euler, it would be sensible to approximate the dominant aspects of the shallow water °ow by a key variable of the 2D Euler model, namely the streamfunction $\tilde{A}$. This quantity can be diagnosed using a where U is the characteristic velocity.

For the shallow water equations, in the asymptotic limit as the Burger number gets large, $B_u >> 1$, where the Rossby number is kept small, $R_o << 1$, the typical spatial differences in depth become increasingly less important compared to the effect of the characteristic depth within equation (2). In this case, the continuity equation effectively degenerates into a 2D incompressibility condition. This enforces a non-divergent flow described by incompressible 2D Euler as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{k} \times \mathbf{v} + g \nabla h = 0, \tag{5}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{6}$$

Taking the vertical component of the curl of the momentum equation (5) gives the barotropic vorticity equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + \mathbf{v}_\psi \cdot \nabla \left(f + \nabla^2 \psi \right) = 0, \tag{7}$$

where $\mathbf{v}_\psi$ is the rotational, non-divergent part of the wind and the streamfunction ψ is defined by

$$\nabla^2 \psi = \mathbf{k} \cdot \nabla \times \mathbf{v}.$$

achieved using a Helmholtz decomposition, which splits the horizontal wind into a rotational, non-divergent part $\mathbf{v}_{\psi}$ and a divergent, irrotational part $\mathbf{v}_{\chi}$ such that

$$\mathbf{v} = \mathbf{v}_{\psi} + \mathbf{v}_{\chi}. \tag{11}$$

The rotational winds are de¯ned through the streamfunction in the usual way by

$$\mathbf{v}_{\psi} = \mathbf{k} \times \nabla\psi.$$

Operational data assimilation systems separate the mass and wind into parts that approximate the Rossby slow mode and two inertio-gravity modes. Most of these systems use the Helmholtz decomposition to approximate the evolution of the streamfunction ψ on each horizontal level and hence, together with the linear balance equation, separate and identify the mass-wind contribution to the slow Rossby mode. The separation is assumed to be accurate across all Burger regimes.

2.6 Relationship of governing equations to semi-geostrophic shallow water equations

In the shallow water context, the assumption that the streamfunction represents the slow mode becomes increasingly less accurate as the Burger number becomes smaller, $B_u << 1$, while the Rossby number remains small, $R_o << 1$. When the value of the Burger number is smaller than unity it is more appropriate to choose semi-geostrophic shallow water equations as the reference balanced model than inviscid incompressible 2D Euler. This is because it is a more accurate approximation in this regime. The smaller the Burger number, the closer the behaviour of the shallow water equations is to the semi-geostrophic shallow water equations. Further details of the semi-geostrophic shallow water equations can be found in Cullen (2002). It is su±cient to say that all the terms in the continuity equation are important and that this equation no longer reduces into an incompressibility condition. For a small Burger number less than one, the depth ¯elds produced by the standard shallow water equations and the semi-geostrophic shallow water equations are similar, as are the qualitative features of semi-geostrophic and Ertel potential vorticities. In contrast, the rotational wind aspects of the two models di®er substantially.

2.7 Linearisation, linearisation states and the LB method

In data assimilation we are interested in an incremental linearised formulation. We de¯ne our increments as the di®erence between the full heights and linearisation states. In particular, the height and wind increments are de¯ned by

$$\begin{aligned} h' &= h - \overline{h}, \\ \mathbf{v}' &= \mathbf{v} - \overline{\mathbf{v}}, \end{aligned}$$

where the prime variables denote increments and overlined variables represent linearisation states. In this paper, due to limitations of the numerical method, the linearisation states $\overline{h}, \overline{u}, \overline{v}$ are a function of latitude only. The way linearisation states are chosen depends on the situation, but in all cases there is a consistency between the values given to the linearisation states.

The linear equations introduced prior to this section all hold both for increments and for linearisation states, as well as for full ¯elds. In particular, equation (8) can be used to identify ψ' from $\mathbf{v}'$ and $\overline{\psi}$ from $\overline{\mathbf{v}}$.

The streamfunction increment can be considered to represent the balanced part of the incremental flow and the linear balance equation (10) can be used to find the balanced height increment h_b' from the streamfunction increment ψ'. This method for decomposing the flow is referred to here as the LB method. In summary we state the method as

1. calculate the streamfunction increment from the wind increments using equation (8) and assume it is balanced, i.e. $\psi_b' = \psi'$

2.9 The PV method

The following coupled system of partial differential equations (PDE)s, derived from (15) and (10), defines our decomposition of the slow dynamics:

$$\nabla \cdot f \nabla \psi'_b - g \nabla^2 h'_b = 0; \quad (16)$$

$$\nabla^2 \psi'_b - \overline{q} h'_b = \overline{h} q'; \quad (17)$$

which we solve, simultaneously for ψ'_b and h'_b, where $\overline{q}$, $\overline{h}$ and q' are known. The term $\overline{h} q'$ is precalculated from ψ' and h' by just rearranging equation (15) as

$$\overline{h} q' = -\overline{q} h' + \nabla^2 \psi'. \quad (18)$$

In addition, $\nabla^2 \psi'$ is obtained from the full wind increments $\mathbf{v}'$ using the equation $\nabla^2 \psi' = \mathbf{k} \cdot (\nabla \times \mathbf{v}')$.

The coupled system (16), (17) defines a balanced height increment h'_b and a balanced wind increment, given by $\mathbf{v}'_b = \mathbf{k} \times \nabla \psi'_b$. The balanced wind increment is non-divergent and approximates the full rotational wind increment for high Burger number regimes. The rest of the rotational wind is described as having no potential vorticity increment and conserving a departure from linear balance. The unbalanced rotational wind can be obtained in one of two ways, either by subtracting the balanced wind and height from the full rotational wind and height, or by explicitly solving the simultaneous system

$$\nabla \cdot f \nabla \psi'_{ub} - g \nabla^2 h'_{ub} = \nabla \cdot f \nabla \psi' - g \nabla^2 h'; \quad (19)$$

$$\nabla^2 \psi'_{ub} - \overline{q} h'_{ub} = 0; \quad (20)$$

where the unbalanced rotational wind is defined to be $\mathbf{v}'_{ub} = \mathbf{k} \times \nabla \psi'_{ub}$. The unbalanced height is denoted by h'_{ub} and again, on the right hand side, we use known full increments ψ' and h'. The equivalence of the two methods to calculate the unbalanced height and unbalanced rotational wind is readily seen by adding equation (16) to (19) and (17) to (20), to give

$$\nabla \cdot f \nabla \left(\psi'_b + \psi'_{ub}\right) - g \nabla^2 \left(h'_b + h'_{ub}\right) = \nabla \cdot f \nabla \psi' - g \nabla^2 h'; \quad (21)$$

$$\nabla^2 \left(\psi'_b + \psi'_{ub}\right) - \overline{q} \left(h'_b + h'_{ub}\right) = \overline{h} q'. \quad (22)$$

The remaining wind increment, namely the divergent part $\mathbf{v}_{\chi'}$, is stored in the velocity potential χ', which is defined as

$$\nabla^2 \chi' = \nabla \cdot \mathbf{v}'. \quad (23)$$

2.10 Dynamic dependence of linearised potential vorticity on Burger regime

It is now appropriate to describe the effect of different Burger regimes on linearised potential vorticity and linear balance increments. We consider properties of height, vorticity (the Laplacian of the streamfunction) and potential vorticity increments that satisfy both the linearised potential vorticity relationship (15) and the linear balance equation, (10) when the Coriolis term is constant. It is valid to consider relative vorticity perturbations $\nabla^2\psi'$ since the linear balance equation (10) for constant $f = f_0$ is equal to

$$g\nabla^2 h' = f_0\nabla^2\psi'. \tag{24}$$

Let us assume that the height and vorticity perturbations are on a Cartesian grid for consistency in assuming constant f and assume that these perturbations and the reference linearisation states are known.

We use the relationship (24) between the perturbation in relative vorticity and the height to derive a relationship between the potential vorticity perturbation and the height. We consider perturbations in the height and the vorticity that take the form $h' = \hat{h}e^{i(k_1x+k_2y-\sigma t)}$, $\zeta' = \nabla^2\psi' = \hat{\zeta}e^{i(k_1x+k_2y-\sigma t)}$, where k_1 is the wave number in the x direction, k_2 is the wave number in the y direction and σ is the frequency. Also, we assume that the perturbations satisfy (24). Using these two assumptions,

$$\nabla^2\psi' = -\frac{(k_1^2+k_2^2)gh'}{f_0}. \tag{25}$$

If the characteristic length scale L is considered to be equal to $(k_1^2+k_1^2)^{-\frac{1}{2}}$, then the Burger number is equal to $(k_1^2+k_1^2)^{\frac{1}{2}}(g\bar{h})^{\frac{1}{2}}/f$, where the characteristic height scale H is considered to be equal to $\bar{h}$. By using (15), (25), two separate relationships can be determined (Wlasak, 2002): one defines scaled perturbations in potential vorticity in terms of scaled perturbations in height; the other shows how perturbations in scaled relative vorticity perturbations are related to scaled perturbations in potential vorticity. These relationships are given by

$$\frac{q'}{\bar{q}} = -N\frac{h'}{\bar{h}} \qquad \left(1-\frac{1}{N}\right)\frac{q'}{\bar{q}} = \frac{\nabla^2\psi'}{\nabla^2\bar{\psi}+f_0} \tag{26}$$

with

$$N = 1+\frac{f_0\,B_u^2}{f_0+\nabla^2\bar{\psi}}. \tag{27}$$

As the Burger number is always greater than zero, for any given perturbation, N is always greater than 1. For a fixed $q'/\bar{q}$ and $N \gg 1$, $h'/\bar{h}$ will not contribute much to the scaled potential vorticity perturbations; the potential vorticity perturbations are similar to the absolute vorticity perturbations with $q'/\bar{q} \approx \nabla^2\psi'/(f_0+\nabla^2\bar{\psi})$. Moreover, the greater the value of N, the more similar $q'/\bar{q}$ will be to $\nabla^2\psi'/(f_0+\nabla^2\bar{\psi})$. The equation (27) shows that a number of conditions can make N large. One possible way, assuming $(f_0+\nabla^2\bar{\psi})$ to be constant, is to produce a large Burger number. A large Burger number will be obtained when $\bar{h}$ is large or when f_0 is small. In summary, it is expected that for large Burger number $q'/\bar{q}$ will be dominated by $\nabla^2\psi'/(f_0+\nabla^2\bar{\psi})$.

The equations (26) and (27) can also be written as

$$\left(1-\frac{1}{P}\right)\frac{q'}{\bar{q}} = -\frac{h'}{\bar{h}} \qquad \frac{q'}{\bar{q}} = P\frac{\nabla^2\psi'}{f_0+\nabla^2\bar{\psi}} \tag{28}$$

with

$$P = 1 + \frac{f_0 + r^2\widetilde{A}}{f_0 B_u^2}:$$

We assume a solution of the form

$$
\begin{aligned}
h_b'(\lambda,\phi) &= \sum_{k=-M/2}^{k=M/2} \tilde{h}(k,\phi)e^{ik\lambda}, && (30)\\
\psi_b'(\lambda,\phi) &= \sum_{k=-M/2}^{k=M/2} \tilde{\psi}(k,\phi)e^{ik\lambda},\\
\overline{hq}'(\lambda,\phi) &= \sum_{k=-M/2}^{k=M/2} \widetilde{\overline{hq}'}(k,\phi)e^{ik\lambda},
\end{aligned}
$$

with M being an even integer setting a truncation limit to the Fourier approximation, k the wavenumber, and $i = \sqrt{-1}$. Since we have periodicity in the longitudinal direction, λ is discretised as

$$\lambda_j = ja\left(\frac{2\pi}{M+1}\right)\cos\phi, \qquad j = 1,\ldots,M+1, \tag{31}$$

where a is the radius of the earth. The Fourier coefficients $\tilde{h}(k,\phi)$, $\tilde{\psi}(k,\phi)$, $\widetilde{\overline{hq}'}(k,\phi)$ are complex.

Substitution of (30) into (16) and (17), produces a series of coupled systems of second order ODEs in ϕ to be solved for variables $\tilde{h}$, $\tilde{\psi}$. The system is given for each k by

$$-\frac{k^2}{a^2\cos^2\phi}[g\tilde{h} + f\tilde{\psi}] + \frac{g}{a^2\cos\phi}\frac{\partial}{\partial\phi}[\cos\phi\frac{\partial\tilde{h}}{\partial\phi} + f\cos\phi\frac{\partial\tilde{\psi}}{\partial\phi}] = 0 \tag{32}$$

$$-\frac{k^2}{a^2\cos^2\phi}[\tilde{\psi}] + \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}[\cos\phi\frac{\partial\tilde{\psi}}{\partial\phi}] - \overline{q}\tilde{h} = \widetilde{\overline{hq}'}. \tag{33}$$

In this situation, we have $M+1$ different complex coefficients for $\tilde{h}$, $\tilde{\psi}$, $\widetilde{\overline{hq}'}$ which are all functions of latitude λ_j and wave number k. Coefficients $\widetilde{\overline{hq}'}$ are known and $\tilde{h}$, $\tilde{\psi}$ are to be determined.

The system (32)-(33) is to be solved for each wavenumber k considered. The beauty of the separability of the coupled PDE's is now apparent; since $\overline{h}$, $\overline{q}$, $\overline{\psi}$ are functions of latitude only, there is no interaction of wavenumber coefficients and each system of ODEs for each wavenumber is solved independently.

3.3 Boundary conditions

To solve this system we need $\widetilde{\overline{hq}'}(k,\phi)$, which is derived by applying the Fourier transform to the increment $\overline{hq}'(\lambda,\phi)$, given by (18). We obtain used of the form,

$$\int \widetilde{h}_b dS \quad = \quad 0; \tag{36}$$

which is rewritten as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \widetilde{h}_b(0, \phi) \cos \phi d\phi \quad = \quad 0. \tag{37}$$

The global uniqueness condition applied to $\widetilde{\widetilde{A}}$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \widetilde{\widetilde{A}}_b(0, \phi) \cos \phi d\phi \quad = \quad 0; \tag{38}$$

is automatically satisfied due to the imposed anti-symmetric nature of the integrand. Once (32)-(33) has been solved for all wavenumbers considered, we synthesise the complex coefficients using a discrete inverse Fourier transform.

3.4 Scaling

A scaling is introduced to make terms in the discretised operator of approximately the same size. Scaling of equations is important so as to eliminate unnecessary sensitivity to numerical

Additional boundary conditions are applied to both variables. The streamfunction is assumed to be a continuous smooth function and must be zero at the equator due to the balanced streamfunction being anti-symmetric. At the equator, for $k = 0;$ we do not solve the coupled system as its stands but instead solve one equation in which the coupled equations have been added together. This single equation at the equator is discretised using fourth order centered di®erences.

An antisymmetric solution about the equator in balanced streamfunction enforces a symmetric solution in balanced height h_b and there is no need to enforce $\frac{@h_b}{@A} = 0$. Instead a discrete approximation to mass conservation is enforced, so that the balanced height ¯eld represents the same mass as the full height increment di®erence.

A unique solution is given, provided a compatibility condition is enforced on the full potential vorticity increment such that the volume-weighted sum of the discrete increments $\overline{h}q^0$ over the sphere is equal to zero (Swarztrauber,1974). This is automatically achieved due to the antisymmetric nature of the full potential vorticity increment.

4 Results

We investigate whether the PV method provides a better representation of balanced and unbalanced control variables than the LB method. If the coupled PV system is behaving properly, then in high Burger regimes the balanced streamfunction increment $\tilde{A}^0_b$ should be similar to $\tilde{A}^0$. Similarly, at low Burger regimes the balanced height increment should resemble the full height increment. We perform two sets of experiments: the ¯rst testing the PV and LB methods against a primarily balanced °ow de¯ned by the evolution of a Rossby-Haurwitz wave; the second testing the methods with height and wind increments that are essentially unbalanced.

For the ¯rst set of experiments, appropriate height and wind ¯elds need to be generated to test the coupled system. A global SWE model is run, using a Rossby-Haurwitz wave (RH wave) as an initial condition. The RH wave is de¯ned in the Appendix through an analytic expression and is used in standard test cases as an initial condition for testing global SWE models (Williamson et al, 1992). It has the property that the wind pattern is advected meridionally at a constant angular velocity on the sphere when propagated under incompressible 2D Euler equations. When used in the SWE context such behavior occurs in a regime where $B_u >> 1$, the characteristic height H is large and there is little divergent wind. The de¯ning parameters of the wave are given by the wavenumber R

The PV method produces both balanced height and streamfunction increments. Figure 1 compares the balanced streamfunction to the respective full ¯eld over the area (

Figure 3: Balanced Ã (left) and full Ã (right)for RH wave propagated 1 day at low Burger number, with (Á 2 [¼=2; ¡ ¼=2]) £ (µ 2 [0; ¼=2]) (scale denotes grid points)

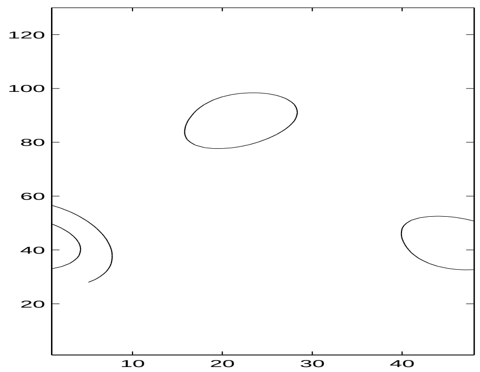
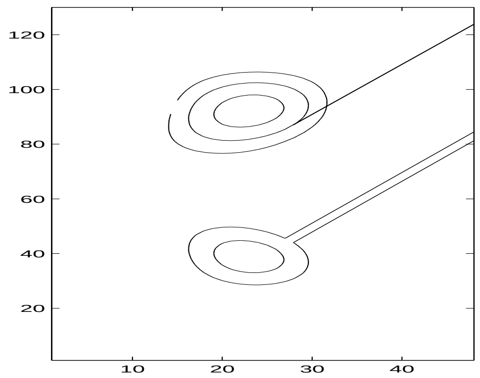


Figure 4: Balanced height (left) and full height (right) for RH wave propagated 1 day at low Burger number, with (Á 2 [¼=2; ¡ ¼=2]) £ (µ 2 [0; ¼=2]) (scale denotes grid points)



produced by a potential vorticity conserving initialisation scheme and subtracting them from the corresponding uninitialised field. The uninitialised field is obtained from a spherical harmonic description of the observed fields at T106 resolution from a NETCDF file *VDG7.13.cdf*, kept at NCAR and found in *ftp://ftp.cgd.ucar.edu/pub/jet/shallow/nminit/*. This experiment shows the strengths and weaknesses of the control variables that we have developed.

Figure 5 shows the height and wind increments used to test the control variables. A stereographic projection is used centred on the North Pole. The increments are composed of many different waves on a wide range of length scales. The wind increments are typically between $-8ms^{-1}$ and $8ms^{-1}$ and the height increments vary between $-60m$ and $60m$. It is also clear from the figure that there is great variability in the flow with waves of both short and long wavelengths present.

If the initialisation is perfect then the increments consist of just the unbalanced flow. A perfect set of control variables would apportion the flow into the two unbalanced variables.

Figure 5: (Top) U and V wind increments produced using test case $INI7C$, (bottom right) height increment using test case $INI7C$, (bottom left) U ¯eld linearisation state

Figure 7: (Top left) Height increment produced using test case $INI7C$. (Top right) balanced height increment produced by LB method. (Bottom left) Balanced height increment using PV method at low B_u. (Bottom right) Balanced height increment using PV method at high B_u

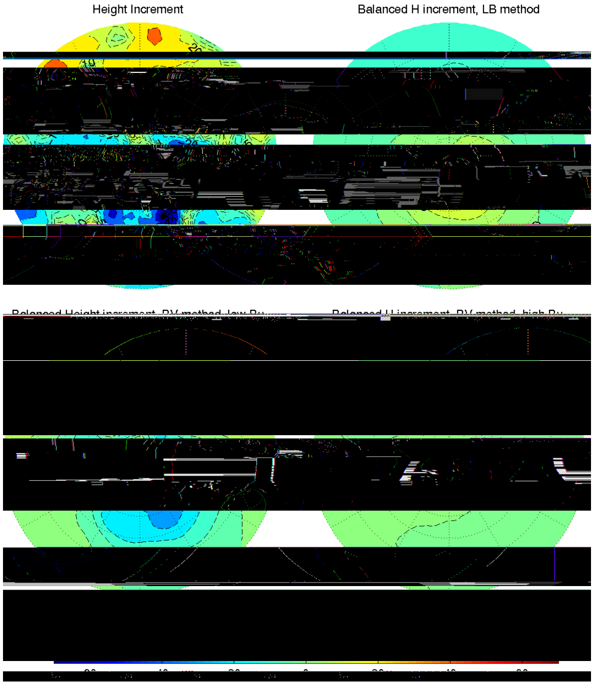
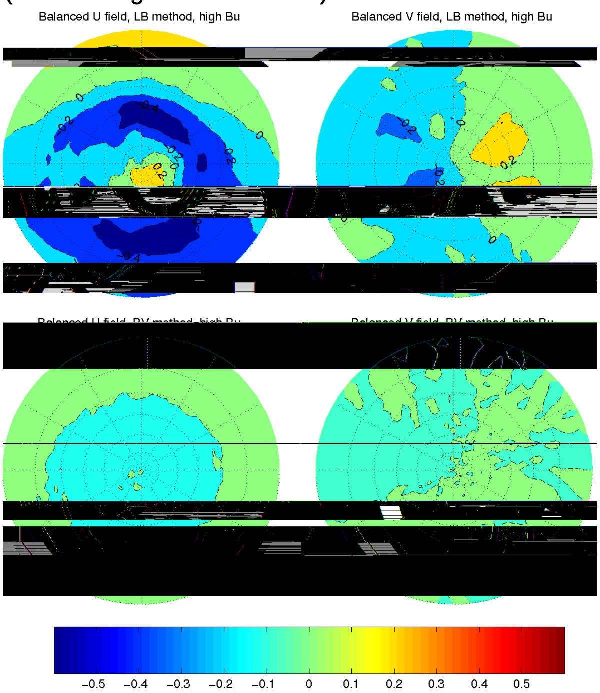


Figure 8: Balanced wind increments produced by using the LB and PV methods at high Bu (mean height H ¼ $11km$)



5 Conclusions

We propose a new PV-based approach to the separation of balanced and unbalanced flow that incorporates flow regime dependence. The benefits of this approach are demonstrated theoretically and experimentally in the context of the shallow water equations. The results concur with findings produced by Cullen (2003) in which a similar technique was applied in a dimensional context to the reformulation of the background error covariance within a four-dimensional data assimilation system. Although Cullen's method had difficulties in dealing with spurious modes produced by the vertical-staggered Lorenz grid used at ECMWF at the time, the findings from both studies are encouraging.

As shown here, the PV-based method at high Burger number produces control variables that are similar to those produced by the customary streamfunction-constrained LB method. At low Burger number the PV method produces control variables in which the full height increments/perturbations dictate the balanced height and wind fields. A difficulty arises in using a linearised potential vorticity increment at very low Burger number. The smaller the

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